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Boundary Element Tearing and Interconnecting Methods for Acoustic and Electromagnetic Scattering

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Abstract

The aim of this work is to analyze the boundary element tearing and interconnecting (BETI) approach for acoustic and electromagnetic scattering problems. Tearing and interconnecting methods are well known domain decomposition methods used for an approximate solution of various physical problems. The theory of this method is mainly based on partial differential equations, which imply elliptic bilinear forms. Since the acoustic and electromagnetic wave equations do not fulfill this requirement, the theory has to be modified. In this work, we establish a rigorous theory for the BETI method applied to the scattering problems. A main part is dedicated to the treatment of the local subproblems, since they are not necessary well posed if the standard approach is directly carried over to scattering problems. Therefore Robin interface conditions are introduced and local Steklov–Poincaré operators are exchanged by local boundary integral equation systems to guarantee well posedness of the local boundary value problems. Further a new combined field integral equation approach is established for exterior scattering problems. This approach differs from others, and it can easily be incorporated in the presented domain decomposition approach. Therefore bounded and unbounded domains can be treated in a unified way. Further an alternative deduction of the tearing and interconnecting approach is presented, which is solely based on algebraic arguments. Therefore it does not depend on the properties of the underlying partial differential equation, what is not the case in the classical deduction. Further we discuss a preconditioning approach for the BETI method in the acoustic case. This approach is carried over from the FETI–H method. Finally, numerical examples are given which confirm the theoretical results.

Zusammenfassung

Ziel dieser Arbeit ist es, die „boundary element tearing and interconnecting“–Methode (BETI) für akustische und elektromagnetische Streuprobleme zu analysieren. Die „tearing and interconnecting“–Methode ist eine etablierte Gebietszerlegungsmethode, welche für vielfältige physikalische Probleme verwendet wird. Jedoch basiert die Herleitung dieser Methode auf der Annahme, dass die zugrunde liegenden partiellen Differentialgleichungen zu elliptischen Bilinearformen führen. Da die akustische und die elektromagnetische Wellengleichung nicht in diese Kategorie fallen, kann die ursprüngliche Theorie nicht direkt angewendet werden. Teil dieser Arbeit ist es, die zugrunde liegende Theorie dementsprechend anzupassen. Ein wichtiger Teil dieser Anpassung ist die Umformulierung der lokalen Lösungsstrategie, da diese im üblichen Ansatz nicht zu notwendigerweise wohlgestellten Problemen führt. Auf Grund dessen werden die Neumann–Transmissionsbedingungen des klassischen Ansatzes durch sogenannte Robin–Transmissionsbedingungen ersetzt und lokale Steklov–Poincaré Operatoren werden zu lokalen Randintegralgleichungssystemen umformuliert, sodass die eindeutige Lösbarkeit des resultierenden Systems garantiert werden kann. Weiters wird eine neue kombinierte Feldintegralgleichung für Aussenraumprobleme vorgestellt, welche sich auf einfache Weise in die Gebietszerlegungsformulierung einbinden lässt, da Innen– und Aussenraumprobleme auf die gleiche Art und Weise behandelt werden können. Weiters wird eine alternative Herleitung der „tearing and interconnecting“–Methode vorgestellt, welche rein auf algebraischen Argumenten basiert. Der Vorteil dieser Herleitung im Unterschied zur klassischen Herleitung ist, dass sie in keiner Weise auf Eigenschaften der zugrunde liegenden partiellen Differentialgleichung angewiesen ist. Im Falle des akustischen Streuproblems wird zusätzlich eine Vorkonditionierung des resultierenden BETI–Gleichungssystem diskutiert, welche auf die FETI–H Methode zurückgeht. Abschließend werden einige numerische Beispiele angeführt, um die theoretischen Ergebnisse zu untermauern.

CONTENTS

1 INTRODUCTION

Motivation

Numerical simulations are an important industrial tool to enhance existing products or to decrease the developing costs of new products. The usage of numerical simulations and the requirements on them increased heavily in the last decades. It is often no longer sufficient to simulate only one physical model because, with increasing regularity, the different physical models are coupled together and simulated. Examples include the cooling of electromagnetic devices [26] as well as the acoustic noises induced by vibrating structures [69]. On the other hand, the structures simulated by a single physical model are increasing in size and complexity. Hence, simulations have to be parallelized on multiple computing cores. This can range from a few cores on one CPU to several thousand cores in high performance clusters. For both cases domain decomposition approaches are used, since they deliver a natural framework to couple different physical models and to split huge problems into smaller ones, which can be distributed to different computing cores.

Acoustic and especially electromagnetic phenomena belong to the most simulated physical problems. Acoustic problems range from the simulation of the acoustic noise inside a car to the simulation of sonic detection and ranging (SODAR) instruments which are used in meteorology. The field of electromagnetic problems is even larger. The magnetic resonance imaging (MRI) in medicine, the radio detection and ranging (RADAR) application or the global positioning system (GPS) are only a few examples of electromagnetic applications simulated by numerical methods. Difficulties in the simulation of such applications/systems appear often, if the physical parameters are not uniform in the whole simulated area. Possible unbounded simulation areas are also a challenging part in the simulation of acoustic and electromagnetic phenomena.

The aim of this thesis is to derive and to analyze a domain decomposition method which is suitable for acoustic and electromagnetic scattering problems in bounded and unbounded domains, and which can also deal with jumping parameters.

Partial differential equations and boundary element methods

Several physical phenomena are described by partial differential equations (PDEs), e.g. heat transfer, structural mechanics, acoustics and electromagnetism are just some of them. In this thesis we are concerned with two particular PDEs which

describe the nature of acoustic and electromagnetic waves. These are elliptic PDEs of second order. For this class of PDEs several numerical methods are established. Among the most well known are the finite difference method (FDM) [130], the finite volume method (FVM) [63], the finite element method (FEM) [23] and the boundary element method (BEM) [125]. In this work we will solely consider the boundary element method, although a coupling to or an exchange by finite element methods would be partially possible.

One of the first articles about boundary integral equations for Lipschitz domains was written by Costabel [50]. Comprehensive overviews on boundary integral equation methods are given by McLean [102] as well as by Hsiao and Wendland [89] and for acoustic and electromagnetic scattering problems by Colton and Kress [47] and by Nédélec [109]. In the books of Sauter and Schwab [121] and Steinbach [125] the boundary element method is also explained in detail. In short, the boundary element method is based on representation formulae, which yield the solution of partial differential equations only from the Cauchy data on the boundary. To deduce such a representation formula for a PDE, it is necessary to know the fundamental solution of the partial differential operators. These fundamental solutions exist for a wide range of linear PDEs, e.g. all three dimensional elliptic and scalar PDEs with constant coefficients of second order [121], the time harmonic Maxwell equations [109], etc.. Since only one part of the Cauchy data is prescribed within the boundary value problem, the remaining one has to be computed by applying a limiting process on the representation formula towards the boundary of the computational domain. This results in a boundary integral equation which can be solved for the missing Cauchy datum. The main feature of the boundary element method is that only quantities and operators have to be discretized, which are only defined on the boundary of the domain. In many cases this is a big advantage compared to other methods, for example if dealing with moving boundaries, unbounded domains, or even if just a surface mesh is available. Of course there are also disadvantages, for example, in general the boundary element method can not be used directly for non–linear PDEs or for PDEs with variable coefficients. Another important issue for a numerical method is the computational complexity, i.e. storage requirements and number of operations. In the boundary element method only the boundary is discretized, therefore the number of degrees of freedom for a three–dimensional problem behaves like $\mathcal{O}(N^2)$, where N is a measure for the number of elements in one spatial direction. In contrast, the number of elements in the finite element method behaves like $\mathcal{O}(N^3)$. A drawback of the boundary element method is, that it usually leads to fully populated matrices which implies a computational complexity of $\mathcal{O}(N^4)$ operations. The matrices in the finite element method are sparse, implying a complexity of $\mathcal{O}(N^3)$ operations. To remove this disadvantage, fast boundary element methods, which approximate the fully populated matrices were introduced. This leads to a computational complexity of $\mathcal{O}(N^2 \log(N^2)^\alpha)$ operations for fast boundary element methods, which gives for 3D problems an asymptotic complexity advantage in comparison to most other methods. The most well known fast boundary element methods are the fast multipole method

[76], the Panel–Clustering method [79] and $\mathcal{H}$–matrices in combination with the adaptive cross approximation (ACA) [15]. The latter method will be used for some of the numerical examples given in this thesis.

Acoustic and electromagnetic scattering

Under the assumption of a time–harmonic excitation acoustic and electromagnetic quantities, such as the sound pressure or the electric field density often behave like waves. Examples are the movement of sound in air or the signal of a satellite. In these cases the acoustic waves can be described by the scalar wave equation

$$-\Delta U(x) - k^2(x)U(x) = F(x)$$

with the wave number $k(x)$. This equation is also called the Helmholtz equation named after Hermann Ludwig Ferdinand von Helmholtz, a German physician. Electromagnetic waves are described by the electromagnetic wave equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) - k^2(x)\mathbf{U}(x) = \mathbf{F}(x).$$

These equations can be used to describe a couple of different physical problems, such as interior source problems, exterior scattering problems, transmission problems and so on. Comprehensive overviews on different acoustic and electromagnetic wave problems and solution strategies are given by Colton and Kress [47], Monk [106] and Nédélec [109]. The boundary element method is one of the methods used to solve this kind of problems numerically. It is especially suitable for exterior scattering problems and transmission problems, due to an easy handling of unbounded computational domains. Unfortunately, the standard boundary integral formulations of the exterior scattering problem do not admit a unique solution for all wave numbers, which does not reflect the physical behaviour. Brakhage and Werner [24] were the first to present a regularized indirect formulation for the acoustic scattering problem, which was well posed for all wave numbers. Panich followed with a similar approach for the electromagnetic scattering problem [114]. A direct approach for the acoustic problem was given shortly afterwards by Burton and Miller [41]. All of these approaches were developed for smooth domains and in a L_2–analytical setting. To extend these ideas to Lipschitz domains and the natural functional analytic setting, so–called regularized combined field integral equations (CFIE) were developed among others by Buffa and Hiptmair for the acoustic and the electromagnetic case [36, 37], by Engleder and Steinbach for the acoustic case [60] and by Steinbach and Windisch for the electromagnetic case [127]. The transmission problem is often tackled by a coupled finite and boundary element method, using finite elements in the interior and boundary elements in the exterior. The coupling of the two methods is often realized by the symmetric coupling approach introduced by Costabel [49]. However, this classical approach was once more haunted by spurious modes. Regularized formulations were developed, for example, by Hiptmair and Meury for the acoustic and electromagnetic case [86,87]. In this thesis we present a flexible framework which can

be used for interior and exterior boundary value problems and by a suitable reformulation even for transmission problems with multiple subdomains. The aim is to state a well posed approach in the natural functional analytic setting. Other boundary element approaches for the transmission problem with multiple subdomains are, for example, given by Hanckes and Hiptmair [85] and von Petersdorff [138] who use a multiple trace formulation based on the Calderon projector. For exterior scattering problems, mixed formulations with multiple subdomains are, for example, given by Bendali, Boubendir and Fares in [21, 22].

Domain decomposition methods

Most likely the first work on domain decomposition methods is the alternating Schwarz method [122], which was originally used as a theoretical tool to rigorously prove Dirichlet's minimization principle [71]. Since then several domain decomposition methods have been developed, which can be divided into overlapping and non–overlapping methods. The non–overlapping methods can again be divided into primal and dual methods. Primal methods use the physical quantities on the boundaries as degrees of freedom [25, 55], dual methods in contrast use Lagrangian multipliers which enforce continuity on interfaces as degrees of freedom. Such methods include Mortar methods [17], hybrid methods [124] or tearing and interconnecting methods like the finite element tearing and interconnecting (FETI) method [67, 68]. For an comprehensive overview on domain decomposition methods we refer to the monographs [101, 117, 136].

In this work we use a boundary element tearing and interconnecting (BETI) method, which is the counterpart of the FETI method and uses boundary elements instead of finite elements to solve the local problems. This method was introduced by Langer and Steinbach [95], who also introduced the coupled FETI/BETI method [96]. Several modifications of these methods exist, such as the dual–primal FETI (FETI–DP) method which uses a mixture of primal and dual unknowns [64] and the all–floating tearing and interconnecting method [112] which allows a unified treatment of floating subdomains.

Tearing and interconnecting methods are mainly used for problems which lead to elliptic bilinear forms or operators, because the standard deduction of the method is using a reformulation as a minimizing problem. De La Bourdonnaye, Farhat, Macedo and Tezaur extended the applicability of the FETI method to the Helmholtz equation by using a reformulation as a saddle point problem [53,66], referred to as the FETI–H method. Of course also other domain decomposition methods are used for acoustic scattering problems. FETI like domain decomposition methods were given by Bendali, Boubendir and Fares [16] and by Bourdonnaye, Farhat, Macedo, Magoulès and Roux in [54]. A slightly more different approach is the ultra–weak variational formulation method (UWVF), which was applied to the acoustic scattering problem by Buffa and Monk in [39]. For the electromagnetic scattering problem in conjunction with domain decomposition methods, less literature is available. In [97], Li and Jin

applied the FETI–DP approach to an electromagnetic scattering problem and called the method FETI–DPEM. In this thesis we modify the classical BETI method, such that it is applicable for the acoustic and electromagnetic scattering problem. Although several ideas of the FETI-H method can be used, the well posedness of the local and global problems must be proven in an alternative way. We further use an alternative algebraic deduction of the basic tearing and interconnecting approach, which does not rely on the underlying partial differential equation.

Outline of Contents

In the second chapter, a short introduction to the physical modeling of acoustic and electromagnetic scattering is given. Under the assumption of a time harmonic excitation, mathematical models are deduced that describe the behaviour of acoustic and electromagnetic waves. Further assumptions on the material parameters and on the wave inducing excitation lead to different well posed mathematical problems.

In the third chapter, some basics from functional analysis for the mathematical description of acoustic and electromagnetic problems are provided. To analyze the acoustic scattering problem, only rather standard theorems such as Cea's lemma or Gårding inequalities are needed. For the electromagnetic scattering a more complicated framework has to be established. Only the necessary parts of these theories are given, a more comprehensive discussion can be found in [29, 38, 44, 84, 142].

In chapter 4, geometrical assumptions on the physical scatterers are made in order to give an analytical framework of Sobolev spaces suitable for acoustic and electromagnetic scattering. For the acoustic scattering standard Sobolev spaces can be used, hence the introduction of these spaces is rather short. In the case of electromagnetic scattering appropriate Sobolev spaces must be used, cf. [31, 32]. An introduction and short motivation of these spaces in the domain and on the boundary are given. Due to the unavoidable use of generalized Gårding inequalities, related Hodge– and Helmholtz–type splittings are given and discussed, cf. [29].

Chapter 5 is about local acoustic scattering problems. First interior/exterior Dirichlet and Neumann boundary value problems are studied on their well/ill posedness. Moreover, Robin boundary value problems are considered due to their non–constrained well posedness. Thereafter, a representation formula for the Helmholtz equation is given and corresponding potential operators are supplied. Based on the representation formula, boundary integral equations and operators are deduced and analyzed. With the domain decomposition in mind, the Dirichlet–to–Neumann map including the Steklov–Poincaré operator and other solution strategies for interior boundary value problems are introduced and discussed. Exterior boundary value problems are discussed thereafter in a separate section, because of a different nature of the problem. A Galerkin boundary element method is discussed for an approximate solution of the prior introduced problems. In addition, error estimates for these approximate solutions are stated and verified by a numerical example. To conclude this chapter

we describe a preconditioning strategy for the local problems and give a numerical example to show the efficiency of the proposed preconditioner.

Dirichlet and Neumann boundary value problems for the electromagnetic wave equation are analyzed in chapter 6. Because of their special properties, Robin boundary value problems are once again introduced and discussed. Afterwards the Stratton–Chu representation formula is given and related potential operators are defined. Boundary integral equations and operators are afterwards introduced in a way such that in addition to electromagnetic scattering problems also eddy current problems can be treated. A stable formulation for interior Robin boundary value problems, which is also suitable for the domain decomposition approach, is given. Furthermore, a regularized formulation of the exterior scattering problem leading to another direct combined field integral equation is introduced. To solve these problems approximately, a Galerkin boundary element scheme based on Raviart–Thomas elements is presented. A numerical example confirms the approximation estimates of the boundary element approach, and a possible preconditioning strategy is discussed.

Geometrical assumptions on the domain decomposition are the starting point of chapter 7. Thereafter a classical Dirichlet domain decomposition approach is deduced. This domain decomposition approach leads to artificial local Neumann boundary value problems, which can be haunted by spurious modes. Therefore, Neumann interface conditions are partially exchanged by Robin interface conditions to achieve unique solvability of the artificially constructed local boundary value problems. We explain how and when this exchange works and give an algorithm which can be used for the implementation of the exchange procedure. Afterwards a boundary element approximation of spaces defined on the skeleton of the domain decomposition is introduced and a discrete variational formulation of the decomposed problem is given. The idea of the tearing and interconnecting approach is explained thereafter and an algebraic deduction of this approach is provided. This approach is then applied to the boundary element discretization of the variational formulation and a preconditioning strategy for the resulting system is discussed. Finally, we provide several examples to confirm convergence estimates, and to test the proposed preconditioner.

The eighth chapter starts by discussing properties of appropriate Sobolev spaces for the electromagnetic wave equation on the skeleton. Thereafter the same classical Dirichlet domain decomposition approach as for the Helmholtz equation is deduced. In the spirit of the acoustic scattering problem, interface conditions are adjusted in order to get well posed local boundary value problems. Afterwards a boundary element discretization of the variational formulation of the decomposed problem is given. Since the tearing and interconnecting approach does not depend on the underlying partial differential equation, the results of the Helmholtz case can be applied directly to the discretized variational formulation. Finally, we give a numerical example to demonstrate the feasibility of the boundary element tearing and interconnecting approach for electromagnetic scattering problems.

In the last chapter we give a short conclusion and discuss some open questions.

2 PHYSICAL BACKGROUND

2.1 Linear Acoustics

The propagation of acoustic waves is described by a hyperbolic partial differential equation of second order, the so called wave equation. In this section we first summarize the deduction of the wave equation based on linear acoustics [47, 58]. The modeling of nonlinear acoustics leads to more complicated partial differential equations of higher order, which will not be discussed in this work, see, for example, [80].

Let $\mathbf{v}(x,t)$ be the velocity vector of a fluid particle in an inviscid and inhomogeneous fluid with the spatial coordinates $x \in \mathbb{R}^3$ at time t, $p(x,t)$ is the pressure, $\rho(x,t)$ is the density and $S(x,t)$ is the specific entropy. Without external forces the conservation of momentum leads to the Euler equation

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} + \frac{1}{\rho}\,\mathbf{grad}\, p = 0. \tag{2.1}$$

The equation of continuity

$$\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{v}) = 0 \tag{2.2}$$

describes the conversation of mass. The equation of state

$$p = f(\rho, S) \tag{2.3}$$

describes the relation between pressure and density, where f is a function depending on the properties of the fluid. Finally we have the adiabatic hypothesis

$$\frac{\partial}{\partial t}S + \mathbf{v} \cdot \mathbf{grad}\, S = 0. \tag{2.4}$$

For waves with small amplitudes we can assume that $\mathbf{v}$, p, ρ and S are small perturbations of the static state $\mathbf{v}_0 = \mathbf{0}$, $p = p_0 = \mathit{constant}$, $\rho = \rho_0(x)$ and $S = S_0(x)$ with $p_0 = f(\rho_0, S_0)$. The perturbed quantities can be written by

$$\begin{aligned} \mathbf{v}(x,t) &= \varepsilon\mathbf{v}_1(x,t) + \dots, \\ p(x,t) &= p_0 + \varepsilon p_1(x,t) + \dots, \\ \rho(x,t) &= \rho_0(x) + \varepsilon\rho_1(x,t) + \dots, \\ S(x,t) &= S_0(x) + \varepsilon S_1(x,t) + \dots \end{aligned} \tag{2.5}$$

with $0 < \varepsilon \ll 1$, cf. [47]. Inserting (2.5) into the physical equations and neglecting all terms of quadratic or higher order in ε, cf. [47], this leads to the linearized equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{v}_1(x,t) + \frac{1}{\rho_0(x)}\,\mathbf{grad}\, p_1(x,t) = 0,\\
&\frac{\partial}{\partial t}\rho_1(x,t) + \rho_0(x)\,\mathrm{div}\,(\mathbf{v}_1(x,t)) = 0,\\
&\qquad\qquad \frac{\partial}{\partial t}p_1(x,t) = c^2(x)\left(\frac{\partial}{\partial t}\rho_1(x,t) + \mathbf{v}_1(x,t)\cdot\mathbf{grad}\,\rho_0(x)\right),
\end{aligned}$$

where the speed of sound $c(x)$ is defined by

$$c^2(x) := \frac{\partial}{\partial \rho} f(\rho_0(x), S_0(x)).$$

From this we deduce that p_1 satisfies

$$\frac{\partial^2}{\partial t^2}p_1(x,t) = c^2(x)\rho_0(x)\,\mathrm{div}\left(\frac{1}{\rho_0(x)}\,\mathbf{grad}\, p_1(x,t)\right).$$

Under the additional assumptions that all terms involving $\mathbf{grad}\,\rho_0$ are negligible we finally obtain the wave equation

$$\frac{\partial^2}{\partial t^2}p_1(x,t) = c^2(x)\Delta p_1(x,t).$$

By assuming a time harmonic behaviour, i.e.

$$p_1(x,t) = \mathrm{Re}\left(U(x)e^{-i\omega t}\right),$$

we conclude that $U(x)$ satisfies the Helmholtz equation

$$\Delta U(x) + \frac{\omega^2}{c^2(x)}U(x) = 0.$$

Until now we have only deduced an equation, which describes how time harmonic acoustic waves of small amplitude behave in a slowly varying inhomogeneous medium. We still have to describe how the wave is initiated. For this we assume that the wave motion is caused by an incident field $U^i(x)$. We further assume that the inhomogeneous region has compact support, i.e., $c(x) = c_0$ for $x \in \mathbb{R}^3 \setminus \Omega$ with Ω compact. This leads to the system

$$\Delta U(x) + \widetilde{k}^2 n(x) U(x) = 0 \qquad \text{for } x \in \mathbb{R}^3, \tag{2.6}$$

$$U^i(x) + U^s(x) = U(x), \tag{2.7}$$

$$\lim_{r\to\infty} r\left(\frac{\partial U^s(x)}{\partial r} - ikU^s(x)\right) = 0, \tag{2.8}$$

where $\widetilde{k} = \omega/c_0 \geq 0$ is the wave number and

$$n(x) := \frac{c_0^2}{c^2(x)}$$

is the refractive index. $U^i(x)$ is hereby a solution of the Helmholtz equation, i.e.

$$\Delta U^i(x) + \widetilde{k}^2 n(x) U^i(x) = 0,$$

and $U^s(x)$ is the scattered field, which is assumed to fulfill the Sommerfeld radiation condition (2.8), see [123]. In $\mathbb{R}^3 \setminus \Omega$ we assume that $n(x) = 1$, otherwise we will usually anticipate $n(x) > 0$. This corresponds to a scattering problem, which is the major topic of this thesis. The mathematical theory also allows us to include the case

$$n(x) = n_1(x) + in_2(x)$$

with $n_1(x) > 0$ and $n_2(x) \geq 0$. This reflects the physical behaviour of an acoustic wave in an absorbing medium. From now on we shorten the notation by $k(x) = \widetilde{k}\sqrt{n(x)}$. Based on the equations (2.6)–(2.8) different acoustic problems can be modelled.

The exterior scattering problem

For the direct acoustic scattering by an impenetrable object the model is reduced to a boundary value problem of the scattered field $U^s(x)$. If the scatterer is assumed to be a sound–soft obstacle, then the exterior Dirichlet trace of the total field $U(x)$ vanishes on the boundary $\Gamma = \partial\Omega$, i.e. $\gamma_0^c U(x) = 0$. This leads to the Dirichlet boundary value problem

$$\begin{aligned} \Delta U^s(x) + k^2 U^s(x) &= 0 && \text{for } x \in \mathbb{R}^3 \setminus \overline{\Omega}, \\ \gamma_0^c U^s(x) &= g_d(x) && \text{for } x \in \Gamma, \\ \lim_{r\to\infty} r\left(\frac{\partial U^s(x)}{\partial r} - ikU^s(x)\right) &= 0, \end{aligned}$$

where the Dirichlet datum is given by $g_d(x) := -\gamma_0^c U^i(x)$ and k is constant. In the case of a sound–hard obstacle, the exterior Neumann trace of the total field $U(x)$ vanishes on the boundary Γ, i.e. $\gamma_1^c U(x) = 0$. This leads to the Neumann boundary problem

$$\begin{aligned} \Delta U^s(x) + k^2 U^s(x) &= 0 && \text{for } x \in \mathbb{R}^3 \setminus \overline{\Omega}, \\ \gamma_1^c U^s(x) &= g_n(x) && \text{for } x \in \Gamma, \\ \lim_{r\to\infty} r\left(\frac{\partial U^s(x)}{\partial r} - ikU^s(x)\right) &= 0, \end{aligned}$$

with $g_n(x) := -\gamma_1^c U^i(x)$. Again k is assumed to be constant. A practical problem which can be modeled by the exterior scattering problem is, for example, the sound radiation of an airplane.

The interior scattering problem

An acoustic field inside a closed domain $\Omega \subset \mathbb{R}^3$ can be computed if the acoustic pressure $U(x)$ is known on the boundary, i.e. $\gamma_0 U(x) = g_d(x)$. This can be stated by the interior Dirichlet problem

$$\begin{aligned} \Delta U(x) + k^2(x)U(x) &= F(x) && \text{for } x \in \Omega, \\ \gamma_0 U(x) &= g_d(x) && \text{for } x \in \Gamma, \end{aligned}$$

where $F(x)$ represents interior acoustic sources. If the velocity field is known, i.e. $\gamma_1 U(x) = g_n(x)$, then the acoustic field can be recovered by solving the interior Neumann problem

$$\begin{aligned} \Delta U(x) + k^2(x)U(x) &= F(x) && \text{for } x \in \Omega, \\ \gamma_1 U(x) &= g_n(x) && \text{for } x \in \Gamma. \end{aligned}$$

In most cases we will assume that no interior sources are present, i.e. $F(x) \equiv 0$. If $F(x) \not\equiv 0$ we call the problem interior source problem, otherwise we call it interior scattering problem. A practical example for the interior scattering problem is the simulation of the acoustic noise in a car cabin excited by the vibrations of the engine.

The transmission problem

The transmission problem describes the scattering of an incident wave by a penetrable obstacle Ω. It can be viewed as an exterior scattering problem combined with an interior source problem. The problems are coupled by transmission conditions on the boundary Γ. The mathematical formulation of this problem is given by:

$$\begin{aligned} \Delta U(x) + k^2(x)U(x) &= F(x) && \text{for } x \in \Omega, \\ \Delta U^s(x) + k_0^2 U^s(x) &= 0 && \text{for } x \in \Omega^c, \\ \gamma_0^c U^s(x) - \gamma_0 U(x) &= g_d(x) && \text{for } x \in \Gamma, \\ \gamma_1^c U^s(x) - \gamma_1 U(x) &= g_n(x) && \text{for } x \in \Gamma, \\ \lim_{r\to\infty} r\left(\frac{\partial U^s(x)}{\partial r} - ikU^s(x)\right) &= 0. \end{aligned}$$

As for the interior scattering problem we assume that no internal acoustic sources are present, i.e. $F(x) \equiv 0$. We further assume that k_0 is constant. This model, for example, can be used to simulate the scattering of an acoustic wave by a submarine.

The well/ill posedness of the presented acoustic scattering problems will be discussed in Chapter 5.

2.2 Electromagnetism

The Maxwell equations, which are valid in the whole space, are the starting point of nearly any model deduction for electromagnetic phenomena. We will use their differential form to deduce the electromagnetic wave equation. These differential equations contain several physical quantities, including the electric field intensity $\widetilde{\mathbf{E}}$, the electric displacement field (electric flux) $\widetilde{\mathbf{D}}$, the magnetic field intensity $\widetilde{\mathbf{H}}$, the magnetic induction field (magnetic flux) $\widetilde{\mathbf{B}}$, the current density function $\widetilde{\mathbf{j}}$ and the charge density ρ. The first equation

$$\mathbf{curl}\,\widetilde{\mathbf{E}}(x,t) = -\frac{\partial}{\partial t}\widetilde{\mathbf{B}}(x,t) \tag{2.9}$$

is called Faraday's induction law and describes that a change (in time) of the magnetic flux induces an electric field. Ampère's law

$$\mathbf{curl}\,\widetilde{\mathbf{H}}(x,t) = \frac{\partial}{\partial t}\widetilde{\mathbf{D}}(x,t) + \widetilde{\mathbf{j}}(x,t) \tag{2.10}$$

states that electric currents induce magnetic curls. In this case changes of the electric displacement field are also treated as electric currents. The last two equations

$$\operatorname{div}\,\widetilde{\mathbf{D}}(x,t) = \widetilde{\rho}(x,t), \tag{2.11}$$

$$\operatorname{div}\,\widetilde{\mathbf{B}}(x,t) = 0 \tag{2.12}$$

are known as the electric and magnetic Gauß law. They describe that sources of electric fields are electric charges and that magnetic fields do not contain sources. Another important relation is the conservation of charges, which is stated by the continuity equation

$$\operatorname{div}\,\widetilde{\mathbf{j}}(x,t) = -\frac{\partial}{\partial t}\widetilde{\rho}(x,t), \tag{2.13}$$

which follows from a combination of the divergence of equation (2.10) and the time derivative of equation (2.11). Taking the divergence of the equations (2.9) and (2.10) leads to

$$\frac{\partial}{\partial t}\operatorname{div}\,\widetilde{\mathbf{B}}(x,t) = \frac{\partial}{\partial t}(\operatorname{div}\,\widetilde{\mathbf{D}}(x,t) - \widetilde{\rho}(x,y)) = 0.$$

Hence, the equations (2.11) and (2.12) always hold, if they hold in one moment in time.

In what follows we assume that all fields are time harmonic, i.e. for a generic field $\mathbf{F}$ this implies

$$\mathbf{F}(x,t) = \mathbf{F}(x)e^{i\omega t}$$

with some frequency ω. This simplification inserted in Maxwell's equations leads to

$$\mathbf{curl}\,\widetilde{\mathbf{E}}(x) = -i\omega\widetilde{\mathbf{B}}(x), \tag{2.14}$$

$$\mathbf{curl}\,\widetilde{\mathbf{H}}(x) = i\omega\widetilde{\mathbf{D}}(x) + \widetilde{\mathbf{j}}(x), \tag{2.15}$$

$$\operatorname{div}\widetilde{\mathbf{D}}(x) = \widetilde{\rho}(t), \tag{2.16}$$

$$\operatorname{div}\widetilde{\mathbf{B}}(x) = 0. \tag{2.17}$$

The fields are now complex quantities. By using the continuity equation (2.13) we can eliminate the charge density $\widetilde{\rho}$ via $-i\omega\widetilde{\rho} = \operatorname{div}\widetilde{\mathbf{j}}$.

Remark 2.1. *For a general periodic excitation, a Fourier series expansion approach can be used to analyze the electromagnetic scattering problem, cf. [9].*

For a further simplification of system (2.14)–(2.17) we have to investigate the material properties that establish a connection between the electric quantities $\widetilde{\mathbf{D}}$ and $\widetilde{\mathbf{E}}$, and the magnetic quantities $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{H}}$. In this thesis we distinguish between two cases:

- In vacuum, the fundamental relations

  $$\widetilde{\mathbf{D}} = \varepsilon_0\widetilde{\mathbf{E}},$$
  $$\widetilde{\mathbf{B}} = \mu_0\widetilde{\mathbf{H}}$$

  are valid. Hereby is

  $$\varepsilon_0 := \frac{10^7}{4\pi c^2}\frac{Am}{Vs} \approx 8.855 \cdot 10^{-12}\frac{Am}{Vs}$$

  the dielectric permittivity of free space and

  $$\mu_0 := 4\pi 10^{-7}\frac{Vs}{Am} \approx 1.2566 \cdot 10^{-6}\frac{Vs}{Am}$$

  is the magnetic constant in free space. Further holds $\frac{1}{\sqrt{\varepsilon_0\mu_0}} = c$, where c is the speed of light.

- For inhomogeneous, isotropic materials we have the relations

  $$\widetilde{\mathbf{D}} = \varepsilon\widetilde{\mathbf{E}}, \qquad \widetilde{\mathbf{B}} = \mu\widetilde{\mathbf{H}}, \tag{2.18}$$

  with ε and μ as scalar, positive and bounded functions in space. We assume that these functions are piecewise constant.

In conductive materials the electric field induces currents. For a moderate electric field density we can describe this by Ohm's law

$$\widetilde{\mathbf{j}} = \sigma\widetilde{\mathbf{E}} + \widetilde{\mathbf{j}}_a, \tag{2.19}$$

with σ as the electric conductivity. In general this is a positive semi definite matrix function in space. But in the case of isotropic materials we can assume that σ is a positive scalar function, and in vacuum holds $\sigma \equiv 0$. The quantity $\widetilde{\mathbf{j}}_a$ describes the applied current density.

By using the assumptions (2.18) and (2.19) we obtain the time harmonic Maxwell system

$$\begin{aligned}
\mathbf{curl}\,\widetilde{\mathbf{E}}(x) &= -i\omega\mu\widetilde{\mathbf{H}}(x),\\
\mathbf{curl}\,\widetilde{\mathbf{H}}(x) &= i\omega\varepsilon\widetilde{\mathbf{E}}(x) + \sigma\widetilde{\mathbf{E}}(x) + \widetilde{\mathbf{j}}_a(x),\\
\operatorname{div}(\varepsilon\widetilde{\mathbf{E}}(x)) &= -\frac{1}{i\omega}\operatorname{div}(\sigma\widetilde{\mathbf{E}} + \widetilde{\mathbf{j}}_a(x)),\\
\operatorname{div}(\mu\widetilde{\mathbf{H}}(x)) &= 0.
\end{aligned}$$

By normalizing the fields

$$\mathbf{E} = \varepsilon_0^{1/2}\widetilde{\mathbf{E}}, \qquad \mathbf{H} = \mu_0^{1/2}\widetilde{\mathbf{H}}$$

and by setting

$$\varepsilon_r = \frac{1}{\varepsilon_0}\left(\varepsilon - \frac{i\sigma}{\omega}\right), \qquad \mu_r = \frac{\mu}{\mu_0},$$

we finally get

$$\mathbf{curl}\,\mathbf{E}(x) = -i\widetilde{k}\mu_r\mathbf{H}(x), \tag{2.20}$$

$$\mathbf{curl}\,\mathbf{H}(x) = i\widetilde{k}\varepsilon_r\mathbf{E}(x) + \frac{1}{i\widetilde{k}}\widetilde{\mathbf{F}}(x), \tag{2.21}$$

$$\operatorname{div}(\varepsilon_r\mathbf{E}(x)) = \frac{1}{\widetilde{k}^2}\operatorname{div}\widetilde{\mathbf{F}}(x), \tag{2.22}$$

$$\operatorname{div}(\mu_r\mathbf{H}(x)) = 0, \tag{2.23}$$

where the excitation is given by $\widetilde{\mathbf{F}} = i\widetilde{k}\mu_0\widetilde{\mathbf{j}}_a$ and $\widetilde{k} = \omega\sqrt{\varepsilon_0\mu_0} \geq 0$ is the wave number. For $\widetilde{k} > 0$ the equations (2.22) and (2.23) can be deduced by taking the divergence of (2.20) and (2.21). By eliminating $\mathbf{H}$ in (2.20) and (2.21) we get the partial differential equation

$$\mathbf{curl}(\frac{1}{\mu_r}\,\mathbf{curl}\,\mathbf{E}) - \widetilde{k}^2\varepsilon_r\mathbf{E} = -\widetilde{\mathbf{F}}. \tag{2.24}$$

Of course an elimination of the electric field density would also have been possible, but since this approach is more common in the literature we will use the electric field as the main quantity. Formula (2.24) holds for arbitrary inhomogeneous, isotropic materials. By setting $k = \sqrt{\varepsilon_r\mu_r}\widetilde{k}$ and $\mathbf{F} = \mu_r\widetilde{\mathbf{F}}$ we can further simplify this to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = -\mathbf{F}. \tag{2.25}$$

Note that $\sigma = 0$ implies $k \in \mathbb{R}$. The case $\sigma = 0$ corresponds to a scattering problem, which is the main focus of this thesis. For $\sigma > 0$ we would get an eddy current problem, this case will also be partially covered in this thesis.

For piecewise constant coefficients ε_r or μ_r describing different materials we do not have continuity of the normal component of the electric field density, rather only tangential continuity is preserved, see [106]. This means that for two domains Ω_i and Ω_j there holds

$$\mathbf{n} \times (\mathbf{E}_i(x) - \mathbf{E}_j(x)) \times \mathbf{n} = \mathbf{0} \qquad \text{for } x \in \Gamma_{ij} \tag{2.26}$$

where Γ_{ij} is the interface between the two domains. In general, we have for the magnetic field

$$(\mathbf{H}_i(x) - \mathbf{H}_j(x)) \times \mathbf{n} = \mathbf{j}_s(x) \qquad \text{for } x \in \Gamma_{ij},$$

where the tangential vector field $\mathbf{j}_s$ is called surface current density. If Γ_{ij} is not a thin conductive layer, or if $\mathbf{F}(x)$ has no singularities which induce surface currents on Γ_{ij}, then $\mathbf{j}_s(x) = 0$ holds, cf. [106]. Hence, in most cases

$$(\mathbf{H}_i(x) - \mathbf{H}_j(x)) \times \mathbf{n} = \mathbf{0} \qquad \text{for } x \in \Gamma_{ij}$$

holds. In unbounded domains the electric field $\mathbf{E}(x)$ has in addition to fulfill the Silver–Müller radiation condition

$$\lim_{r\to\infty} r\,(\mathbf{curl}\,\mathbf{E}^s(x) \times \mathbf{n} - ik\mathbf{E}^s(x)) = \mathbf{0}, \tag{2.27}$$

see [106, 107].

As for the acoustic scattering problem, we have to claim further conditions to get mathematically well posed problems.

The exterior scattering problem

For a perfect conductor, i.e. $\sigma \to \infty$, we can deduce from Ohm's law $\mathbf{E}(x) = \mathbf{0}$ inside the conductor. Together with equation (2.26) this gives us the perfect conducting boundary condition $\mathbf{n} \times \mathbf{E}(x) \times \mathbf{n} = \mathbf{0}$ for $x \in \Gamma$. If we assume an excitation by an incoming field $\mathbf{E}^i$, then we have to solve for the scattered field $\mathbf{E}^s$ the Dirichlet boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^s(x) - k^2\mathbf{E}^s(x) = \mathbf{0} \qquad \text{for } x \in \mathbb{R}^3 \setminus \overline{\Omega}, \tag{2.28}$$

$$\mathbf{n} \times \mathbf{E}^s(x) \times \mathbf{n} = \mathbf{g}_d(x) \qquad \text{for } x \in \Gamma, \tag{2.29}$$

$$\lim_{r\to\infty} r\,(\mathbf{curl}\,\mathbf{E}^s(x) \times \mathbf{n} - ik\mathbf{E}^s(x)) = \mathbf{0} \tag{2.30}$$

with $\mathbf{E}^s = \mathbf{E} - \mathbf{E}^i$ and $\mathbf{g}_d(x) = -\mathbf{n} \times \mathbf{E}^i(x) \times \mathbf{n}$. A related practical problem is the RADAR detection of an airplane. In this case an incoming wave is induced by a RADAR station which is refracted by the airplane.

The interior scattering problem

If the magnetic field density $\mathbf{H}(x)$ is known on the boundary Γ, the electric field $\mathbf{E}(x)$ can be computed by solving the Neumann boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}(x) - k^2(x)\mathbf{U}(x) &= \mathbf{F}(x) \qquad \text{for } x \in \Omega, \\ \mathbf{curl}\,\mathbf{E}(x) \times \mathbf{n} &= \mathbf{g}_n(x) \qquad \text{for } x \in \Gamma, \end{aligned}$$

where $\mathbf{F}$ describes an interior source. As for the acoustic scattering we assume in this thesis that $\mathbf{F}(x) = \mathbf{0}$.

The well/ill posedness of the presented acoustic scattering problems will be discussed in Chapter 6.

3 FUNCTIONAL ANALYTIC BASICS

In this chapter we introduce abstract definitions and theorems as used in the next chapters. Most of them are rather standard and can be found in the most pertinent textbooks such as [125,142]. The more advanced theorems, including nonconforming subspaces, can be found, e.g., in [29,38,44,84].

Definition 3.1. *Let W be a complex vector space and $a(\cdot,\cdot)$ be a mapping from $W \times W$ to $\mathbb{C}$. $a(\cdot,\cdot)$ is called a sesquilinear form if it fulfills*

$$\begin{aligned} a(x_1 + x_2, y) &= a(x_1, y) + a(x_2, y), \\ a(x, y_1 + y_2) &= a(x, y_1) + a(x, y_2), \\ a(\lambda x, y) &= \lambda a(x, y), \\ a(x, \lambda y) &= \overline{\lambda} a(x, y) \end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in W$ and $\lambda \in \mathbb{C}$.

Definition 3.2. *Let W be a complex vector space with topology. A mapping $f : W \to \mathbb{C}$ which fulfills*

$$\begin{aligned} f(x_1 + x_2) &= f(x_1) + f(x_2), \\ f(\lambda x) &= \overline{\lambda} f(x) \end{aligned}$$

for all $x, x_1, x_2 \in W$ and $\lambda \in \mathbb{C}$ is called an antilinear functional on W. The space of all continuous antilinear functionals on W is called the dual space of W. We denote this space by W'. If W is a normed space, we shall write

$$\langle g, x \rangle = g(x)$$

for the value of the functional $g \in W'$ at the vector $x \in W$.

Definition 3.3. *A sesquilinear form $a(\cdot,\cdot)$ on a Hilbert space X is called bounded if*

$$|a(x, y)| \leq c_2 \|x\|_X \|y\|_X$$

is satisfied for all $x, y \in X$. It is called X–elliptic if

$$|a(x, x)| \geq c_1 \|x\|_X^2$$

is satisfied for all $x \in X$.

In the upcoming chapters we will often switch between an operator formulation and an equivalent variational formulation. The following lemma enables us to do so.

Lemma 3.4. *[125] For any linear and bounded operator $A : X \to X'$ the duality product $\langle \cdot, \cdot \rangle$ induces a sesquilinear form $a(\cdot, \cdot)$, i.e.,*

$$a(x, y) := \langle Ax, y \rangle \qquad \text{for all } x, y \in X. \tag{3.1}$$

On the other hand, for each sesquilinear form $a(\cdot, \cdot)$ there exists a linear and bounded operator $A : X \to X'$, such that (3.1) holds.

The solvability of variational formulations for elliptic operators is shown by the lemma of Lax–Milgram. Since we are mainly dealing with complex valued operators, we state the complex version of this lemma. The proof of this version can be found in [26].

Lemma 3.5 (Lax–Milgram lemma)**.** *Let X be a complex Hilbert space and $f \in X'$. Further let $a : X \times X \to \mathbb{C}$ be a bounded X–elliptic sesquilinear form. Then the variational problem*

Find $x \in X$ such that

$$a(x, y) = f(y) \tag{3.2}$$

holds for all $y \in X$.

has a unique solution $x \in X$ satisfying

$$\|x\|_X \leq \frac{1}{c_1} \|f\|_{X'},$$

where c_1 is the ellipticity constant as used in Definition 3.3.

In practice we have to restrict ourselves to finite dimensional problems, which give approximate solutions of the original problem. Cea's lemma provides error estimates for such an approximate solution.

Lemma 3.6 (Cea's lemma)**.** *[125] Let X be a complex Hilbert space, $f \in X'$ and let $a : X \times X \to \mathbb{C}$ be a bounded and X–elliptic sesquilinear form. Further let $X_h \subset X$ be a finite dimensional subspace. The discrete variational problem*

Find $x_h \in X_h$ such that

$$a(x_h, y_h) = f(y_h) \tag{3.3}$$

for all $y_h \in X_h$.

has a unique solution x_h, satisfying the stability estimate

$$\|x_h\|_X \leq \frac{1}{c_1} \|f\|_{X'}$$

and the quasi–optimal error estimate

$$\|x - x_h\|_X \leq \frac{c_2}{c_1} \inf_{y_h \in X_h} \|x - y_h\|_X,$$

where x is the unique solution of (3.2), and where c_1 and c_2 are the constants as used in Definition 3.3.

However, in the case of the Helmholtz and the Maxwell equations it is necessary to use weaker assumptions than ellipticity.

Definition 3.7. *Let X be a Hilbert space. The operator $A : X \to X'$ is called coercive, if a compact operator $C : X \to X'$ exists, such that the Gårding inequality*

$$|\langle (A+C)x, x\rangle| \geq c_1 \|x\|_X^2 \qquad \text{for all } x \in X$$

is fulfilled for a fixed $c_1 > 0$.

Remark 3.8. *In the literature, see, e.g. [89, 102], a Gårding inequality is often formulated only for the real part of the duality product, i.e.,*

$$\mathrm{Re}(\langle (A+C)x, x\rangle) \geq c_1 \|x\|_X^2 \qquad \text{for all } x \in X.$$

In several cases this more specific formulation will be sufficient for our needs, in these cases we will use the latter definition.

For coercive operators a similar solvability result as for elliptic operators can be obtained. The main difference is that injectivity has to be assumed in addition.

Theorem 3.9. *[125] Let $A : X \to X'$ be a bounded, injective and coercive operator and $f \in X'$. Then the equation $Ax = f$ has a unique solution $u \in X$ satisfying*

$$\|x\|_X \leq c \|f\|_{X'}$$

for some positive constant $c > 0$.

To obtain stability results and error estimates for discrete variational formulations related to coercive operators, we need the following version of Cea's lemma.

Lemma 3.10 (Cea's lemma)**.** *[125] Let $X_h \subset X$ be a finite dimensional subspace. If the sesquilinear form $a(\cdot,\cdot)$ fulfills the discrete inf–sup condition*

$$\sup_{0 \neq y_h \in X_h} \frac{|a(x_h, y_h)|}{\|y_h\|_X} \geq \gamma \|x_h\|_X , \qquad \gamma > 0 \tag{3.4}$$

for all $x_h \in X_h$, then the discrete problem (3.3) *has a unique solution which fulfills the quasi–optimal error estimate*

$$\|x - x_h\|_X \leq \left(1 + \frac{c_2}{\gamma}\right) \inf_{y_h \in X_h} \|x - y_h\|_X .$$

Definition 3.11. *A sequence of conforming subspaces $\{X_{h_i}\}_{i\in\mathbb{N}} \subset X$ with $X_{h_i} \subset X_{h_j}$ for $i > j$ is called approximating, if*

$$\lim_{i\to\infty} \inf_{x_h \in X_{h_i}} \|x - x_h\|_X = 0$$

holds for every $x \in X$.

Lemma 3.12. *[125] Let $A : X \to X'$ be a bounded linear operator which is coercive and injective. Let $\{X_{h_i}\}_{i\in\mathbb{N}} \subset X$ be an approximating sequence of conforming trial spaces. Then there exists a $j \in \mathbb{N}$ such that the discrete stability condition* (3.4) *is satisfied for all $i > j$.*

To analyze the vectorial wave equation it is necessary to use generalized Gårding inequalities. These are based on decompositions of Hilbert spaces.

Definition 3.13. *A decomposition $X = U \oplus V$ is called X–stable if for all $x \in X$ there exists a unique representation $x = u + v$ with $u \in U$ and $v \in V$ such that*

$$c^{-1} \|x\|_X \leq \|u\|_X + \|v\|_X \leq c \|x\|_X$$

is satisfied for a fixed $c > 0$ independent of x.

In this section all splittings $X = U \oplus V$ are assumed to be X–stable.

Definition 3.14. *[29] Let $\{X_{h_i}\}_{i\in\mathbb{N}} \subset X$ be an approximating sequence of conforming trial spaces. The family of finite dimensional subspaces $\{X_{h_i}\}_{i\in\mathbb{N}} \subset X$ fulfills the gap property with respect to the splitting $X = U \oplus V$ if discrete splittings $X_{h_i} = U_{h_i} \oplus V_{h_i}$ exist such that*

$$\lim_{i\to\infty} \sup_{u_h\in U_{h_i}} \inf_{u\in U} \frac{\|u_h - u\|_X}{\|u_h\|_X} = 0,$$
$$\lim_{i\to\infty} \sup_{v_h\in V_{h_i}} \inf_{v\in V} \frac{\|v_h - v\|_X}{\|v_h\|_X} = 0$$

holds.

Note that we did not enforce $V_{h_i} \subset V$ or $U_{h_i} \subset U$. In such a case the gap property is trivially satisfied. For a given decomposition $X = U \oplus V$ we introduce the twisting operator $\mathcal{X} : X \to X$ by

$$\mathcal{X}(u, v) := (-u, v) \tag{3.5}$$

where $u + v = x \in X$, $u \in U$ and $v \in V$.

Definition 3.15. *Let $a : X \times X \to \mathbb{C}$ be a bounded sesquilinear form. If a compact sesquilinear form C exists, such that*

$$\mathrm{Re}\,(a(u, \mathcal{X}u) + C(u, \mathcal{X}u)) \geq c \|u\|_X^2$$

is satisfied for all $x \in X$, then $a(\cdot, \cdot)$ fulfills a generalized Gårding inequality.

Theorem 3.16. *[29, Theorem 2.1] Let $A : X \to X'$ be a bounded and injective operator which satisfies a generalized Gårding inequality. Then, for $f \in X'$ the equation $Ax = f$ has a unique solution $u \in X$ which fulfills*

$$\|x\|_X \leq c \, \|f\|_{X'} \, .$$

Remark 3.17. *Instead of a twisting operator $\mathcal{X}$ it would be possible to use more general operators, see also the discussion in [29]. However, since this will be sufficient for our purposes, we will stay with twisting operators as introduced in* (3.5).

Theorem 3.18. *[29, Theorem 3.7] Let the sesquilinear form $a : X \times X \to \mathbb{C}$ be injective satisfying a generalized Gårding inequality. If the family of finite dimensional subspaces $\{X_{h_i}\}_{i \in \mathbb{N}} \subset X$ is approximating and satisfies the gap property, then there exists a $j < \infty$, such that for all $i > j$ the discrete inf–sup condition*

$$\sup_{y_h \in X_{h_i}} \frac{|a(x_h, y_h)|}{\|y_h\|} \geq \gamma \, \|x_h\|_X$$

holds for all $x_h \in X_{h_i}$ with some positive constant $\gamma > 0$. Cea's lemma implies that the discrete problem (3.3) *has a unique solution x_h, which satisfies the quasi–optimal error estimate*

$$\|x - x_h\| \leq c \cdot \inf_{y_h \in X_h} \|x - y_h\|_X \, .$$

Remark 3.19. *In all theorems involving Gårding or generalized Gårding inequalities, injectivity was assumed to ensure unique solvability. Instead, injectivity can be replaced by assuming surjectivity, since all proofs are based on Fredholm's alternative, see, e.g., [102].*

4 SOBOLEV SPACES

After formulating a few geometrical assumptions, we will first introduce standard Sobolev spaces as used for the mathematical analysis of boundary integral equations related to the Helmholtz equation, see, for example, [1, 102, 121]. Afterwards we define suitable Sobolev spaces to be used for the analysis of the electromagnetic wave equation, which are not as common as those for the Helmholtz equation. Therefore the discussion in this section, which is primarily based on the papers [31,32], will be more detailed.

4.1 Geometrical assumptions

The partial differential equation (2.25), which describes the electromagnetic scattering, is not of strongly elliptic type, cf. [89]. Therefore, it is not advisable to use standard Sobolev spaces for its mathematical analysis. While for the acoustic scattering problem we may assume $\Omega \subset \mathbb{R}^3$ to be a Lipschitz domain, for the electromagnetic scattering problem we assume that Ω is a Lipschitz polyhedron. Note that ordinary Lipschitz domains are rather common, we refer to [1, 102]. However, because they are not as common, Lipschitz polyhedrons will be defined in a more detailed way.

Definition 4.1 (Lipschitz polyhedron)**.** *A domain $\Omega \subset \mathbb{R}^3$ is called a Lipschitz polyhedron, if it fulfills the following requirements:*

- *Ω is bounded by a finite number N_Γ of polygons,*
- *Ω is a Lipschitz domain,*
- *Ω is simply connected.*

Although it is possible to generalize the upcoming theory of Sobolev spaces to piecewise smooth Lipschitz domains and to multi–connected domains, see, for example, [31, 32] and [28], we will restrict ourselves to Lipschitz polyhedrons, because it would be more technical work with little gain. Nevertheless, we give the definition of piecewise smooth Lipschitz domains as it is done in [51] where they are called curved linear polyhedrons.

Notation 4.2. *In this work we use the symbol Γ for the boundary of Ω, so $\Gamma = \partial\Omega$. For the outgoing normal vector we use the symbol $\mathbf{n}$. This normal vector is defined almost everywhere on the boundary Γ. We further use Ω for bounded and Ω^c for unbounded domains.*

Definition 4.3 (Piecewise smooth Lipschitz domain)**.** *A domain Ω is called a piecewise smooth Lipschitz domain if it has a piecewise smooth boundary, so that for every boundary point in $\partial\Omega$ the domain Ω can be locally transformed by a $\mathcal{C}^\infty$–diffeomorphism into a neighbourhood of a boundary point of a Lipschitz polyhedron.*

Notation 4.4. *If Ω is a Lipschitz polyhedron we denote by Γ^i the polygonal subboundaries of Ω. An edge between two neighbouring subboundaries is defined by $e_{ij} := \overline{\Gamma^i} \cap \overline{\Gamma^j}$.*

Numerical estimates for an approximate solution are often based on regularity assumptions on the sought function. This on the other hand requires regularity assumptions on the domain Ω. An often used class of regular domains are $\mathcal{C}^{k,\kappa}$ domains. This class represents domains with boundaries which posses a local parametrization which in turn have a k–th Hölder continuous derivative, see [89].

4.2 Sobolev spaces for the Helmholtz equation

In this section we introduce the standard Sobolev spaces as far as it is necessary for our purposes, further details can be found in [1, 102]. Within this section we assume that Ω is an arbitrary Lipschitz domain.

Definition 4.5. *$\mathcal{C}^k(\Omega)$ is the space of bounded k–times continuously differentiable functions on $\Omega \subset \mathbb{R}^3$. Accordingly $\mathcal{C}^\infty(\Omega) := \bigcap_{k\geq 0} \mathcal{C}^k(\Omega)$ defines the space of bounded infinitely times differentiable functions. $\mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega) \subset \mathcal{C}^\infty(\Omega)$ is the space of infinitely times differentiable functions with compact support. In addition we define the space of all functions from $\mathcal{C}^\infty(\Omega)$ with compact support by*

$$\mathcal{C}_{comp}^\infty := \mathcal{C}_0^\infty(\mathbb{R}^n)|_\Omega := \{U|_\Omega : U \in \mathcal{C}_0^\infty(\mathbb{R}^n)\}.$$

With $\mathcal{C}(\Gamma)$ we denote the continuous functions on the boundary Γ.

There are two common ways to define Sobolev spaces. The more descriptive one uses weak partial derivatives.

Definition 4.6. *For $s \in \mathbb{N}$, a multi index $\alpha \in \mathbb{N}_0^3$ and an open non–empty subset $\Omega \subset \mathbb{R}^3$, we define the Sobolev space $W_2^s(\Omega)$ by*

$$W_2^s(\Omega) := \{U \in L^2(\Omega) : \partial^\alpha U \in L^2(\Omega) \text{ for } |\alpha| \leq s\},$$

where $\partial^\alpha U$ has to be interpreted as distribution on Ω, cf. [102, p. 73]. The corresponding norm is defined by

$$\|U\|_{W_2^s(\Omega)}^2 := \left(\sum_{|\alpha|\leq s} \int_\Omega |\partial^\alpha U(x)|^2 \mathrm{d}x \right).$$

This definition can be extended to $s \in \mathbb{R}^+$ by using the Sobolev–Slobodeckii norm

$$\|U\|^2_{W_2^s(\Omega)} := \|U\|^2_{W_2^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha U(x) - \partial^\alpha U(y)|^2}{|x-y|^{3+2\kappa}} \mathrm{d}x\mathrm{d}y$$

with $s = k+\kappa$, $k \in \mathbb{N}_0$ and $\kappa \in (0,1)$. The spaces $W_2^s(\Omega)$ are Hilbert spaces, see [102].

Another way to define Sobolev spaces is by using the Fourier transformation, see [102]. This is often a more practical definition for the analysis of Sobolev spaces. These spaces are denoted by $H^s(\Omega)$. For $s > 0$ and appropriate assumptions on Ω both definitions are equivalent, i.e.,

$$W^s(\mathbb{R}^3) = H^s(\mathbb{R}^3), \qquad W^s(\Omega) = H^s(\Omega), \qquad H^0(\Omega) = L^2(\Omega)$$

holds, see [102, 140]. The dual space of $H^s(\Omega)$ is denoted by $\widetilde{H}^{-s}(\Omega)$ for all $s \in \mathbb{R}$.

Remark 4.7. *If we say that a function u is in the dual space of X, we mean that $\langle u, \cdot \rangle$ is in the dual space X'.*

For unbounded domains $\Omega^c = \mathbb{R} \setminus \overline{\Omega}$, it is not always suitable to use the space $H^s(\Omega^c)$, since also very regular functions can fail to be in this space, because they may have an unbounded support. So we use the space of locally $H^s(\Omega^c)$–regular functions.

Definition 4.8. *For $s \in \mathbb{R}$, the space $H^s_{loc}(\Omega^c)$ contains all linear functionals U on $\mathcal{C}^\infty(\Omega^c)$, such that $\phi U \in H^s(\Omega^c)$ for all $\phi \in \mathcal{C}^\infty_{comp}(\Omega^c)$.*

The dual space of $H^{-s}_{loc}(\Omega^c)$ is for all $s \in \mathbb{R}$ given by

$$H^s_{comp}(\Omega^c) := \bigcup_K \{U \in H^s_{loc}(\Omega^c) : \mathrm{supp}(U) \subset K\},$$

for all compact $K \subset \Omega^c$.

Remark 4.9. *The same definitions for bounded domains Ω would lead to three coinciding spaces $H^s(\Omega)$, $H^s_{loc}(\Omega)$ and $H^s_{comp}(\Omega)$.*

For a more detailed explanation see [121, p. 48]. Since several boundary integral operators related to the Helmholtz equation fail to be elliptic in some sense, we will be forced to use so–called Gårding inequalities intensively. These in turn are based on the following theorem.

Theorem 4.10 (Lemma of Rellich)**.** *Let Ω be a bounded Lipschitz domain, then the embeddings $H^s(\Omega) \subset H^t(\Omega)$ and $H^s_{loc}(\Omega^c) \subset H^t_{loc}(\Omega^c)$ are compact for $s > t$.*

Proof. See [121, p. 47 ff]. □

4.2.1 Sobolev spaces on the boundary

Since we primarily deal with boundary integral operators in the following chapters, it is essential to introduce Sobolev spaces on the boundary $\Gamma = \partial\Omega$. The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ are defined by

$$\begin{aligned} L_2(\Gamma) &:= \overline{\mathcal{C}(\Gamma)}^{\|\cdot\|_{L_2(\Gamma)}}, \\ H^s(\Gamma) &:= \overline{\mathcal{C}(\Gamma)}^{\|\cdot\|_{H^s(\Gamma)}} \qquad \text{for } s \in (0,1) \end{aligned}$$

The norm $\|\cdot\|_{H^s(\Gamma)}$ is in the case of a Lipschitz boundary Γ and for $s \in (0,1)$ equivalent to the Sobolev–Slobodeckii norm on the boundary, which is given by

$$\|u\|^2_{W_2^s(\Gamma)} := \|u\|^2_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^{2+2s}} \mathrm{d}s_x \mathrm{d}s_y.$$

The intrinsic definition of $\|\cdot\|_{H^s(\Gamma)}$ can be found in [102]. The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ are Hilbert spaces.

For $s \in (-1,0)$ is the space $H^s(\Gamma)$ defined by duality with respect to the $L_2(\Gamma)$–inner product

$$\langle u, v\rangle_{L_2(\Gamma)} := \int_\Gamma u(x) \cdot \overline{v}(x)\, \mathrm{d}s_x.$$

More precisely for $s < 0$ we define the norm

$$\|u\|_{H^s(\Gamma)} := \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{\langle u, v\rangle_{L_2(\Gamma)}}{\|v\|_{H^{-s}(\Gamma)}}.$$

The closure of $L_2(\Gamma)$ with respect to $\|\cdot\|_{H^s(\Gamma)}$ is denoted by $H^s(\Gamma)$.

The $L_2(\Gamma)$ inner product can be extended to a duality pairing on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, which will be denoted by $\langle \cdot, \cdot \rangle_\Gamma$, for details see [89, 102].

For Lipschitz polyhedrons we will also need the space $H^1(\Gamma)$, for our needs we use the definition

$$H^1(\Gamma) := \left\{ \phi \in L^2(\Gamma) : \phi|_{\Gamma^j} \in H^1(\Gamma^j) \text{ and } \phi_i|_{e_{ij}} = \phi_j|_{e_{ij}} \text{ in } H^{1/2}(e_{ij}) \right\}.$$

$H^{-1}(\Gamma)$ is again defined via duality,

$$H^{-1}(\Gamma) := [H^1(\Gamma)]'.$$

In domain decomposition approaches, equations on interfaces appear naturally. Thus we want to give a short definition of Sobolev spaces on open manifolds.

Definition 4.11 (Sobolev spaces on open manifolds)**.** *For an open part $\Gamma_0 \subset \Gamma$ of the boundary, Sobolev spaces of the order $s \geq 0$ are defined by*

$$H^s(\Gamma_0) := \{u = \widetilde{u}|_{\Gamma_0} : \widetilde{u} \in H^s(\Gamma)\},$$
$$\widetilde{H}^s(\Gamma_0) := \{u = \widetilde{u}|_{\Gamma_0} : \widetilde{u} \in H^s(\Gamma) \text{ and } \operatorname{supp} \widetilde{u} \subset \overline{\Gamma}_0\}$$

with the norm

$$\|u\|_{H^s(\Gamma_0)} := \inf\{\|\widetilde{u}\|_{H^s(\Gamma)} : \widetilde{u} \in H^s(\Gamma) \text{ and } \widetilde{u}_{|\Gamma_0} = u\}.$$

Sobolev spaces of negative order on open manifolds are once more defined via duality. For $s > 0$, we define

$$H^{-s}(\Gamma_0) := [\widetilde{H}^s(\Gamma_0)]',$$
$$\widetilde{H}^{-s}(\Gamma_0) := [H^s(\Gamma_0)]'.$$

The Dirichlet trace operator γ_0 is for smooth functions $U \in \mathcal{D}(\overline{\Omega})$ defined by

$$\gamma_0 U := U|_\Gamma.$$

Theorem 4.12. *If Ω is a $\mathcal{C}^{k-1,1}$ domain and $1/2 < s \leq k$, then γ_0 has a unique extension to a bounded linear operator*

$$\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\Gamma).$$

This extension has a unique right inverse.

Proof. See [102].

The Neumann trace operator γ_1 is for smooth functions $U \in \mathcal{D}(\overline{\Omega})$ defined by

$$\gamma_1 U := \mathbf{n} \cdot \boldsymbol{\nabla} U|_\Gamma.$$

This definition can be easily extended to functions in $H^2(\Omega)$, but not for general functions in $H^1(\Omega)$. Fortunately, solutions of the potential equation $\Delta U = F$ with F in the space $\widetilde{H}^{-1}(\Omega)$ have a well–defined Neumann trace, which depends on U and F. If F is in $L^2(\Omega)$ then γ_1 is a continuous linear operator

$$\gamma_1 : H(\Delta, \Omega) := \{U \in L^2(\Omega) : \Delta U \in L^2(\Omega)\} \to H^{-1/2}(\Gamma)$$

which only depends on U, see, for example, [74]. Nevertheless, if F is clear from the context we will neglect the dependency from F in the notation and just write $\gamma_1 U$. For a more detailed explanation see, for example, [102].

Notation 4.13. *In the upcoming, bounded domains Ω will be denoted as interior domains. The complement $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$ will be called exterior domain. For the exterior domain the traces are defined as for the interior domain, we only exchange the Sobolev spaces by there localized versions, cf. Definition 4.8. Trace operators for the exterior domain will be marked by the superscript $\cdot^c$. Since we use the outgoing normal vector in both cases, we have $\gamma_1^c U = \gamma_1 U$ for every function $U \in \mathcal{C}^1(\mathbb{R}^3)$.*

4.3 Sobolev spaces for the Maxwell equation

In this section we give a short introduction to Sobolev spaces for the electromagnetic wave equation. We will use the pioneering works [31, 32] as guideline, while also using some ideas from [26] and [105]. Important tools for our needs are the so called Helmholtz– or Hodge–type decompositions which we will discuss in Subsection 4.3.2. This will be mainly based on [84] and [105]. We assume now, that $\Omega \subset \mathbb{R}^3$ is a Lipschitz–polyhedron, keeping in mind that the theory can be extended to the more general case of piecewise smooth Lipschitz domains.

Remark 4.14. *Appropriate Sobolev spaces for the electromagnetic wave equation on ordinary Lipschitz domains are discussed in [34].*

In the case of the electromagnetic wave equation we are dealing with vector–valued physical quantities. Hence we have to carry over the definition of Sobolev spaces for scalar functions to vector–valued functions. This can be done in a canonical way by

$$\mathbf{H} := [H]^3 := \{\mathbf{U} : U_i \in H, i = 1, 2, 3\}, \quad \|\mathbf{U}\|_{\mathbf{H}}^2 := \sum_{l=1}^{3} \|U_l\|_H^2 .$$

Remark 4.15. *We mark vector–valued functions and other vector–valued quantities by bold letters, the only exception are points in $\mathbb{R}^3$ and several operators, since they have a distinguished notation. Functions in the domain are be denoted by upper case letters, whereas functions on the boundary are denoted by lower case letters.*

For further considerations we have to investigate certain properties of functions $\mathbf{U}$ satisfying the partial differential equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} + k^2\mathbf{U} = \mathbf{0},$$

with $k \in \mathbb{C} \setminus \{0\}$.

Theorem 4.16 (Green's first formula)**.** *For sufficiently smooth vector–valued functions $\mathbf{U}$ and $\mathbf{V}$ there holds*

$$\begin{aligned}\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\mathrm{d}x = &\int_\Omega \mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\overline{\mathbf{V}}(x)\mathrm{d}x \\ &- \int_\Gamma (\mathbf{curl}\,\mathbf{U}_{|\Gamma(x)} \times \mathbf{n}(x)) \cdot (\mathbf{n}(x) \times (\overline{\mathbf{V}}_{|\Gamma(x)} \times \mathbf{n}(x)))\mathrm{d}s_x.\end{aligned} \tag{4.1}$$

Proof. From the identity

$$\mathrm{div}\,(\mathbf{W} \times \mathbf{V}) = \mathbf{V} \cdot \mathbf{curl}\,\mathbf{W} - \mathbf{W} \cdot \mathbf{curl}\,\mathbf{V}$$

it follows that

$$\begin{aligned}\int_\Omega (\mathbf{V}(x)\cdot\mathbf{curl}\,\mathbf{W}(x)-\mathbf{W}(x)\cdot\mathbf{curl}\,\mathbf{V}(x))\mathrm{d}x &= \int_\Omega \mathrm{div}\,(\mathbf{W}(x)\times\mathbf{V}(x))\mathrm{d}x\\ &= \int_\Gamma (\mathbf{W}(x)\times\mathbf{V}(x))\cdot\mathbf{n}(x)\,\mathrm{d}s_x\\ &= -\int_\Gamma (\mathbf{W}(x)\times\mathbf{n}(x))\cdot\mathbf{V}(x)\mathrm{d}s_x\\ &= -\int_\Gamma (\mathbf{W}(x)\times\mathbf{n}(x))\cdot(\mathbf{n}(x)\times(\mathbf{V}(x)\times\mathbf{n}(x)))\mathrm{d}s_x.\end{aligned}$$

Setting $\mathbf{W}=\mathbf{curl}\,\mathbf{U}$ and replacing $\mathbf{V}$ by $\overline{\mathbf{V}}$ proves the theorem. □

Motivated by Green's first formula we define the energy space for the Maxwell equations.

Definition 4.17. *The energy space related to the Maxwell equations is defined by*

$$\mathbf{H}(\mathbf{curl},\Omega):=\{\mathbf{V}\in\mathbf{L}_2(\Omega):\mathbf{curl}\,\mathbf{V}\in\mathbf{L}_2(\Omega)\}$$

with the energy norm

$$\|\mathbf{V}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}:=\|\mathbf{V}\|^2_{\mathbf{L}_2(\Omega)}+\|\mathbf{curl}\,\mathbf{V}\|^2_{\mathbf{L}_2(\Omega)}.$$

In the same manner, as in Definition 4.17 we define the auxiliary space

$$\mathbf{H}(\mathrm{div}\,,\Omega):=\{\mathbf{V}\in\mathbf{L}_2(\Omega):\mathrm{div}\;\mathbf{V}\in L_2(\Omega)\},$$

endowed with its graph norm

$$\|\mathbf{V}\|^2_{\mathbf{H}(\mathrm{div}\,,\Omega)}:=\|\mathbf{V}\|^2_{\mathbf{L}_2(\Omega)}+\|\mathrm{div}\;\mathbf{V}\|^2_{L_2(\Omega)}.$$

Furthermore, the trace operators are also motivated by (4.1). The Dirichlet trace operator is given by

$$\boldsymbol{\gamma}_D\mathbf{U}:=\mathbf{n}\times(\mathbf{U}_{|\Gamma}\times\mathbf{n})=\mathbf{n}\times\boldsymbol{\gamma}_\times\mathbf{U}$$

with

$$\boldsymbol{\gamma}_\times\mathbf{U}:=\mathbf{U}_{|\Gamma}\times\mathbf{n},$$

and the Neumann trace operator by

$$\boldsymbol{\gamma}_N\mathbf{U}:=\mathbf{curl}\,\mathbf{U}_{|\Gamma}\times\mathbf{n}.$$

To prove mapping properties of the trace operators we must first introduce tangential Sobolev spaces.

Definition 4.18. *The space of tangential* $\mathbf{L}_2$*–integrable functions is defined by*

$$\mathbf{L}_{2,t}(\Gamma) := \{\mathbf{u} \in \mathbf{L}_2(\Gamma) : \mathbf{u} \cdot \mathbf{n} = 0\}.$$

Remark 4.19. *The space* $\mathbf{L}_{2,t}(\Gamma)$ *can be identified by the space of two–dimensional quadratic integrable vector fields.*

For Sobolev spaces of higher order we use a piecewise definition, since for $s \geq 1/2$ we lose regularity at the edges, cf. [26],

$$\mathbf{H}^s_{pw,t}(\Gamma) := \{\mathbf{u} \in \mathbf{L}_{2,t}(\Gamma) : \mathbf{u} \in \mathbf{H}^s(\Gamma^k)\,, k = 1, \ldots, N_\Gamma\}$$

with the norm

$$\|\mathbf{u}\|^2_{\mathbf{H}^s_{pw,t}(\Gamma)} := \sum_{k=1}^{N_\Gamma} \|\mathbf{u}\|^2_{\mathbf{H}^s(\Gamma^k)}\,.$$

For the trace operators $\boldsymbol{\gamma}_D$ and $\boldsymbol{\gamma}_\times$ we have the mapping properties

$$\begin{aligned}\boldsymbol{\gamma}_D &: \boldsymbol{\mathcal{D}}(\overline{\Omega}) \to \mathbf{H}^{1/2}_{pw,t}(\Gamma),\\ \boldsymbol{\gamma}_\times &: \boldsymbol{\mathcal{D}}(\overline{\Omega}) \to \mathbf{H}^{1/2}_{pw,t}(\Gamma),\end{aligned}$$

see [26]. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{H}^1(\Omega)$ we can extend these two operators to $\mathbf{H}^1(\Omega)$. But they are not surjective from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}_{pw,t}(\Gamma)$, and have different open images. Hence, we have to introduce two additional spaces which are based on the functionals

$$\begin{aligned}\mathsf{N}^{\|}_{lk}(\mathbf{u}) &:= \int_{\Gamma^l}\int_{\Gamma^k} \frac{|\mathbf{u}(x)\cdot\mathbf{t}_{lk}(x) - \mathbf{u}(y)\cdot\mathbf{t}_{lk}(y)|^2}{|x-y|^3}\mathrm{d}s_x\mathrm{d}s_y,\\ \mathsf{N}^{\perp}_{lk}(\mathbf{u}) &:= \int_{\Gamma^l}\int_{\Gamma^k} \frac{|\mathbf{u}(x)\cdot\mathbf{t}_{l}(x) - \mathbf{u}(y)\cdot\mathbf{t}_{k}(y)|^2}{|x-y|^3}\mathrm{d}s_x\mathrm{d}s_y,\end{aligned}$$

which are defined for two neighbouring polygons with indices l and k and the common edge e_{lk}. Hereby is $\mathbf{t}_{lk}$ the unit vector with the direction of e_{lk} and $\mathbf{t}_l$ is for the plane Γ_l defined by $\mathbf{t}_l := \mathbf{t}_{lk} \times \mathbf{n}$.

Definition 4.20. *We define*

$$\begin{aligned}\mathbf{H}^{1/2}_{\|}(\Gamma) &:= \left\{\mathbf{u} \in \mathbf{H}^{1/2}_{pw,t}(\Gamma) : \mathsf{N}^{\|}_{lk}(\mathbf{u}) < \infty \text{ for all edges } e_{lk}\right\},\\ \mathbf{H}^{1/2}_{\perp}(\Gamma) &:= \left\{\mathbf{u} \in \mathbf{H}^{1/2}_{pw,t}(\Gamma) : \mathsf{N}^{\perp}_{lk}(\mathbf{u}) < \infty \text{ for all edges } e_{lk}\right\}\end{aligned}$$

with the corresponding norms

$$\begin{aligned}\|\mathbf{u}\|^2_{\mathbf{H}^{1/2}_{\|}(\Gamma)} &:= \sum_{k=1}^{N_\Gamma} \|\mathbf{u}\|^2_{\mathbf{H}^{1/2}(\Gamma^k)} + \sum_{e_{lk}} \mathsf{N}^{\|}_{lk}(\mathbf{u}),\\ \|\mathbf{u}\|^2_{\mathbf{H}^{1/2}_{\perp}(\Gamma)} &:= \sum_{k=1}^{N_\Gamma} \|\mathbf{u}\|^2_{\mathbf{H}^{1/2}(\Gamma^k)} + \sum_{e_{lk}} \mathsf{N}^{\perp}_{lk}(\mathbf{u}).\end{aligned} \tag{4.2}$$

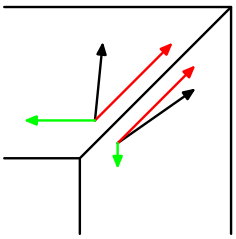


Figure 4.1: $\mathbf{H}_{\parallel}^{1/2}(\Gamma)$

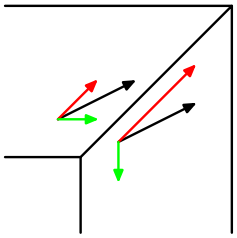


Figure 4.2: $\mathbf{H}_{\perp}^{1/2}(\Gamma)$

Theorem 4.21. *The spaces* $\mathbf{H}_{\parallel}^{1/2}(\Gamma)$ *and* $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ *aren't closed subspaces of* $\mathbf{H}_{pw,t}^{1/2}(\Gamma)$ *with respect to the norm defined in* (4.2)*, but they are Hilbert spaces.*

Proof. See [31]. □

Theorem 4.22. *The trace operators* $\boldsymbol{\gamma}_D : \mathbf{H}^1(\Omega) \to \mathbf{H}_{\parallel}^{1/2}(\Gamma)$ *and* $\boldsymbol{\gamma}_\times : \mathbf{H}^1(\Omega) \to \mathbf{H}_{\perp}^{1/2}(\Gamma)$ *are continuous, linear, surjective and posses continuous right inverses* R_D *and* $R_\times$*, respectively.*

Proof. See [31]. □

$\mathbf{H}_{\parallel}^{1/2}(\Gamma)$ describes a space with tangential continuity over the edge, see Figure 4.1. $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ in contrast provides perpendicular continuity, see Figure 4.2. The corresponding dual spaces are

$$\mathbf{H}_{\parallel}^{-1/2}(\Gamma) := [\mathbf{H}_{\parallel}^{1/2}(\Gamma)]', \qquad \mathbf{H}_{\perp}^{-1/2}(\Gamma) := [\mathbf{H}_{\perp}^{1/2}(\Gamma)]'.$$

Rellich's embedding theorem can be extended to those trace spaces.

Lemma 4.23. *The embeddings* $\mathbf{H}_{\perp}^{1/2}(\Gamma) \hookrightarrow \mathbf{L}_t^2(\Gamma)$ *and* $\mathbf{H}_{\parallel}^{1/2}(\Gamma) \hookrightarrow \mathbf{L}_t^2(\Gamma)$ *are compact.*

Proof. The right inverse R_D of $\boldsymbol{\gamma}_D$ is continuous from $\mathbf{H}_{\parallel}^{1/2}(\Gamma)$ to $\mathbf{H}^1(\Omega)$. The embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{H}^{1/2}(\Omega)$ is compact. Finally the trace operator $\boldsymbol{\gamma}_D$ is continuous from $\mathbf{H}^{1/2}(\Omega)$ to $\mathbf{L}_2(\Gamma)$ due to its definition. The second proposition can be proved analogous, see also [84]. □

4.3.1 Surface derivatives

Surface derivatives play a key role in the investigation of Sobolev spaces and of boundary integral operators for the electromagnetic wave equation. As a starting point we introduce the surface gradient $\boldsymbol{\nabla}_\Gamma$ and the surface curl $\mathbf{curl}_\Gamma$. These operators can be defined in several ways, we have chosen the following one.

Definition 4.24. *Since we deal with polyhedrons, the surface gradient* $\boldsymbol{\nabla}_\Gamma : H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$ *can be defined face by face*

$$(\boldsymbol{\nabla}_\Gamma \phi)_i = \boldsymbol{\nabla}_2(\phi_i) \qquad \text{on } \Gamma^i,$$

whereas $\boldsymbol{\nabla}_2$ *is the classical two dimensional differential operator with respect to the plane spanned by* Γ^i*, cf. [32]. The surface curl* $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$ *is given by* $\mathbf{curl}_\Gamma \cdot := \boldsymbol{\nabla}_\Gamma \cdot \times \mathbf{n}$*. Alternative definitions of these two operators can be found in [31].*

The operators $\boldsymbol{\nabla}_\Gamma$ and $\mathbf{curl}_\Gamma$ are linear and continuous from $H^1(\Gamma)$ to $\mathbf{L}_t^2(\Gamma)$. Based on these operators we can define the scalar surface derivatives by duality.

Definition 4.25. *The surface divergence* $\mathrm{div}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ *is defined via the duality*

$$\langle \mathrm{div}_\Gamma \mathbf{u}, \phi \rangle_\Gamma = -\langle \mathbf{u}, \boldsymbol{\nabla}_\Gamma \phi \rangle_\Gamma$$

for all $\phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma)$*. In the same way we define the scalar surface curl* $\mathrm{curl}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ *by*

$$\langle \mathrm{curl}_\Gamma \mathbf{u}, \phi \rangle_\Gamma = \langle \mathbf{u}, \mathbf{curl}_\Gamma \phi \rangle_\Gamma$$

for all $\phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma)$*.*

The upcoming natural trace spaces are based on extensions of the vector–valued surface derivatives to more and less regular spaces such as $H^{3/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. More precisely they can be extended to linear and continuous operators

$$\begin{aligned} \boldsymbol{\nabla}_\Gamma : H^{3/2}(\Gamma) \to \mathbf{H}_\parallel^{1/2}(\Gamma), \qquad \boldsymbol{\nabla}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}_\parallel^{-1/2}(\Gamma), \\ \mathbf{curl}_\Gamma : H^{3/2}(\Gamma) \to \mathbf{H}_\perp^{1/2}(\Gamma), \quad \mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}_\perp^{-1/2}(\Gamma), \end{aligned}$$

with $H^{3/2}(\Gamma) := \left\{ \phi \in H^1(\Gamma) : \boldsymbol{\nabla}_\Gamma \phi \in \mathbf{H}_\parallel^{1/2}(\Gamma) \right\}$, see [31].

For their scalar counterparts,

$$\begin{aligned} \mathrm{div}_\Gamma : \mathbf{H}_\parallel^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma), \quad \mathrm{div}_\Gamma : \mathbf{H}_\parallel^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ \mathrm{curl}_\Gamma : \mathbf{H}_\perp^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma), \quad \mathrm{curl}_\Gamma : \mathbf{H}_\perp^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \end{aligned}$$

holds. These are the basic ingredients required to prove the mapping properties of the trace operators. We do not provide a proof for the mapping properties of the surface differential operators, but we think it is essential to state these mapping properties, in order to get a rough idea how these spaces are constructed. For a detailed deduction

see [31, 32]. Based on these mapping properties we can define the final trace spaces. Let's start with a short motivation and consider

$$\begin{aligned}\int_\Omega (\mathbf{V}(x)\cdot\mathbf{curl}\,\mathbf{W}(x)-\mathbf{W}(x)\cdot\mathbf{curl}\,\mathbf{V}(x))\mathrm{d}x &= \int_\Omega \mathrm{div}\,(\mathbf{W}(x)\times\mathbf{V}(x))\mathrm{d}x\\ &= -\int_\Gamma \boldsymbol{\gamma}_\times\mathbf{W}(x)\cdot\boldsymbol{\gamma}_D\mathbf{V}(x)\mathrm{d}s_x,\end{aligned} \tag{4.3}$$

which was used when deducing Green's first formula (4.1) for sufficiently smooth functions. For $\mathbf{W}\in H^1(\Omega)$ we have $\boldsymbol{\gamma}_\times\mathbf{W}\in\mathbf{H}_\perp^{1/2}(\Gamma)$ which encourages us to extend $\boldsymbol{\gamma}_D$ to a mapping from $\mathbf{H}(\mathbf{curl},\Omega)$ to $\mathbf{H}_\perp^{-1/2}(\Gamma)$, which is a continuous but not surjective extension.

Taking (4.3), setting $\mathbf{V}=\mathbf{grad}\,\phi$ and using Stoke's formulae for the surface (see [109]) we get

$$\begin{aligned}\int_\Omega \mathbf{curl}\,\mathbf{W}(x)\cdot\mathbf{grad}\,\phi(x)\,\mathrm{d}x &= \int_\Omega \mathbf{curl}\,\mathbf{W}(x)\cdot\mathbf{grad}\,\phi(x)-\mathbf{W}(x)\cdot\mathbf{curl}\,\mathbf{grad}\,\phi(x)\mathrm{d}x\\ &= \int_\Gamma \boldsymbol{\gamma}_D\mathbf{W}(x)\cdot\boldsymbol{\gamma}_\times\,\mathbf{grad}\,\phi(x)\,\mathrm{d}s_x\\ &= \int_\Gamma \boldsymbol{\gamma}_D\mathbf{W}(x)\cdot(\boldsymbol{\nabla}_\Gamma\,\phi(x)\times\mathbf{n}(x))\mathrm{d}s_x\\ &= -\int_\Gamma \boldsymbol{\gamma}_D\mathbf{W}(x)\cdot\mathbf{curl}_\Gamma\,\phi(x)\,\mathrm{d}s_x\\ &= -\int_\Gamma \mathrm{curl}_\Gamma(\boldsymbol{\gamma}_D\mathbf{W}(x))\gamma_0\phi(x)\,\mathrm{d}s_x.\end{aligned}$$

Hence we can derive the inequality

$$\begin{aligned}\left|\int_\Gamma \mathrm{curl}_\Gamma(\boldsymbol{\gamma}_D\mathbf{W}(x))\gamma_0\phi(x)\,\mathrm{d}s_x\right| &= \left|\int_\Omega \mathbf{curl}\,\mathbf{W}(x)\cdot\mathbf{grad}\,\phi(x)\,\mathrm{d}x\right|\\ &\le \|\mathbf{curl}\,\mathbf{W}\|_{\mathbf{L}_2(\Omega)}\,\|\mathbf{grad}\,\phi\|_{\mathbf{L}_2(\Omega)}\\ &\le \|\mathbf{W}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,\|\phi\|_{H^1(\Omega)}\\ &\le c\,\|\mathbf{W}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,\|\gamma_0\phi\|_{H^{1/2}(\Gamma)},\end{aligned}$$

which indicates that the surface curl of the Dirichlet trace should be in $H^{-1/2}(\Gamma)$.

Definition 4.26. *The trace spaces are defined by*

$$\begin{aligned}\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma) &:= \left\{\mathbf{u}\in\mathbf{H}_\parallel^{-1/2}(\Gamma):\mathrm{div}_\Gamma\,\mathbf{u}\in H^{-1/2}(\Gamma)\right\},\\ \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) &:= \left\{\mathbf{u}\in\mathbf{H}_\perp^{-1/2}(\Gamma):\mathrm{curl}_\Gamma\,\mathbf{u}\in H^{-1/2}(\Gamma)\right\}\end{aligned}$$

with the corresponding norms

$$\|\mathbf{u}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := \|\mathbf{u}\|^2_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma\,\mathbf{u}\|^2_{H^{-1/2}(\Gamma)}\,,$$
$$\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} := \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\Gamma)} + \|\mathrm{curl}_\Gamma\,\mathbf{u}\|^2_{H^{-1/2}(\Gamma)}\,.$$

Based on these definitions we can state one of the major theorems for Maxwell Sobolev trace spaces.

Theorem 4.27 (Trace theorem)**.** *The Dirichlet traces*

$$\boldsymbol{\gamma}_D : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma),$$
$$\boldsymbol{\gamma}_\times : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$$

are linear, continuous and surjective.

Proof. See [31, 32]. □

The twisting operator $\mathbf{R} = \cdot \times \mathbf{n}$ can be extended to a linear continuous and isometric operator

$$\mathbf{R} : \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma),$$

see [26, p. 27].

To investigate the Neumann trace operator $\boldsymbol{\gamma}_N$, we first have to introduce the space

$$\mathbf{H}(\mathbf{curl}^2,\Omega) := \{\mathbf{V} \in \mathbf{H}(\mathbf{curl},\Omega) : \mathbf{curl}\,\mathbf{curl}\,\mathbf{V} \in \mathbf{L}_2(\Omega)\}$$

with the norm

$$\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl}^2,\Omega)} := \|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}\|^2_{\mathbf{L}_2(\Omega)}\,.$$

Since the mapping $\mathbf{curl} : \mathbf{H}(\mathbf{curl}^2,\Omega) \to \mathbf{H}(\mathbf{curl},\Omega)$ is linear, continuous and surjective, we deduce from Theorem 4.27

$$\boldsymbol{\gamma}_N : \mathbf{H}(\mathbf{curl}^2,\Omega) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma).$$

Another significant result is the duality of the Dirichlet and Neumann trace spaces. This provides a framework for the analysis of boundary integral operators, which is quite similar to the framework in the Helmholtz case.

Theorem 4.28. *The Dirichlet and the Neumann trace spaces are dual to each other, i.e.*

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) = [\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)]'.$$

Proof. See [32]. □

To deduce the representation formula for solutions of the electromagnetic wave equation, it is, in contrast to its acoustic counterpart, necessary to define another trace operator.

Definition 4.29. *The trace operator* $\gamma_n : \boldsymbol{\mathcal{D}}(\overline{\Omega})^3 \to L_2(\Gamma)$ *is defined by*

$$\gamma_n \mathbf{U} = \mathbf{U}|_\Gamma \cdot \mathbf{n}.$$

Theorem 4.30. *The operator* γ_n *can be extended to a linear and continuous operator*

$$\gamma_n : \mathbf{H}(\operatorname{div}, \Omega) \to H^{-1/2}(\Gamma).$$

Proof. See [26]. □

With the mapping properties established, we can state an auxiliary proposition which is needed to state the upcoming representation formula, see (6.6), in a compact form.

Lemma 4.31. *The identities*

$$\gamma_n \circ \mathbf{curl}\, \mathbf{U} = \operatorname{curl}_\Gamma \circ \boldsymbol{\gamma}_D \mathbf{U} = \operatorname{div}_\Gamma \circ \boldsymbol{\gamma}_\times \mathbf{U}$$

hold for $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$*, in the sense of* $H^{-1/2}(\Gamma)$*.*

Proof. See [82]. □

Finally, we formulate Green's formulae with the natural spaces.

Theorem 4.32 (Green's first formula)**.** *For* $\mathbf{U} \in \mathbf{H}(\mathbf{curl}^2, \Omega)$ *and* $\mathbf{V} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *there holds*

$$\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\mathrm{d}x = \int_\Omega \mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\overline{\mathbf{V}}(x)\mathrm{d}x - \int_\Gamma \boldsymbol{\gamma}_N \mathbf{U}(x) \cdot \boldsymbol{\gamma}_D \overline{\mathbf{V}}(x)\mathrm{d}s_x.$$

Proof. See [32]. □

Theorem 4.33 (Green's second formula)**.** *For* $\mathbf{U} \in \mathbf{H}(\mathbf{curl}^2, \Omega)$ *and* $\mathbf{V} \in \mathbf{H}(\mathbf{curl}^2, \Omega)$ *there holds*

$$\begin{aligned}\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\mathrm{d}x - \int_\Omega \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{V}}(x) \cdot \mathbf{U}(x)\mathrm{d}x \\ = -\int_\Gamma \boldsymbol{\gamma}_N \mathbf{U}(x) \cdot \boldsymbol{\gamma}_D \overline{\mathbf{V}}(x)\mathrm{d}s_x + \int_\Gamma \boldsymbol{\gamma}_N \overline{\mathbf{V}}(x) \cdot \boldsymbol{\gamma}_D \mathbf{U}(x)\mathrm{d}s_x.\end{aligned}$$

Proof. This can be deduced by applying Green's first formula twice. □

4.3.2 Decompositions

In this section we introduce splittings for the space $\mathbf{H}(\mathbf{curl}, \Omega)$ and the trace spaces $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma}, \Gamma)$ and $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)$, as proposed in [84, 105]. These splittings are needed to establish generalized Gårding inequalities for various variational formulations of boundary integral integral equations which are related to the Maxwell equations.

Lemma 4.34. *There exists a continuous lifting operator*

$$L : \mathbf{H}(\mathrm{div}\ 0, \Omega) := \{\mathbf{U} \in \mathbf{L}_2(\Omega), \mathrm{div}\ \mathbf{U} = 0\} \to \mathbf{H}^1(\Omega)$$

that satisfies $\mathrm{div}\ L\mathbf{U} = 0$ *and* $\mathbf{curl}\, L\mathbf{U} = \mathbf{U}$ *for all* $\mathbf{U} \in \mathbf{H}(\mathrm{div}\ 0, \Omega)$.

Proof. See [6, Lemma 3.5]. □

Based on this lifting operator we can define the projection operator

$$\begin{aligned} P : \mathbf{H}(\mathbf{curl}, \Omega) &\to \mathbf{H}^1(\Omega), \\ P\mathbf{U} &:= L(\mathbf{curl}\, \mathbf{U}). \end{aligned}$$

Lemma 4.35. *The operator P is a projection and satisfies*

- $\mathbf{curl}\, P\mathbf{U} = \mathbf{curl}\, \mathbf{U}$ *for all* $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$,
- $P\mathbf{U} = \mathbf{0}$ *for all* $\mathbf{U} \in \mathbf{H}(\mathbf{curl}\, \mathbf{0}, \Omega)$,
- $\|P\mathbf{U}\|_{\mathbf{H}^1(\Omega)} \leq C \|\mathbf{curl}\, \mathbf{U}\|_{\mathbf{L}^2(\Omega)}$ *for all* $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$.

Proof. All claims except of the projection property follow from Lemma 4.34. From $P\mathbf{U} = \mathbf{U} + \mathbf{Z}$ with $\mathbf{curl}\, \mathbf{Z} = \mathbf{0}$ it follows $P^2\mathbf{U} = P\mathbf{U} + P\mathbf{Z} = P\mathbf{U}$. □

Due to $\mathrm{Ker}(P) = \mathrm{Ker}(\mathbf{curl}) \cap \mathbf{H}(\mathbf{curl}, \Omega)$ we can define

$$\begin{aligned} \mathbf{X}(\mathbf{curl}, \Omega) &:= P(\mathbf{H}(\mathbf{curl}, \Omega)), \\ \mathbf{N}(\mathbf{curl}, \Omega) &:= \mathrm{Ker}(\mathbf{curl}) \cap \mathbf{H}(\mathbf{curl}, \Omega). \end{aligned}$$

This provides a stable and direct Helmholtz–type splitting

$$\mathbf{H}(\mathbf{curl}, \Omega) = \mathbf{X}(\mathbf{curl}, \Omega) \oplus \mathbf{N}(\mathbf{curl}, \Omega).$$

Since P is continuous and $\mathbf{X}(\mathbf{curl}, \Omega) \subset \mathbf{H}^1(\Omega)$ we get the following corollary.

Corollary 4.36. *The embedding* $\mathbf{X}(\mathbf{curl}, \Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ *is compact.*

A similar splitting can be found for the Neumann trace space $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$. Let $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and set $\omega := \operatorname{div}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)$. We define $\Psi \in H^1(\Omega)/\mathbb{R}$ as the solution of the Neumann problem

$$\begin{aligned} \Delta \Psi &= 0 \qquad \text{in } \Omega, \\ \gamma_1 \Psi &= \omega \qquad \text{on } \Gamma. \end{aligned}$$

Note that $\omega = \operatorname{div}_\Gamma \boldsymbol{\lambda}$ fulfills the required solvability condition of the above Neumann problem since

$$\int_\Gamma 1 \cdot \operatorname{div}_\Gamma \lambda(x) \mathrm{d}s_x = \int_\Gamma \boldsymbol{\nabla}_\Gamma 1 \cdot \lambda(x) \mathrm{d}s_x = 0.$$

Hence we can define $\mathbf{W} := \mathbf{grad}\, \Psi \in \mathbf{H}(\operatorname{div} 0, \Omega)$. Again by using the lifting operator L we can define a continuous operator $J : H^{-1/2}(\Gamma) \to \mathbf{H}^1(\Omega)$ by $J\omega := L\mathbf{W}$. Continuity follows from

$$\|J\omega\|_{H^1(\Omega)} \leq C \|\mathbf{W}\|_{L^2(\Omega)} \leq C \|\omega\|_{H^{-1/2}(\Gamma)}.$$

Now it is possible to define a similar projection as before by

$$P^\Gamma := \boldsymbol{\gamma}_\times \circ J \circ \operatorname{div}_\Gamma : \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \mapsto \mathbf{H}_\perp^{1/2}(\Gamma).$$

Corollary 4.37. *The operator P^Γ is a projection and satisfies*

- $\operatorname{div}_\Gamma P^\Gamma \boldsymbol{\lambda} = \operatorname{div}_\Gamma \boldsymbol{\lambda}$ *for all* $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$.
- $P^\Gamma \boldsymbol{\lambda} = \mathbf{0}$ *for* $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ *and* $\operatorname{div}_\Gamma \boldsymbol{\lambda} = 0$.
- $\left\|P^\Gamma \boldsymbol{\lambda}\right\|_{\mathbf{H}_\perp^{1/2}(\Gamma)} \leq C \|\operatorname{div}_\Gamma \boldsymbol{\lambda}\|_{H^{-1/2}(\Gamma)}$ *for all* $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$.

Proof. The first claim follows from recent definitions and Lemma 4.31, more precisely

$$\begin{aligned} \operatorname{div}_\Gamma P^\Gamma \boldsymbol{\lambda} &= \operatorname{div}_\Gamma \boldsymbol{\gamma}_\times J \operatorname{div}_\Gamma \lambda = \operatorname{div}_\Gamma \boldsymbol{\gamma}_\times J\omega \\ &= \operatorname{div}_\Gamma \boldsymbol{\gamma}_\times L\mathbf{W} = \gamma_n \mathbf{curl}\, L\mathbf{W} = \gamma_n \mathbf{W} = \gamma_1 \Psi = \operatorname{div}_\Gamma \boldsymbol{\lambda}. \end{aligned}$$

The other properties follow immediately. □

Now we define

$$\begin{aligned} \mathbf{X}(\operatorname{div}_\Gamma, \Gamma) &:= P^\Gamma(\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)), \\ \mathbf{N}(\operatorname{div}_\Gamma, \Gamma) &:= \operatorname{Ker}(\operatorname{div}_\Gamma) \cap \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma), \end{aligned}$$

and finally we end up with the direct splitting

$$\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) = \mathbf{X}(\operatorname{div}_\Gamma, \Gamma) \oplus \mathbf{N}(\operatorname{div}_\Gamma, \Gamma).$$

Again it is possible to establish some extra regularity for one of the subspaces.

Corollary 4.38. *The embedding* $\mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \hookrightarrow \mathbf{L}_t^2(\Gamma)$ *is compact.*

Proof. This follows immediately from the compact embedding $\mathbf{H}_\perp^{1/2}(\Gamma) \hookrightarrow \mathbf{L}_t^2(\Gamma)$. □

Another important result is the relation of the splitting of $\mathbf{H}(\mathbf{curl}, \Omega)$ and the splittings of the trace spaces.

Theorem 4.39. *For any* $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *the Helmholtz–type splitting* $\mathbf{U} = \mathbf{U}^\perp \oplus \mathbf{U}^0$ *with* $\mathbf{U}^\perp \in \mathbf{X}(\mathbf{curl}, \Omega)$ *and* $\mathbf{U}^0 \in \mathbf{N}(\mathbf{curl}, \Omega)$ *implies a valid Hodge–type decomposition of the Dirichlet trace* $\boldsymbol{\gamma}_\times \mathbf{U}$ *by* $(\boldsymbol{\gamma}_\times \mathbf{U})^\perp := \boldsymbol{\gamma}_\times \mathbf{U}^\perp \in \mathbf{H}_\perp^{1/2}(\Gamma)$ *and* $(\boldsymbol{\gamma}_\times \mathbf{U})^0 := \boldsymbol{\gamma}_\times \mathbf{U}^0 \in \mathbf{N}(\mathrm{div}_\Gamma, \Gamma)$.

Proof. We start with $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and the decomposition $\mathbf{U} = \mathbf{U}^0 + \mathbf{U}^\perp$ with $\mathbf{U}^0 \in \mathbf{N}(\mathbf{curl}, \Gamma)$ and $\mathbf{U}^\perp \in \mathbf{X}(\mathbf{curl}, \Gamma)$. Since $\gamma_n \circ \mathbf{curl} = \mathrm{div}_\Gamma \circ \boldsymbol{\gamma}_\times$ we conclude that $\boldsymbol{\gamma}_\times \mathbf{U}^0$ is in the kernel of div_Γ. From $\mathbf{X}(\mathbf{curl}, \Omega) \subset \mathbf{H}^1(\Omega)$ and Theorem 4.22 we conclude that $\boldsymbol{\gamma}_\times \mathbf{U}^\perp \in \mathbf{H}_\perp^{1/2}(\Gamma)$. □

A similar splitting can be derived for the space $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, i.e.

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \mathbf{X}(\mathrm{curl}_\Gamma, \Gamma) \oplus \mathbf{N}(\mathrm{curl}_\Gamma, \Gamma),$$

where $\mathbf{N}(\mathrm{curl}_\Gamma, \Gamma)$ is in the kernel of curl_Γ and $\mathbf{X}(\mathrm{curl}_\Gamma, \Gamma)$ is compactly embedded in $\mathbf{L}_t^2(\Gamma)$. Additionally, Theorem 4.39 remains valid if exchanging the trace $\boldsymbol{\gamma}_\times$ by $\boldsymbol{\gamma}_D$ and div_Γ by curl_Γ.

In the next chapters we will use so–called 'sign flip' operators, which will play an important role in establishing generalized Gårding inequalities. These on the other hand are needed to prove unique solvability of various boundary integral formulations.

Definition 4.40. *The sign flip operator* $\mathcal{X}$ *is for* $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *defined by*

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) &\mapsto (-\mathbf{u}, \mathbf{v}),\\ \mathbf{X}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{N}(\mathrm{curl}_\Gamma, \Gamma) &\to \mathbf{X}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{N}(\mathrm{curl}_\Gamma, \Gamma),\end{aligned}$$

for $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *by*

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) &\mapsto (-\mathbf{u}, \mathbf{v}),\\ \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \times \mathbf{N}(\mathrm{div}_\Gamma, \Gamma) &\to \mathbf{X}(\mathrm{div}_\Gamma, \Gamma) \times \mathbf{N}(\mathrm{div}_\Gamma, \Gamma),\end{aligned}$$

and for $\mathbf{H}(\mathbf{curl}, \Omega)$ *by*

$$\begin{aligned}(\mathbf{U}, \mathbf{V}) &\mapsto (-\mathbf{U}, \mathbf{V}),\\ \mathbf{X}(\mathbf{curl}, \Omega) \times \mathbf{N}(\mathbf{curl}, \Omega) &\to \mathbf{X}(\mathbf{curl}, \Omega) \times \mathbf{N}(\mathbf{curl}, \Omega).\end{aligned}$$

The second sign flip operator we will use is defined by $\mathcal{Y} := -\mathcal{X}$.

5 BOUNDARY INTEGRAL EQUATIONS FOR ACOUSTIC SCATTERING PROBLEMS

In this chapter we discuss boundary value problems of acoustic scattering. First we analyze the solvability properties of interior and exterior problems. Afterwards we state the representation formula, derive the corresponding potentials and boundary integral operators and discuss their properties. Then we introduce the Steklov–Poincaré operator to describe the Dirichlet–to–Neumann map and discuss its properties in detail, since this operator plays a key role in tearing and interconnecting domain decomposition methods. Based on boundary integral operators we present different possibilities to solve the Neumann or Robin boundary value problem and we derive a suitable formulation for the domain decomposition approach. Due to a different situation, the case of a boundary value problem in an unbounded domain is treated in a separate section. There we introduce a new formulation, which can be regarded as a new combined field integral equation. We modify this formulation such that it is also suitable for the tearing and interconnecting approach. Thereafter we discuss the discretization of all formulations and provide some numerical examples. Finally, we present a possible preconditioning strategy together with some numerical examples.

5.1 Boundary value problems

The interior Helmholtz boundary value problem for a bounded domain $\Omega \subset \mathbb{R}^3$ is stated by

$$\begin{aligned} -\Delta U - k^2 U &= 0 && \text{in } \Omega, \\ \gamma_0 U &= g && \text{on } \Gamma_D, \\ \gamma_1 U &= p && \text{on } \Gamma_N, \\ \gamma_1 U + i\eta R \gamma_0 U &= l && \text{on } \Gamma_R, \end{aligned} \tag{5.1}$$

where we assume that $\overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R = \Gamma = \partial\Omega$ and Γ_D, Γ_N and Γ_R are mutually disjoint. Γ_D is called the Dirichlet boundary, Γ_N the Neumann boundary and Γ_R the Robin boundary. Therefore we call the problem a Dirichlet boundary value problem if $\Gamma_D = \Gamma$, and Neumann or Robin boundary problem if $\Gamma_N = \Gamma$ or $\Gamma_R = \Gamma$, respectively. If non of these cases apply we call the problem a mixed boundary value problem.

Definition 5.1. *An operator $R : X \to X'$ is strictly positive if $\langle Ru, u\rangle_\Gamma > 0$ holds for all $0 \neq u \in X$.*

Remark 5.2. *At this point we will use an abstract operator $R : H^{1/2}(\Gamma_R) \to [H^{1/2}(\Gamma_R)]'$ which fulfills several properties. Some possible choices for R will be stated in Section 5.5. For now let us assume that R is self–adjoint, real valued, and strictly positive on $H^{1/2}(\Gamma_R)$. The Robin boundary condition can be denoted as generalized Robin boundary condition due to the additional operator R. These assumptions imply that the generalized Robin boundary condition is stated in the sense of $H^{1/2}(\Gamma_R)'$. In literature this equation is normally stated in $L_2(\Gamma_R)$, since no additional operator R is used.*

Interior Dirichlet and Neumann boundary value problems may suffer from eigen wave numbers. This means that the boundary value problem is not uniquely solvable for some k^2, furthermore the Dirichlet data or respectively the Neumann data have to fulfill certain solvability conditions if k^2 is an eigen wave number. Therefore, we will also consider the Robin boundary value problem which always has a unique solution and the given data can be given arbitrarily. In the following we discuss under which circumstances the Helmholtz equation leads to a unique solvable boundary value problem. The main references we use are [43, 102].

Theorem 5.3. *The Laplace eigenvalue problem*

$$\begin{aligned} -\Delta U_{\lambda_i} &= \lambda_i U_{\lambda_i} && \text{in } \Omega, \\ \gamma_0 U_{\lambda_i} &= 0 && \text{on } \Gamma_D, \\ \gamma_1 U_{\lambda_i} &= 0 && \text{on } \Gamma_N \end{aligned} \tag{5.2}$$

with U_λ not identically zero has infinite, countable many real solutions $\lambda_i \geq 0$. Further holds $\lambda_i \to \infty$ as $i \to \infty$. The corresponding eigenspaces have finite dimension for every λ_i.

Proof. See [5, page 306], [137, page 16] and [102, page 286]. □

Remark 5.4. *If $\Gamma_D = \Gamma$ we denote the Laplace eigenvalues λ_i as introduced in Theorem 5.3 as the Dirichlet eigen wave numbers of the domain Ω. If the domain is clearly determined by the context, we just call them Dirichlet eigen wave numbers. If $\Gamma_N = \Gamma$ the Laplace eigenvalues λ_i are denoted as Neumann eigen wave numbers. If we consider Dirichlet or Neumann eigen wave numbers in the context of an unbounded domain Ω^c, we always mean the eigen wave numbers of the bounded domain $\Omega = \mathbb{R} \setminus \overline{\Omega^c}$, since unbounded domains do not possess (real–valued) eigen wave numbers.*

As a consequence of Theorem 5.3, we can state the following theorem concerning the unique solvability of a mixed boundary value problem.

Theorem 5.5. *The interior boundary value problem*

$$\begin{aligned} -\Delta U - k^2 U &= 0 && \text{in } \Omega, \\ \gamma_0 U &= g && \text{on } \Gamma_D, \\ \gamma_1 U &= p && \text{on } \Gamma_N \end{aligned} \tag{5.3}$$

with $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ *has a unique weak solution* $U \in H^1(\Omega)$ *for each* $g \in H^{1/2}(\Gamma_D)$ *and* $p \in H^{-1/2}(\Gamma_N)$ *if* k^2 *is not an eigenvalue of the Laplace eigenvalue problem* (5.2). *If* k^2 *is an eigenvalue of the eigenvalue problem* (5.2), *then a solution of the boundary value problem* (5.3) *exists only if the solvability condition*

$$\langle \gamma_0 U_{\lambda_i}, p \rangle_{\Gamma_N} = \langle \gamma_1 U_{\lambda_i}, g \rangle_{\Gamma_D} \tag{5.4}$$

is satisfied for the corresponding eigenfunction U_{λ_i}. *The solution* U *is unique in the space* $H^1(\Omega)/\{U_{\lambda_i}\}$.

Proof. See [43, 106]. □

Remark 5.6. *For* $k^2 \in \mathbb{C} \setminus \mathbb{R}^+$ *the boundary value problem (5.3) always admits a unique solution.*

Lemma 5.7. *For any* $\eta \in \mathbb{R}\setminus\{0\}$ *and* $k \in \mathbb{R}$ *there exists a unique solution* $U \in H^1(\Omega)$ *of the Robin–type boundary value problem*

$$\begin{aligned} -\Delta U - k^2 U &= 0 && \text{in } \Omega, \\ \gamma_1 U &= p && \text{on } \Gamma_N, \\ \gamma_1 U + i\eta R \gamma_0 U &= l && \text{on } \Gamma_R \end{aligned} \tag{5.5}$$

with a non–trivial Robin boundary Γ_R, *i.e.* $\operatorname{meas}(\Gamma_R) > 0$, *and* $\Gamma = \overline{\Gamma}_N \cup \overline{\Gamma}_R$.

Proof. The weak formulation of the Robin boundary value problem (5.5) is to find $U \in H^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\nabla} U(x) \cdot \boldsymbol{\nabla} V(x) \mathrm{d}x - k^2 \int_\Omega U(x) V(x) \mathrm{d}x + i\eta \int_{\Gamma_R} (R\gamma_0 U)(x) \gamma_0 V(x) \mathrm{d}s_x \\ = \int_{\Gamma_R} l(x) \gamma_0 V(x) \mathrm{d}s_x + \int_{\Gamma_N} p(x) \gamma_0 V(x) \mathrm{d}s_x \end{aligned}$$

is satisfied for all $V \in H^1(\Omega)$. Since the associated sesquilinear form satisfies a Gårding inequality, i.e. for $V \in H^1(\Omega)$ we have

$$\begin{aligned} \operatorname{Re} \left(\int_\Omega \left[\boldsymbol{\nabla} V(x) \cdot \boldsymbol{\nabla} \overline{V(x)} - k^2 V(x) \overline{V(x)} \right] \mathrm{d}x + i\eta \int_{\Gamma_R} (R\gamma_0 V)(x) \overline{\gamma_0 V(x)} \mathrm{d}s_x \right) \\ = \|V\|^2_{H^1(\Omega)} - (k^2+1) \|V\|^2_{L_2(\Omega)}, \end{aligned}$$

it is sufficient to prove injectivity. Let $U \in H^1(\Omega)$ be any solution of the homogeneous boundary value problem (5.5), i.e. of the variational problem

$$\int_\Omega \boldsymbol{\nabla} U(x) \cdot \boldsymbol{\nabla} V(x) \mathrm{d}x - k^2 \int_\Omega U(x) V(x) \mathrm{d}x + i\eta \int_{\Gamma_R} (R\gamma_0 U)(x) \gamma_0 V(x) \mathrm{d}s_x = 0 \tag{5.6}$$

for all $V \in H^1(\Omega)$. By choosing $V = \overline{U}$ this gives

$$\int_\Omega |\nabla U(x)|^2 \mathrm{d}x - k^2 \int_\Omega |U(x)|^2 \mathrm{d}x + i\eta \int_{\Gamma_R} (R\gamma_0 U)(x)\overline{\gamma_0 U(x)}\mathrm{d}s_x = 0,$$

and therefore, when considering the imaginary part,

$$\int_{\Gamma_R} (R\gamma_0 U)(x)\overline{\gamma_0 U(x)}\mathrm{d}s_x = 0\,.$$

Since R is self–adjoint and strictly positive, $\gamma_0 U(x) = 0$ for $x \in \Gamma_R$ follows. The Robin boundary condition further implies $\gamma_1 U(x) = 0$ for $x \in \Gamma_R$. For $\Gamma_R = \Gamma$ we could use the representation formula, which is introduced later in this chapter to conclude $U(x) = 0$ for $x \in \Omega$. For $\Gamma_R \neq \Gamma$ other arguments have to be used. We use an idea presented in [106, page 92], alternative approaches can be found in [104, page 117] and [138]. Let Ω_* be a neighbouring domain of Ω, such that $\overline{\Omega_*} \cap \overline{\Omega} = \Gamma_R$ and let $\widetilde{U}$ be the trivial extension of U onto Ω_*, i.e.

$$\widetilde{U}_{|\Omega} = U, \qquad \widetilde{U}_{|\Omega_*} = 0.$$

Due to $\gamma_0 U(x) = \gamma_1 U(x) = 0$ for $x \in \Gamma_R$ is $\widetilde{U}$ a weak solution of the Helmholtz equation in $(\overline{\Omega} \cup \overline{\Omega_*})^\circ$. Further a solution of the Helmholtz solution, which is identical zero within an open ball B_ε, is identical zero everywhere, cf. [106, page 92]. Since Ω_* is assumed to be non–trivial, this proves the theorem. □

In contrast to the interior Dirichlet or Neumann boundary value problems, the exterior boundary value problems always allow for a unique solution if we enforce Rellich's radiation condition.

Theorem 5.8. *The exterior boundary value problems*

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega^c, \\ \gamma_0 U &= g \qquad \text{on } \Gamma_D, \\ \lim_{r\to\infty} \int_{B_r} |\gamma_1 U(x) - ik\gamma_0 U(x)|^2 \,\mathrm{d}s_x &= 0 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega^c, \\ \gamma_1 U &= p \qquad \text{on } \Gamma_N, \\ \lim_{r\to\infty} \int_{B_r} |\gamma_1 U(x) - ik\gamma_0 U(x)|^2 \,\mathrm{d}s_x &= 0 \end{aligned}$$

have a unique solution for all $\lambda = k^2 \in \mathbb{R}^+$.

Proof. See [47, 102]. □

Remark 5.9. *In Theorem 5.8 we used the weaker radiation condition of Rellich instead of the radiation condition of Sommerfeld, cf. [123], which is given by*

$$\lim_{r\to\infty} r(\gamma_1 U(x) - ik\gamma_0 U(x)) = 0 \qquad \text{with } r = |x|. \tag{5.8}$$

This condition is sufficient to eliminate incoming solutions and to ensure unique solvability of exterior Helmholtz problems. The equivalence of these two conditions when using a boundary integral approach is discussed, e.g., in [102, page 282].

5.2 Representation formula and integral operators

A basic ingredient for boundary integral equation methods is the fundamental solution, which enables us to deduce a representation formula from Green's second formula. For the Helmholtz equation the fundamental solution is given by

$$g_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} \qquad \text{for } x, y \in \mathbb{R}^3,$$

see [102]. We have two choices for the parameter k, since only k^2 is given in the Helmholtz equation. We assume that $k \geq 0$ for $k^2 \in \mathbb{R}$ and $\mathrm{Im}(k) > 0$ for $k^2 \notin \mathbb{R}$. This fits in the first case to the radiation condition [59], in the second case it will ensure invertibility of several boundary integral operators.

The Newton potential is defined by

$$\mathcal{N}_k(U)(x) := \int_{\mathbb{R}^3} g_k(x,y)U(y)\mathrm{d}y \qquad \text{for } x \in \mathbb{R}^3.$$

It is a continuous operator from $H^s(\mathbb{R}^3)$ to $H^{s+2}_{loc}(\mathbb{R}^3)$, see [121]. Based on the Newton potential we introduce the single and the double layer potential by

$$\Psi_k^S := \mathcal{N}_k \circ \gamma_0',$$
$$\Psi_k^D := \mathcal{N}_k \circ \gamma_1',$$

where γ_0' and γ_1' are the adjoint operators of the Dirichlet and the Neumann trace operators, respectively. For a detailed discussion of the definitions of the potentials Ψ_k^S and Ψ_k^D, and the adjoint trace operators see [121, page 83]. The single layer potential Ψ_k^S is continuous from $H^{-1/2}(\Gamma)$ to $H^1(\mathbb{R})$ and for $w \in L^1(\Gamma)$ we obtain the representation [121]

$$(\Psi_k^S w)(x) = \frac{1}{4\pi}\int_\Gamma \frac{e^{ik|x-y|}}{|x-y|} w(y)\mathrm{d}s_y \qquad \text{for } x \in \mathbb{R}^3 \setminus \Gamma.$$

The double layer potential Ψ_k^D is continuous from $H^{1/2}(\Gamma)$ to $H^1(\mathbb{R}^3\setminus\Gamma)\cap H(\Delta,\Omega\cup\Omega^c)$ and for $v\in L^1(\Gamma)$ we have the representation [121]

$$(\Psi_k^D w)(x) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial \mathbf{n}_y}\frac{e^{ik|x-y|}}{|x-y|}v(y)\mathrm{d}s_y \qquad \text{for } x\in\mathbb{R}^3\setminus\Gamma.$$

These two potentials fulfill the homogeneous Helmholtz equation and the radiation condition (5.8).

For every solution $U\in H^1(\Omega)$ of the homogeneous interior Helmholtz equation, the representation formula

$$U(x) = \Psi_k^S\gamma_1 U(x) - \Psi_k^D\gamma_0 U(x) \qquad \text{for } x\in\Omega \tag{5.9}$$

holds, see [125]. For every distribution $U\in H^1_{loc}(\Omega^c)$ which fulfills the Helmholtz equation in the exterior domain Ω^c and which in addition fulfills the radiation condition (5.8), we have the representation formula

$$U(x) = -\Psi_k^S\gamma_1^c U(x) + \Psi_k^D\gamma_0^c U(x) \qquad \text{for } x\in\Omega^c. \tag{5.10}$$

The average of traces across the boundary Γ is denoted by $\{\gamma\} := \frac{1}{2}(\gamma^c+\gamma)$. The boundary integral operators are defined by

$$\begin{aligned} V_k &:= \{\gamma_0\}\circ\Psi_k^S &:&\quad H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma),\\ K_k^\perp &:= \{\gamma_1\}\circ\Psi_k^S &:&\quad H^{-1/2}(\Gamma)\to H^{-1/2}(\Gamma),\\ K_k &:= \{\gamma_0\}\circ\Psi_k^D &:&\quad H^{1/2}(\Gamma)\to H^{1/2}(\Gamma),\\ D_k &:= -\{\gamma_1\}\circ\Psi_k^D &:&\quad H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma), \end{aligned} \tag{5.11}$$

see [35]. For functions $v,w\in L^\infty(\Gamma)$ we have the following representations for the boundary integral operators, see [102, Section 7]:

$$(V_k w)(x) = \frac{1}{4\pi}\int_\Gamma \frac{e^{ik|x-y|}}{|x-y|}w(y)\mathrm{d}s_y \qquad \text{for } x\in\Gamma,$$

$$(K_k v)(x) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial \mathbf{n}_y}\frac{e^{ik|x-y|}}{|x-y|}v(y)\mathrm{d}s_y \qquad \text{for } x\in\Gamma,$$

$$(K_k^\perp w)(x) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial \mathbf{n}_x}\frac{e^{ik|x-y|}}{|x-y|}w(y)\mathrm{d}s_y \qquad \text{for } x\in\Gamma,$$

$$(D_k v)(x) = -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial \mathbf{n}_y}\frac{e^{ik|x-y|}}{|x-y|}v(y)\mathrm{d}s_y \qquad \text{for } x\in\Gamma.$$

The integral representation of the single layer potential V_k can be interpreted as a weakly singular surface integral, the representations of the double layer potential

K_k and the adjoint double layer potential $K_k^\perp$ as Cauchy singular integrals. The operator D_k is hypersingular and the representation has to be interpreted as a finite part integral, see for example [89]. For globally continuous functions u and v which are differentiable on Γ_k we get, by using integration by parts, for the sesquilinear form induced by D_k the alternative representation, see [109],

$$\langle D_k u, v\rangle_\Gamma = \langle V_k \,\mathbf{curl}_\Gamma\, u, \mathbf{curl}_\Gamma\, v\rangle_\Gamma - k^2\langle V_k \mathbf{n} u, \mathbf{n} v\rangle_\Gamma. \tag{5.12}$$

The jump of a trace is defined by $[\gamma U] := \gamma^c U - \gamma U$. The boundary integral operators satisfy the jump conditions

$$\begin{aligned} [\gamma_0 \Psi_k^S w] &= 0, \qquad [\gamma_0 \Psi_k^D v] = -v, \\ [\gamma_1 \Psi_k^S w] &= w, \qquad [\gamma_1 \Psi_k^D v] = 0, \end{aligned}$$

for $v \in H^{1/2}(\Gamma)$ and $w \in H^{-1/2}(\Gamma)$, cf. [121].

By applying the Dirichlet and the Neumann trace on the representation formulae (5.9) and (5.10) we get two integral equations, respectively. For a solution U of the interior Helmholtz equation we obtain the integral equations

$$\gamma_0 U(x) = (\frac{1}{2} I - K_k)\gamma_0 U(x) + V_k \gamma_1 U(x), \tag{5.13}$$

$$\gamma_1 U(x) = D_k \gamma_0 U(x) + (\frac{1}{2} I + K_k^\perp)\gamma_1 U(x) \tag{5.14}$$

for almost all $x \in \Gamma$, while for a solution U of the exterior Helmholtz equation we get the integral equations

$$\gamma_0^c U(x) = (\frac{1}{2} I + K_k)\gamma_0^c U(x) - V_k \gamma_1^c U(x), \tag{5.15}$$

$$\gamma_1^c U(x) = -D_k \gamma_0^c U(x) + (\frac{1}{2} I - K_k^\perp)\gamma_1^c U(x) \tag{5.16}$$

for almost all $x \in \Gamma$. The two operators

$$\mathcal{C}_{int} = \begin{pmatrix} \frac{1}{2} I - K_k & V_k \\ D_k & \frac{1}{2} I + K_k^\perp \end{pmatrix}, \quad \mathcal{C}_{ext} = \begin{pmatrix} \frac{1}{2} I + K_k & -V_k \\ -D_k & \frac{1}{2} I - K_k^\perp \end{pmatrix}$$

are called Calderon projectors, and they fulfill the projection property $\mathcal{C}_{int}^2 = \mathcal{C}_{int}$ and $\mathcal{C}_{ext}^2 = \mathcal{C}_{ext}$, respectively, see for example [125]. This gives us the following lemma.

Lemma 5.10. *The boundary integral operators fulfill the relations*

$$V_k D_k = (\frac{1}{2} I + K_k)(\frac{1}{2} I - K_k), \tag{5.17}$$

$$D_k V_k = (\frac{1}{2} I + K_k^\perp)(\frac{1}{2} I - K_k^\perp), \tag{5.18}$$

$$K_k V_k = V_k K_k^\perp, \tag{5.19}$$

$$D_k K_k = K_k^\perp D_k. \tag{5.20}$$

Proof. See [125]. □

5.3 Properties of boundary integral operators

Since all of our forthcoming attempts to solve local or global boundary value problems are based on boundary integral operators, it is necessary to discuss their properties in more detail. To use Gårding inequalities to prove unique solvability of systems of boundary integral equations, we have first to investigate, which boundary integral operators fulfill a Gårding inequality by their own. The upcoming Gårding inequalities rely heavily on the fact that the difference of two potentials of the same kind, but with different wave numbers, is a compact operator, e.g. $V_k - V_0$. Most of the compactness results rely in turn on the following theorem.

Theorem 5.11. *The difference $\mathcal{N}^k - \mathcal{N}^0$ is a continuous operator*

$$\mathcal{N}^k - \mathcal{N}^0 : \widetilde{H}^s(\mathbb{R}^3) \to H^{s+4}(\mathbb{R}^3)$$

for all $s \in \mathbb{R}$.

Proof. See [121, page 82]. □

By using the definitions of the boundary integral operators we get the following corollary.

Corollary 5.12. *The operators*

$$\begin{aligned} V_k - V_0 &: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K_k - K_0 &: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K_k^\perp - K_0^\perp &: H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ D_k - D_0 &: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \end{aligned}$$

are compact.

Proof. This follows immediately from Theorem 5.11 and the mapping properties of the Newton potential, since the embedding $H^{s+4}(\mathbb{R}^3)$ in $H^{s+2}(\mathbb{R}^3)$ is compact. □

Another important property to prove coerciveness of systems of boundary integral equations is the connection of the double layer potential and the adjoint double layer potential.

Lemma 5.13. *For all $k \in \mathbb{R}$ there holds*

$$\langle K_k \phi, \psi \rangle_\Gamma = \langle \phi, K_{-k}^\perp \psi \rangle_\Gamma$$

for all $\phi \in H^{1/2}(\Gamma)$ and $\psi \in H^{-1/2}(\Gamma)$.

Proof. This follows from the definition of the double layer potential and the adjoint double layer potential, see (5.11), and the definition of the duality pairing $\langle \cdot, \cdot \rangle_\Gamma$, see Subsection 4.2.1. □

The last two propositions imply that K_k and $K_k^\perp$, for every $k \in \mathbb{C}$, are adjoint up to a compact perturbation.

To prove the coerciveness of the single layer potential for all wave numbers $k \in \mathbb{C}$, we have first to prove the ellipticity of the single layer potential for at least one wave number k.

Lemma 5.14. *For* $\mathrm{Im}(k) > 0$ *or* $0 \leq k^2 < \lambda_0$ *the single layer potential* V_k *is* $H^{-1/2}(\Gamma)$*–elliptic, i.e.*

$$|\langle V_k w, w \rangle_\Gamma| \geq c_1 \|w\|^2_{H^{-1/2}(\Gamma)}$$

with a fixed $c_1 > 0$ *and for all* $w \in H^{-1/2}(\Gamma)$*. Note that* λ_0 *is the minimal Dirichlet eigenvalue of the Laplace operator.*

Proof. See [10, 98, 99]. □

If k does not fulfill the prior assumptions of Lemma 5.14, but k^2 is not an Dirichlet eigenvalue, i.e. the Dirichlet boundary value problem (5.1) has a unique solution, then V_k is still invertible. If $k^2 = \lambda$ is a Dirichlet eigenvalue with the eigenfunction U_λ then is $\gamma_1 U_\lambda \in \mathrm{Ker}(V_k)$ and $\gamma_1 U_\lambda \in \mathrm{Ker}(-\frac{1}{2} I + K_k^\perp)$, see [59].

Lemma 5.15. *For all* $k \in \mathbb{C}$ *the single potential* V_k *fulfills the Gårding inequality*

$$\langle V_k w, w \rangle_\Gamma + C(w, w) \geq c_1 \|w\|^2_{H^{-1/2}(\Gamma)}$$

for all $w \in H^{-1/2}(\Gamma)$ *and a compact sesquilinear form* $C(\cdot, \cdot)$*,* $c_1 > 0$*.*

Proof. This is a direct consequence of Lemma 5.14 and Corollary 5.12. □

Similar results are also obtainable for the hypersingular operator D_k.

Lemma 5.16. *If* $\mathrm{Im}(k) > 0$ *then the hypersingular operator* D_k *is* $H^{1/2}$*–elliptic, i.e.*

$$\langle D_k u, u \rangle_\Gamma \geq c_1 \|u\|^2_{H^{1/2}(\Gamma)}$$

for all $u \in H^{1/2}(\Gamma)$*,* $c_1 > 0$*.*

Proof. See [11, 42]. □

If k does not fulfill the prior assumption of Lemma 5.16, but k^2 is not a Neumann eigenvalue, i.e. the Neumann boundary value problem (5.1) has a unique solution, then D_k is still invertible. If $k^2 = \lambda$ is a Neumann eigenvalue with eigenfunction U_λ then is $\gamma_0 U_\lambda \in \mathrm{Ker}(D_k)$ and $\gamma_0 U_\lambda \in \mathrm{Ker}(\frac{1}{2}I + K_k)$.

Lemma 5.17. *For all $k \in \mathbb{C}$ the hypersingular operator D_k fulfills a Gårding inequality*

$$\langle D_k u, u\rangle_\Gamma + C(u,u) \geq c_1 \|u\|^2_{H^{1/2}(\Gamma)}$$

for all $u \in H^{1/2}(\Gamma)$ and compact sesquilinear form $C(\cdot,\cdot)$, $c_1 > 0$.

Proof. This is a direct consequence of Lemma 5.16 and Corollary 5.12. □

For $k = 0$ it is known that the eigenfunction of the hypersingular operator is independent of the domain Ω, since it is just the constant function. Thus we can introduce the regularized hypersingular operator $\widetilde{D}$, which is induced by

$$\langle \widetilde{D} u, v\rangle_\Gamma := \langle D_0 u, v\rangle_\Gamma + \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma$$

for all $u, v \in H^{1/2}(\Gamma)$. This operator is $H^{1/2}(\Gamma)$–elliptic, see [125]. Another proposition, which is needed to prove well posedness of a combined field integral equation, describes the properties of the imaginary part of the single layer potential and of the hypersingular operator.

Lemma 5.18. *For $k > 0$ holds*

$$\begin{aligned} \mathrm{Im}(\langle V_k u, u\rangle_\Gamma) &\geq 0, \qquad \mathrm{Im}(\langle V_{-k} u, u\rangle_\Gamma) \leq 0, \\ \mathrm{Im}(\langle D_k v, v\rangle_\Gamma) &\leq 0, \qquad \mathrm{Im}(\langle D_{-k} v, v\rangle_\Gamma) \geq 0 \end{aligned}$$

for all $u \in H^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$.

Proof. See [59]. □

From the boundary integral equations (5.15) and (5.16) we can deduce a lemma, which will play a key role in proving injectivity of the local boundary integral system presented later.

Lemma 5.19. *For every $g \in H^{1/2}(\Gamma)$ and $h \in H^{-1/2}(\Gamma)$ the image properties*

$$\begin{aligned} (-\frac{1}{2}I + K_k)g &\in \mathrm{Imag}(V_k), \\ (\frac{1}{2}I + K_k^\perp)h &\in \mathrm{Imag}(D_k) \end{aligned}$$

hold.

Proof. See [59]. □

5.4 Steklov–Poincaré operator

The theory of classical FETI/BETI methods is based upon the Steklov–Poincaré operator which is involved in the Dirichlet–to–Neumann map. Therefore, we introduce this operator, which is related to the Helmholtz equation, in this section and discuss some of its properties.

Let $g \in H^{1/2}(\Gamma)$ be an arbitrary function, and let $U \in H^1(\Omega)$ be the solution of the Dirichlet boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega, \\ \gamma_0 U &= g \qquad \text{on } \Gamma, \end{aligned}$$

which is for the moment assumed to be uniquely solvable. Then the Steklov–Poincaré operator S_k is defined as the mapping

$$\begin{aligned} S_k : g &\mapsto \gamma_1 U, \\ & H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \end{aligned} \tag{5.21}$$

If the Dirichlet problem is uniquely solvable, then the Steklov–Poincaré operator S_k is well defined. If it is well defined and if the Neumann problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega, \\ \gamma_1 U &= p \qquad \text{on } \Gamma \end{aligned}$$

is uniquely solvable, then the Steklov–Poincaré operator is invertible.

Next we show that the Steklov–Poincaré operator related to the Helmholtz equation is coercive if it is well defined.

Theorem 5.20. *The solution operator $\mathcal{E}$ which maps $g \in H^{1/2}(\Gamma)$ to $U \in H^1(\Omega)$ satisfying*

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \textit{in } \Omega, \\ \gamma_0 U &= g \qquad \textit{on } \Gamma \end{aligned}$$

is a continuous operator from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$ if the Dirichlet boundary value problem has a unique solution.

Proof. See [102]. □

The adjoint operator $\mathcal{E}' : \widetilde{H}^{-1}(\Omega) \to H^{-1/2}(\Gamma)$ of $\mathcal{E}$ is defined by

$$\langle g, \mathcal{E}' F \rangle_\Gamma := \langle \mathcal{E} g, F \rangle_\Omega$$

for all $g \in H^{1/2}(\Gamma)$ and $F \in \widetilde{H}^{-1}(\Omega)$.

Lemma 5.21. *Let E_1 be the compact embedding of $H^1(\Omega)$ in $\widetilde{H}^{-1}(\Omega)$. Then the operator $\mathcal{E}' E_1 \mathcal{E}$ is compact from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.*

Proof. $\mathcal{E}$ is continuous from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$, the embedding from $H^1(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$ is compact, and the adjoint operator of $\mathcal{E}$ is continuous from $\widetilde{H}^{-1}(\Omega)$ to $H^{-1/2}(\Omega)$. □

Theorem 5.22. *The Steklov–Poincaré operator S_k as defined in (5.21) is coercive if k^2 is not a Dirichlet eigen wave number, i.e.*

$$\langle S_k g, g\rangle_\Gamma + C(g,g) \geq c_1 \|g\|^2_{H^{1/2}(\Gamma)}$$

for all $g \in H^{1/2}(\Gamma)$ and a compact sesquilinear form $C(\cdot,\cdot)$, $c_1 > 0$.

Proof. We show that the Steklov–Poincaré operator S_k fulfills a Gårding inequality. If k^2 is not a Dirichlet eigen wave number, we can find for every $g \in H^{1/2}(\Gamma)$ a function U such that

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega, \\ \gamma_0 U &= g \qquad \text{on } \Gamma \end{aligned}$$

holds. By using $S_k g = S_k \gamma_0 U = \gamma_1 U$ this gives

$$\begin{aligned} &\langle S_k \gamma_0 U, \gamma_0 U\rangle_\Gamma + (k^2+1) \underbrace{\langle \mathcal{E}' E_1 \mathcal{E} \gamma_0 U, \gamma_0 U\rangle_\Gamma}_{\text{compact}} \\ &\quad = \int_\Gamma \gamma_1 U(x) \gamma_0 \overline{U}(x) \,\mathrm{d}s_x + (k^2+1) \int_\Omega E_1 \mathcal{E}\gamma_0 U(x) \mathcal{E}\gamma_0 \overline{U}(x) \,\mathrm{d}s_x. \end{aligned}$$

By using Green's formula and $\mathcal{E}\gamma_0 U = U$ we get

$$\begin{aligned} &\int_\Gamma \gamma_1 U(x) \gamma_0 \overline{U}(x) \,\mathrm{d}s_x + (k^2+1) \int_\Omega E_1 \mathcal{E}\gamma_0 U(x) \mathcal{E}\gamma_0 \overline{U}(x) \,\mathrm{d}s_x \\ &\quad = \int_\Omega \left[\nabla U(x) \cdot \nabla \overline{U}(x) - k^2 |U(x)|^2\right] \mathrm{d}x + (k^2+1) \int_\Omega |U(x)|^2 \,\mathrm{d}x = \|U\|^2_{H^1(\Omega)} \end{aligned}$$

and the assertion follows by the trace theorem. □

Remark 5.23. *If k^2 is a Dirichlet eigen wave number, then the Dirichlet datum has to fulfill the solvability condition (5.4), otherwise no solution of the interior Neumann boundary value problem exists. On the other side, if the solvability condition is fulfilled, then several solutions with the given Dirichlet data exist. In this case, the Steklov–Poincaré operator can be defined as a mapping from $H^{1/2}_*(\Gamma)$ to $H^{-1/2}(\Gamma)/\{\gamma_1 U_\lambda\}$, where $H^{1/2}_*(\Gamma)$ is the space of all functions in $H^{1/2}(\Gamma)$ which fulfill the solvability condition and U_λ is the eigenfunction of the Dirichlet boundary value problem.*

Remark 5.24. *The Steklov–Poincaré operator S_k is bounded if it is well defined and it is injective if k^2 is not a Neumann eigenvalue. These properties follow directly from the definition. Together with the Gårding inequality this ensures that the Steklov–Poincaré operator S_k is invertible if k^2 is neither a Dirichlet nor a Neumann eigenvalue.*

One realization of the Steklov–Poincaré operator can be achieved by using boundary integral operators. By solving the first boundary integral equation (5.13) we get the representation

$$\gamma_1 U = S_k \gamma_0 U = V_k^{-1}(\frac{1}{2}I + K_k)\gamma_0 U. \tag{5.22}$$

If we insert the representation of $\gamma_1 U$ in (5.22) in the second boundary integral equation (5.14) we obtain the so–called symmetric representation

$$S_k \gamma_0 U = D_k \gamma_0 U + (\frac{1}{2}I + K_k^{\perp})V_k^{-1}(\frac{1}{2}I + K_k)\gamma_0 U.$$

Obviously, these representations are not well defined if k^2 is a Dirichlet eigenvalue, since in this case V_k is not invertible. This fits to the natural properties of the Steklov–Poincaré operator S_k.

The symmetric formulation of the Steklov–Poincaré operator which is related to the exterior Dirichlet problem is given by

$$S_k^c \gamma_0^c U = -(D_k + (\frac{1}{2}I - K_k^{\perp})V_k^{-1}(\frac{1}{2}I - K_k))\gamma_0^c U.$$

Although the exterior Dirichlet boundary value problem has a unique solution and the Dirichlet–to–Neumann map is well defined, the representation by boundary integral operators seems to struggle again if k^2 is a Dirichlet eigenvalue. V_k is not invertible in this case, but as mentioned in Corollary 5.3, it is still invertible on the image of $\frac{1}{2}I - K_k$, further the kernels of V_k and $\frac{1}{2}I - K_k^{\perp}$ coincide. This enables us to use this representation also for exterior boundary value problems. Nevertheless this becomes a crucial point in the numerical analysis as it will be discussed in Section 5.7.

5.5 Robin interface operators R

In this section we discuss possible choices for the operator R as used to describe Robin type boundary conditions in (5.1). Recall, that the operator $R : H^{1/2}(\Gamma_R) \to \widetilde{H}^{-1/2}(\Gamma_R)$ is assumed to be real valued and strictly positive. We will mention three possible choices. The first two fulfill all of these conditions, the third lacks the mapping properties. These operators are:

- The inverse single layer potential V_0^{-1} for the Laplace operator. The operator is obviously real valued. Since V_0 is a $\widetilde{H}^{-1/2}(\Gamma_R)$–elliptic operator, cf. [62], the other properties follow immediately. A drawback is that V_0^{-1} can not be discretized directly.
- The embedding $E_{1/2}$ from $H^{1/2}(\Gamma_R)$ into $\widetilde{H}^{-1/2}(\Gamma_R)$. It can be easily seen that this operator is real valued and strictly positive. The mapping property follows from the fact that $H^{1/2}(\Gamma_R)$ can be compactly embedded in $\widetilde{H}^{-1/2}(\Gamma_R)$. A drawback of this operator is that it is not an operator of the same order as the hypersingular operator D_k. This would lead to additional problems to find a suitable preconditioner for $D_k + i\eta E_{1/2}$.
- The regularized hypersingular operator $\widetilde{D}$, although it does not fulfill the mapping properties, cf. [62], seems to be the most practical one for an implementation. The operator is obviously real valued and also the strictly positiveness would be given if the mapping properties would be correct. One possible approach would be to modify this operator to $\phi'_{\Gamma_R} \circ \widetilde{D} \circ \phi_{\Gamma_R}$ where ϕ_{Γ_R} is a sufficiently smooth function with

  $$\begin{aligned} \phi_{\Gamma_R}(x) &= 0 \qquad \text{for } x \in \Gamma \setminus \Gamma_R, \\ \phi_{\Gamma_R}(x) &> 0 \qquad \text{for } x \in \Gamma_R. \end{aligned}$$

  This would be a strictly positive, real valued operator which fulfills the mapping properties. Unfortunately, the additional function ϕ_{Γ_R} would require additional work to discretize this operator. The most practical alternative is to use the discretization of $\widetilde{D}$ and only use ansatz and trial functions whose support lies completely in Γ_R. This would lead to a real valued positive definite matrix for those degrees of freedom. Unfortunately, this matrix has no exact analytical counterpart.

5.6 Local solution strategies for interior boundary value problems of the Helmholtz equation

In this subsection we discuss the solution of local boundary value problems with boundary integral equations. For simplicity we restrict ourselves to Neumann and Robin boundary value problems, since these are the local problems appearing in the domain decomposition approach. We start the discussion with the Neumann boundary value problem:

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega, \\ \gamma_1 U &= p \qquad \text{on } \Gamma \end{aligned} \tag{5.23}$$

is fulfilled.

If only one local problem needs to be solved, an indirect approach is often the approach of choice, because they are easy to implement and lead to rather fast algorithms. Although they are not useful for the tearing and interconnecting approach, we will shortly mention them. Recall that the single layer potential $\Psi_k^S(\cdot)$ and the double layer potential $\Psi_k^D(\cdot)$ fulfill the Helmholtz equation and the radiation condition (5.8). Therefore, one possibility is to solve the partial differential equation with a single layer approach, i.e. we assume that the solution U has the representation $U = \Psi_k^S w$ with an unknown density function w. After applying the Neumann trace onto this representation we have to solve the boundary integral equation

$$\gamma_1 \Psi_k^S w = (\frac{1}{2} I + K_k^\perp) w = p \qquad \text{on } \Gamma.$$

The double layer approach is based on the representation $U = \Psi_k^D v$ and leads to the boundary integral equation

$$\gamma_1 \Psi_k^D v = D_k v = p \qquad \text{on } \Gamma.$$

Note that these two boundary integral equations are only uniquely solvable, if k^2 is not a Neumann eigen wave number, see Lemma 5.16. Further the density functions w and v do not have a physical meaning, hence indirect approaches are in general not useful for the tearing and interconnecting method, since they rely on Dirichlet to Neumann/Robin maps. That's why direct approaches are often preferred for domain decomposition methods, but indirect approaches can still be used in certain applications, see, for example, [7].

A more reasonable approach for our purposes is the direct approach. This means we solve the second boundary integral equation (5.14) by replacing $\gamma_1 U$ by the given data p, i.e.

$$(\frac{1}{2} I - K_k^\perp) p = D_k \gamma_0 U. \tag{5.24}$$

Now we can compute the unknown Dirichlet datum $\gamma_0 U$, which is uniquely determined if k^2 is not a Neumann eigen wave number. This is nothing else than applying the unsymmetric Steklov–Poincaré operator. Instead of the unsymmetric version we can also use the symmetric one. Equation (5.24) becomes

$$\gamma_1 U = S_k \gamma_0 U = (D_k + (\frac{1}{2} I + K_k^\perp) V_k^{-1} (\frac{1}{2} + K_k)) \gamma_0 U = p. \tag{5.25}$$

However, this operator is still not invertible if k^2 is a Neumann eigen wave number, further more it is not well defined if k^2 is a Dirichlet eigen wave number, cf. Section 5.4. Instead of (5.25) we can consider the system

$$\begin{pmatrix} D_k & \frac{1}{2} I + K_k^\perp \\ -(\frac{1}{2} I + K_k) & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix} \tag{5.26}$$

which is equivalent to equation (5.25) if k^2 is not a Dirichlet eigenvalue, $\gamma_0 U = u$ and $\gamma_1 U = t$. This system has a unique solution if k^2 is not a Neumann eigenvalue (this includes $k^2 \in \mathbb{C}\setminus\mathbb{R}^+$ and therefore $\mathrm{Im}(k) > 0$). When we are considering the solution of a Neumann problem in the next chapters this will be the formulation of choice. Due to the lack of unique solvability for all wave numbers k, we will reformulate the local subproblems within the domain decomposition approach in such a way that we deal with local Robin boundary value problems instead, see Chapter 7 for further details.

The Robin boundary value problem we will consider is of the form:

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega, \\ \gamma_1 U + i\eta R \gamma_0 U &= l \qquad \text{on } \Gamma. \end{aligned} \tag{5.27}$$

Our aim is not to compute U itself, but only the Dirichlet datum $\gamma_0 U$. We already have the Dirichlet–to–Neumann map $\gamma_1 U = S_k \gamma_0 U$, if we add $i\eta R\gamma_0 U$ on both sides we get the equation

$$(S_k + i\eta R)\gamma_0 U = \gamma_1 U + i\eta R \gamma_0 U = l.$$

Note that R is a local operator only acting on $\gamma_0 U_{|\Gamma_R}$. Again this formulation is only well posed if k^2 is not a Dirichlet eigen wave number. Therefore, we use, as for the Neumann problem, a system of boundary integral equations, i.e.

$$\begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^\perp \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} l \\ 0 \end{pmatrix}. \tag{5.28}$$

This problem has for all wave numbers $k \in \mathbb{R}\setminus\{0\}$ a unique solution if $\eta \not\equiv 0$. To prove this, we first prove a Gårding inequality and afterwards we prove the surjectivity of the system. Due to Theorem 3.9 this ensures unique solvability.

Theorem 5.25. *The operator*

$$\begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^\perp \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \tag{5.29}$$

fulfills a Gårding inequality independent of η for all $k \in \mathbb{C}$.

Proof. From

$$\begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^\perp \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} = \begin{pmatrix} \widetilde{D} + i\eta R & \frac{1}{2}I + K_0^\perp \\ -(\frac{1}{2}I + K_0) & V_0 \end{pmatrix} + \underbrace{\begin{pmatrix} D_k - \widetilde{D} & K_k^\perp - K_0^\perp \\ K_0 - K_k & V_k - V_0 \end{pmatrix}}_{\text{compact}}$$

we can deduce that

$$\begin{aligned}
&\operatorname{Re}\left(\left\langle \begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^{\perp} \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix}, \begin{pmatrix} u \\ t \end{pmatrix} \right\rangle_{\Gamma} + C((u,t),(u,t))\right) \\
&\quad = \langle \widetilde{D}u, u\rangle_{\Gamma} + \langle V_0 t, t\rangle_{\Gamma} \\
&\quad \geq c\left(\|u\|^2_{H^{1/2}(\Gamma)} + \|t\|^2_{H^{-1/2}(\Gamma)}\right)
\end{aligned}$$

with a compact sesquilinear form $C((u,t),(u,t))$. Note that $\langle Ru, u\rangle_{\Gamma_R}$ is supposed to be real valued. □

Theorem 5.26. *The operator in* (5.29) *is for all* $k \in \mathbb{R}$ *and* $\eta \not\equiv 0$ *surjective.*

Proof. We prove this in two steps:

- For every $l \in H^{-1/2}(\Gamma)$ there exists a $u \in H^{1/2}(\Gamma)$ and a $t \in H^{-1/2}(\Gamma)$ such that

  $$\begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^{\perp} \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} l \\ 0 \end{pmatrix}$$

  is fulfilled. The Robin boundary value problem (5.5) has a unique solution U. If we set $u = \gamma_0 U$ and $t = \gamma_1 U$ both equations are fulfilled, due to the boundary integral equations (5.13) and (5.14).

- For every $h \in H^{1/2}(\Gamma)$ there exists a $u \in H^{1/2}(\Gamma)$ and a $t \in H^{-1/2}(\Gamma)$ such that

  $$\begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^{\perp} \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} p \\ h \end{pmatrix}$$

  is fulfilled with an arbitrary $p \in H^{-1/2}(\Gamma)$. This can be seen by setting $u = -h$ and t such that $(-\frac{1}{2}I + K_k)h + V_k t = 0$, which is possible according to Corollary 5.19.

The two propositions combined prove the theorem. □

An alternative way to show invertibility, is to prove injectivity (instead of surjectivity) of the system. This way was chosen in [129].

Remark 5.27. *For* $k^2 \in \mathbb{C} \setminus \mathbb{R}$ *we do not need to use Robin boundary conditions because in this case, the Neumann boundary value problem always admits a unique solution. Hence, the proof of Theorem 5.26 also works for the Neumann boundary formulation* (5.26) *if* $\operatorname{Im}(k) > 0$*, since* V_k *is invertible in this case.*

5.7 Exterior Neumann boundary value problem

In contrast to the interior Neumann boundary value problem, the exterior Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega^c, \\ \gamma_1 U &= p \qquad \text{on } \Gamma, \\ \lim_{r\to\infty} \int_{B_r} |\gamma_1 U(x) - ik\gamma_0 U(x)|^2 \, \mathrm{d}s_x &= 0 \end{aligned}$$

always admits a unique solution. Unfortunately, this is not reflected by usual boundary integral approaches. The problem is that for Dirichlet or Neumann eigen wave numbers k^2, density functions w and v exist such that $\Psi_k^S w \equiv 0$ and $\Psi_k^D v \equiv 0$ in Ω^c. This makes standard indirect approaches useless, and also direct approaches fail on the first glance, since D_k is not invertible. But if we take a closer look on the second boundary integral equation (5.16)

$$D_k \gamma_0^c U = -(\frac{1}{2} I + K_k^\perp)\gamma_1^c U, \tag{5.30}$$

we see that the right hand side is in the image of D_k, even if D_k is not invertible, cf. Lemma 5.19. This implies that the boundary integral equation (5.30) has a solution for all wave numbers k. Of course this solution suffers uniqueness if k^2 is a Neumann eigen wave number. Let us assume now that $k^2 = \lambda$ is a Neumann eigen wave number, hence D_k is not invertible. Further, let u_λ be in the kernel of D_k. All solutions of (5.30) are given by $u = \gamma_0^c U + \alpha u_\lambda$ (we assume for simplicity that the dimension of the eigenspace is one). If we plug this in the representation formula (5.10) for the solution of the exterior boundary value problem we get

$$-\Psi_k^S(\gamma_1 U) + \Psi_k^D(\gamma_0 U + \alpha u_\lambda) = -\Psi_k^S(\gamma_1 U) + \Psi_k^D(\gamma_0 U) = U \qquad \text{in } \Omega^c$$

since $\Psi^D(u_\lambda) \equiv 0$ in Ω^c, cf. [137]. Thus in some cases, this approach seams still feasible. Practical results show that this approach works quite well if the computed datum is used only for the representation formula, since the eigenfunctions do not change the results in this case (up to numerical errors), see [61]. But if the Cauchy data are the point of interest, then this approach provides false results. For V_k and the first boundary integral equation (5.15) we would obtain the same results.

Several modified approaches were introduced to handle this problem. One of the first was the indirect approach by Brakhage and Werner [24], who introduced a formulation based on a complex linear combination of the single and double layer potentials. A similar direct approach was introduced by Burton and Miller [41]. These approaches work quite well, but the proofs are only able to establish well posedness in a L^2–setting and for smooth boundaries. The idea of a linear combination of the two potentials was generalized by so–called combined field integral equations (CFIE)

see [37, 40, 59, 83]. However, these approaches do not seem to fit into our analytical domain decomposition approach, because we want to use similar approaches on all subdomains, to simplify the global formulation and it's verification.

Therefore, the idea is to use the following formulation, which is just rewriting of the boundary integral equations (5.15) and (5.16), to solve the local exterior Neumann problem:

Find $u \in H^{1/2}(\Gamma)$ and $t \in H^{-1/2}(\Gamma)$ such that

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp \\ \frac{1}{2}I - K_k & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} -p \\ 0 \end{pmatrix} \tag{5.31}$$

holds for the given Neumann data $p \in H^{-1/2}(\Gamma)$.

Note that we changed the sign of the boundary integral equations (5.15) and (5.16) and that we are still using the outgoing normal vector (so $t = \gamma_1^c U$). This will be advantageous for the coupling in the domain decomposition approach. As for the interior problem the system (5.31) fulfills a Gårding inequality, but surjectivity/injectivity is not given if k^2 is an Dirichlet eigen wave number. Nevertheless, it can be shown that the system is always solvable for a right hand side $(p, 0)^\top$ and that u is unique in this case. Since we are only interested in u and not in t, this seems to be sufficient.

Lemma 5.28. *The boundary integral operator as considered in* (5.31) *is injective in* u*. In particular, the homogeneous system*

$$V_k t + (\frac{1}{2}I - K_k)u = 0, \quad D_k u + (-\frac{1}{2}I + K_k^\perp)t = 0$$

implies $u = 0$ *for all wave numbers* k*.*

Proof. Let $(u, t) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be a solution of the homogeneous system

$$V_k t + (\frac{1}{2}I - K_k)u = 0, \quad D_k u + (-\frac{1}{2}I + K_k^\perp)t = 0.$$

When applying the single layer potential V_k to the hypersingular boundary integral equation, and when using the Calderon relation (5.17) we obtain

$$\begin{aligned} 0 &= V_k D_k u + V_k(-\frac{1}{2}I + K_k^\perp)t \\ &= (\frac{1}{2}I - K_k)(\frac{1}{2}I + K_k)u + (-\frac{1}{2}I + K_k)V_k t \\ &= (\frac{1}{2}I - K_k)(\frac{1}{2}I + K_k)u + (-\frac{1}{2}I + K_k)(-\frac{1}{2}I + K_k)u \\ &= (\frac{1}{2}I - K_k)\left[(\frac{1}{2}I + K_k) + (\frac{1}{2}I - K_k)\right]u \\ &= (\frac{1}{2}I - K_k)u \end{aligned}$$

and therefore

$$V_k t = 0.$$

When applying the hypersingular boundary integral operator D_k to the weakly singular boundary integral equation, we obtain in a similar way

$$\begin{aligned} 0 &= D_k V_k t + D_k(\frac{1}{2}I - K_k)u \\ &= (\frac{1}{2}I - K_k^\perp)(\frac{1}{2}I + K_k^\perp)t + (\frac{1}{2}I - K_k^\perp)D_k u \\ &= (\frac{1}{2}I - K_k^\perp)(\frac{1}{2}I + K_k^\perp)t + (\frac{1}{2}I - K_k^\perp)(\frac{1}{2}I - K_k^\perp)t \\ &= (\frac{1}{2}I - K_k^\perp)t \end{aligned}$$

and therefore

$$D_k u = 0.$$

Summarizing the above we conclude

$$V_k t = 0, \quad (\frac{1}{2}I - K_k^\perp)t = 0, \quad D_k u = 0, \quad (\frac{1}{2}I - K_k)u = 0.$$

$D_k u = 0$ is only satisfied for a non–trivial u if $\lambda = k^2$ is an eigenvalue of the Neumann eigenvalue problem. But this implies that

$$(\frac{1}{2}I + K_k)u = 0,$$

and therefore $u = 0$ follows. If k^2 is not a Dirichlet eigen wave number, $V_k t = 0$ implies also $t = 0$. Otherwise, t can be a multiple of the Neumann trace of the eigenfunction, i.e. $t = c\gamma_1 U_\lambda$ for some $c \in \mathbb{R}$. □

Although the numerical tests seem to work in almost all cases (for further details see Section 5.8), it is not possible to establish a rigorous numerical analysis for this approach, since the unique solvability of u will be disturbed in the discrete case. Therefore, we try to find a regularized version of this formulation. If we just want to solve an exterior Neumann problem we can make use of the knowledge of the Neumann datum. For example we can solve the modified system

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp \\ \frac{1}{2}I - K_k & V_k + iV_0 \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} -p \\ iV_0 p \end{pmatrix} \tag{5.32}$$

which is equivalent to (5.31), since $t = p$ is known. Instead of V_0 we can also use any other real valued $H^{-1/2}(\Gamma)$–elliptic operator. The operator as given in (5.32) fulfills a Gårding inequality. The proof works analogous to the proof of Theorem 5.25. The surjectivity of the operator can be proved by a reinterpretation of equation (5.32) as an interior Robin boundary value problem.

Lemma 5.29. *The operator*

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp \\ \frac{1}{2}I - K_k & V_k + iV_0 \end{pmatrix} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$$

is surjective.

Proof. The Robin boundary value problem

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega, \\ \gamma_0 U + iV_0\gamma_1 U &= l \qquad \text{on } \Gamma \end{aligned} \tag{5.33}$$

has a unique solution $U \in H^1(\Omega)$ for all $l \in H^{1/2}(\Gamma)$. This can be proved analogously as Lemma 5.7. If we want to compute the Neumann trace $\gamma_1 U$ of the unique solution U only, it is possible to use a similar formulation as in (5.28), which is also deduced in an analogous way. More precisely the corresponding boundary integral formulation is

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp \\ \frac{1}{2}I - K_k & V_k + iV_0 \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ l \end{pmatrix} \tag{5.34}$$

where l is the Robin datum as given in (5.33). With the interpretation as interior Robin boundary value problem in mind, the surjectivity of the operator as given in (5.34) can be proved as in Theorem 5.26. □

The coerciveness and the surjectivity of the operator as given in Lemma 5.29 give us the unique solvability of the equation posed in (5.32). Note that $V_k + iV_0$ can be inverted since $\mathrm{Im}(\langle V_k u, u\rangle_\Gamma) \geq 0$, see Lemma 5.18. Hence the sum $V_k + iV_0$ is $H^{-1/2}(\Gamma)$–elliptic. By eliminating t in equation (5.32) we get a CFIE which is closely related to the one in [60], although the deduction is rather different. However, in a domain decomposition method the local Neumann datum p is not known. In every iteration of the global solving routine (see Section 7.7) we only know the tested version of p, i.e. we only know $\langle p, \phi_i\rangle_\Gamma$ for all test functions $\phi_i \in S_h^1(\Gamma)$. Hence, it is not possible to compute $V_0 p$ for the right hand side correctly. To get rid of this problem, we first exchange the operator V_0 in (5.32) by the inverse of the modified hypersingular operator $\widetilde{D}^{-1}$, which is also an $H^{-1/2}(\Gamma)$–elliptic operator, i.e.

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp \\ \frac{1}{2}I - K_k & V_k + i\widetilde{D}^{-1} \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} -p \\ i\widetilde{D}^{-1}p \end{pmatrix}. \tag{5.35}$$

The idea is to replace $i\widetilde{D}^{-1}(t-p)$ by a new unknown $s \in H^{1/2}(\Gamma)$. This leads to the 3×3 system

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp & \\ \frac{1}{2}I - K_k & V_k & I \\ & -I & -i\widetilde{D} \end{pmatrix} \begin{pmatrix} u \\ t \\ s \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ -p \end{pmatrix}. \tag{5.36}$$

Since only g without additional operators appears on the right hand side, this seems to be a much more practical approach for the domain decomposition algorithm. Note that we exchanged V_0 by $\widetilde{D}^{-1}$ only to get rid of inverse operators in (5.36).

Lemma 5.30. *The operator as defined in* (5.36) *is invertible for all wave numbers* $k \in \mathbb{R}$.

Proof. This follows directly from the bijectivity of $\widetilde{D}$ and from the unique solvability of equation (5.35). □

Remark 5.31. *The solution of* (5.36) *can be constructed by taking the unique solution* U *of the exterior Neumann boundary value problem with Neumann data* $-p$ *and setting* $u = \gamma_0 U$, $\gamma_1 U = t = p$ *and* $s = 0$. *Hence, for a right hand side of the form* $(p, 0, p)^\top$, *we always obtain* $s = 0$.

However, it is still not possible to use (5.36) in the domain decomposition approach, since it is only feasible to couple the exterior Neumann trace just once with the interior Neumann trace. Otherwise, we can not prove the coerciveness of the global formulation. Hence, we have to apply some linear combinations to get rid of the second appearance of the Neumann data p on the right hand side. By subtracting the first line from the last line in (5.36) we get

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp & \\ \frac{1}{2}I - K_k & V_k & I \\ -D_k & -\frac{1}{2}I - K_k^\perp & -i\widetilde{D} \end{pmatrix} \begin{pmatrix} u \\ t \\ s \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ 0 \end{pmatrix}. \tag{5.37}$$

By replacing u by $u + s$ in (5.37) we finally obtain

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp & -D_k \\ \frac{1}{2}I - K_k & V_k & \frac{1}{2}I + K_k \\ -D_k & -\frac{1}{2}I - K_k^\perp & -i\widetilde{D} + D_k \end{pmatrix} \begin{pmatrix} u + s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ 0 \end{pmatrix}. \tag{5.38}$$

The injectivity of this system follows immediately from the injectivity of the system in (5.36). To show that a discretized version of (5.38) is also uniquely solvable, we have to prove again a Gårding inequality.

Theorem 5.32. *The associated bilinear form of the operator as defined in* (5.38) *fulfills a Gårding inequality.*

Proof. Since the differences $\widetilde{D} - D_0$, $V_k - V_0$, $K_k - K_0$ and $K_k^\perp - K_0^\perp$ are compact, we can replace the wave number k by 0. In the case of the hypersingular operator we will use the regularized version $\widetilde{D}$ instead. The corresponding associated bilinear

form of this regularized operator is a compact perturbation of the original one and is given by

$$\begin{aligned} a((u,t,s),(\lambda,\mu,\tau)) :=& \langle \widetilde{D}u,\lambda\rangle_\Gamma + \langle(-\frac{1}{2}+K_0^\perp)t,\lambda\rangle_\Gamma + \langle -\widetilde{D}s,\lambda\rangle_\Gamma \\ &+ \overline{\langle(\frac{1}{2}I+K_0)u,\mu\rangle_\Gamma} + \overline{\langle V_0 t,\mu\rangle_\Gamma} + \overline{\langle(\frac{1}{2}I+K_0)s,\mu\rangle_\Gamma} \\ &+ \langle -\widetilde{D}u,\tau\rangle_\Gamma + \langle(-\frac{1}{2}-K_0^\perp)t,\tau\rangle_\Gamma + \langle -i\widetilde{D}+\widetilde{D}s,\tau\rangle_\Gamma. \end{aligned}$$

For $(\lambda,\mu,\tau)=(u,t,s)$ we get the inequality

$$\begin{aligned} |a((u,t,s),(u,t,s))| &= |\langle \widetilde{D}(u-s),(u-s)\rangle_\Gamma - i\langle \widetilde{D}s,s\rangle_\Gamma + \langle V_0 t,t\rangle_\Gamma| \\ &\geq c_1|\,\|u-s\|^2_{H^{1/2}(\Gamma)} + i\,\|s\|^2_{H^{1/2}(\Gamma)}\,| + c_2\,\|t\|^2_{H^{-1/2}(\Gamma)} \\ &\geq \frac{c_1}{2\sqrt{2}}(\|u\|^2_{H^{1/2}(\Gamma)} + \|s\|^2_{H^{1/2}(\Gamma)}) + c_2\,\|t\|^2_{H^{-1/2}(\Gamma)}\,. \end{aligned}$$

The ellipticity of this regularized bilinear form implies the coerciveness of the original bilinear form. □

Remark 5.33. *From Remark 5.31 and equation* (5.38) *we obtain* $s=0$.

5.8 Boundary element methods

In this section we discuss the boundary element discretization of the boundary integral equations related to local interior and exterior boundary value problems and state their convergence properties.

We consider a discretization of the boundary Γ in N planar triangles τ_l and M nodes x_k such that

$$\overline{\Gamma} = \bigcup_{l=1}^{N} \overline{\tau}_l.$$

By

$$h_l = \left(\int_{\tau_l} \mathrm{d}s_x\right)^{1/2}$$

we define the local mesh width and by

$$h = \max_{l=1,\ldots,N} h_l$$

we define the global mesh width. Further we assume that the mesh is quasi–uniform, i.e.

$$\frac{\max_{l=1,\dots,N} h_l}{\min_{l=1,\dots,N} h_l} \leq c$$

with a fixed constant $c \geq 1$ independent of N. Let $Z_h = S_h^0(\Gamma) = \operatorname{span}\{\psi_k^0\}_{k=1}^N$ be the space of piecewise constant basis functions

$$\psi_k^0(x) = \begin{cases} 1 & \text{for } x \in \tau_k, \\ 0 & \text{else.} \end{cases}$$

For this space there holds the following approximation property, cf. [125].

Theorem 5.34. *Assume $u \in H^s(\Gamma)$ for some $s \in [0,1]$. For $\sigma \in [-1,0]$ there holds the approximation property*

$$\inf_{v_h \in S_h^0(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c h^{s-\sigma} |u|_{H^s(\Gamma)} .$$

Further let

$$W_h = S_h^1(\Gamma) = \operatorname{span}\{\phi_k^1\}_{k=1}^M$$

be the space of piecewise linear continuous basis functions with

$$\phi_k^1(x) = \begin{cases} 1 & \text{for } x = x_k, \\ 0 & \text{for } x = x_l \neq x_k, \\ \text{linear} & \text{else.} \end{cases}$$

For this trial space we have a similar approximation theorem, cf. [125].

Theorem 5.35. *Assume $u \in H^s(\Gamma)$ for some $s \in [\sigma, 2]$ and $\sigma \in [0,1]$. Then there holds the approximation property*

$$\inf_{v_h \in S_h^1(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c h^{s-\sigma} |u|_{H^s(\Gamma)} .$$

5.8.1 Fast boundary element techniques

Due to fully populated matrices, standard boundary element methods have quadratic computational complexity. This was a major drawback of the boundary element method, if used for solving challenging industrial problems. Fortunately, several fast boundary element methods were developed in the last decades, which drastically decreased the computational effort and memory requirements. The first methods were the fast multipole method [75, 76] and the Panel–Clustering method [79]. These

methods are based on suitable series expansions of the kernel. Here we use $\mathcal{H}$–matrices [15, 77, 78, 120]. These matrices are organized in a hierarchical manner, such that parts of the matrix can be approximated by low rank matrices. These $\mathcal{H}$–matrices offer a complete arithmetic, such as inversion or LU decomposition. There are different methods to create the low rank blocks, on which the $\mathcal{H}$–matrices are based. One is the hybrid cross approximation [19], which is based on an expansion of the fundamental solution. The method of our choice is the Adaptive Cross Approximation (ACA) [13], which constructs sparse matrices on an algebraic base. Due to the construction algorithm, this method allows a kind of black box implementation for various kernels. Within this work we use such a black box implementation, which is provided by the AHMED software package [12].

5.8.2 The interior Neumann/Robin problem

Within the domain decomposition approach we have in general to solve interior Robin boundary value problems, see Chapter 7. In some cases interior Neumann boundary value problems can be used instead, for example if it is known that the local wave number is not a Neumann eigen wave number, for example if $\mathrm{Im}(k) > 0$ is satisfied. To simplify the upcoming notation of mixed Neumann/Robin boundary value problems, we unify the notation of Neumann and Robin boundary conditions to

$$\gamma_1 U + i\eta(x) R \gamma_0 U = l \qquad \text{on } \Gamma$$

with $\Gamma = \overline{\Gamma}_N \cup \overline{\Gamma}_R$ and $\eta(x) \equiv 0$ for $x \in \Gamma_N$ and $\eta(x) \equiv c$ for $x \in \Gamma_R$ with $c \in \mathbb{R} \setminus \{0\}$. For readability, we further shorten the notation from $\eta(x)$ to η. If η is assumed to be identical zero, this means that $\Gamma = \Gamma_N$. On the other hand, for $\eta \not\equiv 0$ the Robin boundary part Γ_R is non–trivial.

As discussed in Section 5.6, we will use the boundary integral equation system (5.28)

$$\begin{pmatrix} D_k + i\eta R & \frac{1}{2}I + K_k^{\perp} \\ -(\frac{1}{2}I + K_k) & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} l \\ 0 \end{pmatrix} \tag{5.39}$$

with $u \in H^{1/2}(\Gamma)$ and $t, l \in H^{-1/2}(\Gamma)$, to solve the local Neumann ($\eta \equiv 0$) or (mixed) Robin ($\eta \not\equiv 0$) boundary value problem. Note that this equation is uniquely solvable if $\eta \not\equiv 0$ and $k \in \mathbb{R}$ or if $\eta \equiv 0$, $\mathrm{Im}(k) \geq 0$ and k^2 is not a Neumann eigen wave number. Further the operator as given in (5.39) fulfills a Gårding inequality, cf. Theorem 5.25. If we discretize the system (5.39) by using S_h^0 to approximate $t \in H^{-1/2}(\Gamma)$ and S_h^1 to approximate $u \in H^{1/2}(\Gamma)$ we obtain the discrete problem:

Find $(u_h, t_h) \in W_h \times Z_h$ such that

$$\begin{aligned} \langle (D_k + i\eta R) u_h, v_h \rangle_\Gamma + \langle (\frac{1}{2}I + K^{\perp}) t_h, v_h \rangle_\Gamma &= \langle l, v_h \rangle_\Gamma, \\ \langle -(\frac{1}{2}I + K_k) u_h, w_h \rangle_\Gamma + \langle V_k t_h, w_h \rangle_\Gamma &= 0 \end{aligned} \tag{5.40}$$

for all $(v_h, w_h) \in W_h \times Z_h$.

The corresponding matrix formulation is

$$\begin{pmatrix} D_{k,h} + i\eta R_h & \frac{1}{2}M_h^\top + K_{k,h}^\top \\ -(\frac{1}{2}M_h + K_{k,h}) & V_{k,h} \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} \underline{l} \\ \underline{0} \end{pmatrix}.$$

By using Theorem 3.10 we can deduce the following theorem.

Theorem 5.36. *Let $\eta \not\equiv 0$ and $k \in \mathbb{R}$ or let $\eta \equiv 0$, $\mathrm{Im}(k) \geq 0$ and k^2 not be a Neumann eigen wave number. Further let (u,t) be the unique solution of (5.39). Then, for a sufficient small mesh width h, the discrete problem (5.40) has a unique solution (u_h, t_h). Further we have the quasi–optimal estimate*

$$\begin{aligned} &\|u - u_h\|_{H^{1/2}(\Gamma)} + \|t - t_h\|_{H^{-1/2}(\Gamma)} \\ &\qquad \leq c \left(\inf_{v_h \in W_h} \|u - v_h\|_{H^{1/2}(\Gamma)} + \inf_{w_h \in Z_h} \|t - w_h\|_{H^{-1/2}(\Gamma)} \right). \end{aligned}$$

For $u \in H^2(\Gamma)$ and $t \in H^1_{pw}(\Gamma)$ we obtain

$$\|u - u_h\|^2_{H^{1/2}(\Gamma)} + \|t - t_h\|^2_{H^{-1/2}(\Gamma)} \leq ch^3(\|u\|^2_{H^2(\Gamma)} + \|t\|^2_{H^1_{pw}(\Gamma)}).$$

By using the Aubin–Nitsche trick we get

$$\|u - u_h\|_{L_2(\Gamma)} \leq c(u,t)h^2.$$

Proof. The proof can be found in [125]. □

Numerical example

As a numerical example we consider the Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega = (0,1)^3, \\ \gamma_1 U &= p \qquad \text{on } \Gamma. \end{aligned} \tag{5.41}$$

As an exact solution we use the fundamental solution

$$\hat{U}(x) = \frac{e^{ik|x-\hat{x}|}}{|x-\hat{x}|}$$

of the Helmholtz equation, with the pole in $\hat{x} = (2.0, 0.0, 2.0)^\top$. For this example we consider the wave numbers $k = 1.0$ and 4.0 which are well separated from the Neumann eigen wave numbers of Ω. Hence, the Neumann boundary value problem (5.41) has a unique solution. The boundary element discretization of the coupled variational formulation (5.40) is done with respect to a globally uniform boundary mesh of N_i plane triangular elements with M_i nodes and by using piecewise constant

N_i	M_i	It	rel. L_2–error	eoc
12	8	19	0.1215612	
48	26	48	0.0311467	1.964
192	98	116	0.0076578	2.024
768	386	194	0.0018961	2.014
3072	1538	326	0.0004708	2.010
12288	6146	> 400	0.0001133	2.055

(a) $k = 1.0$

N_i	M_i	It	rel. L_2–error	eoc
12	8	19	1.9399842	
48	26	52	0.5320741	1.867
192	98	135	0.1633048	1.704
768	386	229	0.0371847	2.135
3072	1538	371	0.0036690	3.341
12288	6146	> 400	0.0007019	2.386

(b) $k = 4.0$

Table 5.1: Iteration numbers and errors for the interior scattering problem on the cube.

basis functions ψ_m^i and piecewise linear continuous basis functions ϕ_n^i. In particular, since no eigen wave numbers appear, we consider $\eta \equiv 0$ in the linear system (5.40). This linear system is solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. The results are given in Table 5.1. For the wave number $k = 1.0$ we see a convergence order as predicted in the theory. In the case $k = 4.0$ the error is rather large for the coarser meshes. Afterwards the convergence rate is faster than the expected theoretical convergence rate. This is due to the so–called pollution effect. This effect states that if we do not use enough elements per wavelength, the approximate solution is much worse then the best approximation of the trial space. For a detailed discussion of this effect see for example [73]. Hence, the convergence rate is larger than suspected if we start to use enough elements, but goes afterwards down again to the anticipated convergence rate.

5.8.3 Exterior Neumann scattering problem

In Section 5.7, we presented two viable boundary integral formulations, which also fit in the proposed tearing and interconnecting approach, to solve the exterior Neumann scattering problem. The first formulation

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^{\perp} \\ \frac{1}{2}I - K_k & V_k \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} -p \\ 0 \end{pmatrix} \tag{5.42}$$

possesses for all wave numbers k a unique solution $u \in H^{1/2}(\Gamma)$. But $t \in H^{-1/2}(\Gamma)$ is not uniquely determined, if k^2 is a Dirichlet eigen wave number. Since we are only interested in u, we want to test if this formulation is viable in practice or not.

The discrete formulation of equation (5.42) is given by:

Find $(u_h, t_h) \in W_h \times Z_h$ such that

$$\begin{aligned} \langle D_k u_h, v_h\rangle_\Gamma + \langle(-\frac{1}{2} + K^\perp)t_h, v_h\rangle_\Gamma &= \langle -p, v_h\rangle_\Gamma, \\ \langle(\frac{1}{2}I - K_k)u_h, w_h\rangle_\Gamma + \langle V_k t_h, w_h\rangle_\Gamma &= 0 \end{aligned} \tag{5.43}$$

for all $(v_h, w_h) \in W_h \times Z_h$.

The corresponding matrix formulation is

$$\begin{pmatrix} D_{k,h} & -\frac{1}{2}M_h^\top + K_{k,h}^\top \\ \frac{1}{2}M_h - K_{k,h} & V_{k,h} \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} -\underline{p} \\ \underline{0} \end{pmatrix}. \tag{5.44}$$

If equation (5.42) is uniquely solvable and the mesh width h is sufficient small, then the discrete system (5.43) admits a unique solution. Again, the proof can be done by utilizing the coercivity of the operator given in (5.42). In this case we have, in addition, the same convergence estimates as for the interior problem. If k^2 gets close to a Dirichlet eigenvalue, the iteration numbers to solve problem (5.43) with a GMRES solver increase moderately. See Figure 5.1 for the iteration numbers to solve problem (5.43) with/without preconditioner on the unit cube $(0,1)^3$ for wave numbers k between 0 and 10. The used block diagonal preconditioner, based on operators of opposite order, will be discussed in Section 5.9. For this experiment we used a triangulation with 386 nodes and 768 elements. However, if k^2 gets close to an eigenvalue, not only the iteration numbers are increasing, the solution t_h also gets disturbed by a multiple of the eigensolution. Since the analytic eigensolution is $(0, \gamma_1 U_\lambda)^\top$ and we are only interested in the Dirichlet part of the solution, the numerical solutions keep acceptable in this experiment. In Figure 5.2 we see that the error of the Neumann data explodes in the surrounding of a Dirichlet eigenvalue, while the error of the Dirichlet data keeps rather low.

Nevertheless, we want to use a formulation which can be proven to be stable for all wave numbers k. Therefore we will use the regularized formulation (5.32)

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp \\ \frac{1}{2}I - K_k & V_k + iV_0 \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} -p \\ iV_0 p \end{pmatrix} \tag{5.45}$$

to solve the exterior Neumann boundary value problem. This equation has for all wave numbers k a unique solution, see Section 5.7. The discrete formulation of the regularized equation (5.45) is then given by

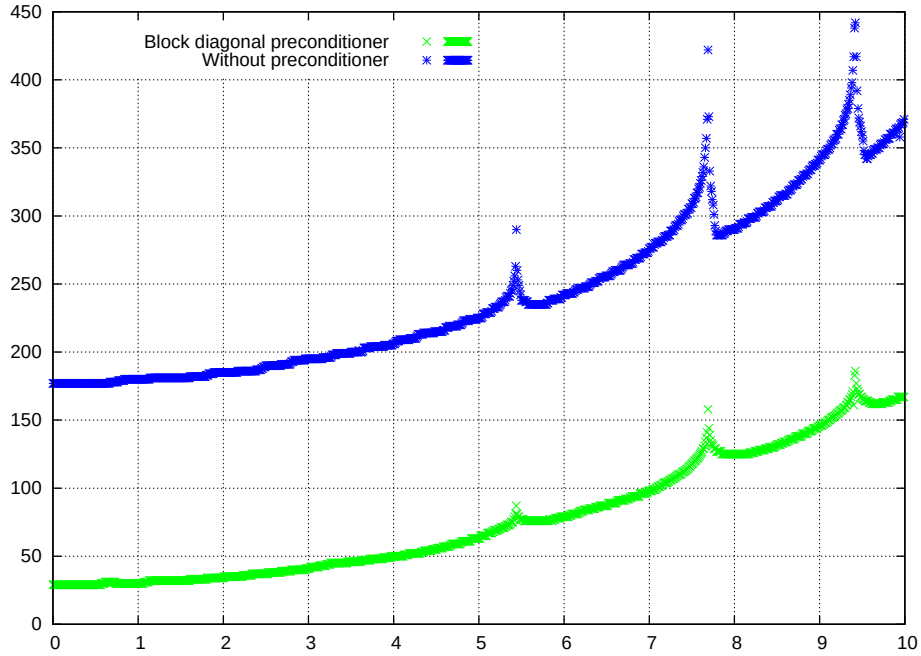


Figure 5.1: Iteration numbers for the unmodified exterior block system.

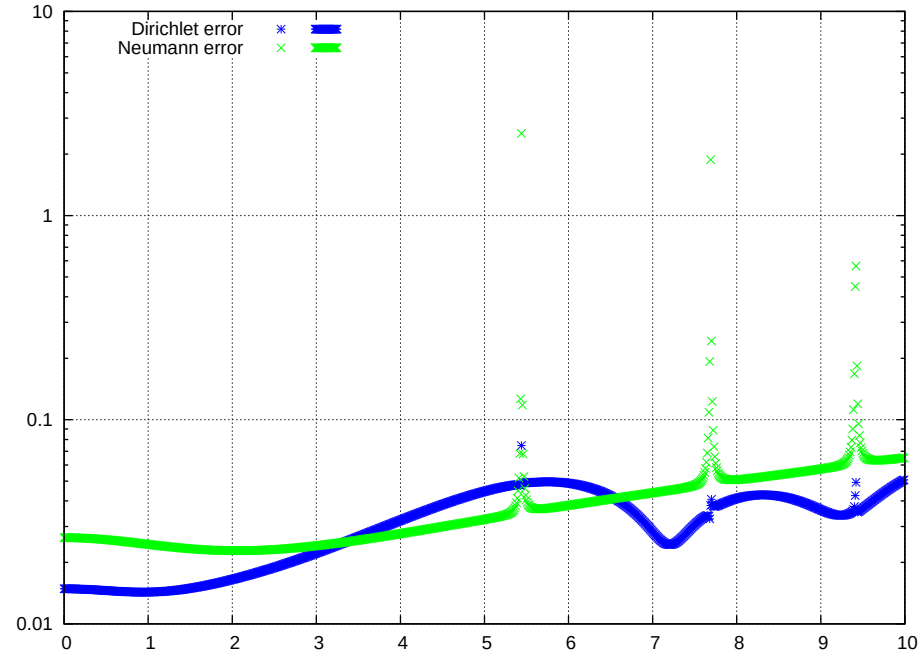


Figure 5.2: Error of Neumann and Dirichlet data for the unmodified exterior block problem.

Find $(u_h, t_h) \in W_h \times Z_h$ such that

$$\begin{aligned} \langle D_k u_h, v_h\rangle_\Gamma + \langle(-\frac{1}{2} + K^\perp)t_h, v_h\rangle_\Gamma &= \langle p, v_h\rangle_\Gamma, \\ \langle(\frac{1}{2}I - K_k)u_h, w_h\rangle_\Gamma + \langle(V_k + iV_0)t_h, w_h\rangle_\Gamma &= \langle V_0 p, w_h\rangle_\Gamma \end{aligned} \tag{5.46}$$

for all $(v_h, w_h) \in W_h \times Z_h$.

The corresponding matrix formulation is

$$\begin{pmatrix} D_{k,h} & -\frac{1}{2}M_h^\top + K_{k,h}^\top \\ \frac{1}{2}M_h - K_{k,h} & V_{k,h} + iV_{0,h} \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} -\underline{p} \\ i\underline{V_0 p} \end{pmatrix}.$$

This discrete problem has a unique solution if the mesh width h is small enough, see Lemma 3.12. Further we gain the same convergence properties as for the interior problem, see Theorem 5.36.

The matrix $V_{k,h} + iV_{0,h}$ is invertible, as already discussed in the last section. Hence, we can eliminate the artificial variable $\underline{t}$ to obtain the CFIE–like formulation

$$\begin{aligned} (D_{k,h} + (\frac{1}{2}M_h^\top - K_{k,h}^\top)(V_{k,h} + iV_{0,h})^{-1}(\frac{1}{2}M_h^\top - K_{k,h}^\top))\underline{u} \\ = -\underline{p} - i(\frac{1}{2}M_h^\top - K_{k,h}^\top)(V_{k,h} + iV_{0,h})^{-1}\underline{V_0 p}. \end{aligned}$$

If we just want to solve an exterior acoustic scattering problem, this is the formulation of choice. But this formulation is not suitable for the domain decomposition approach, since in the iterative algorithm, which will be used to solve the global problem, only the vector $\underline{p}$ is known and not the analytic function p. Hence, $\underline{V_0 p}$ can not be computed.

Therefore we reformulate equation (5.45), i.e.,

$$\begin{pmatrix} D_k & -\frac{1}{2}I + K_k^\perp & \\ \frac{1}{2}I - K_k & V_k & I \\ & -I & -i\widetilde{D} \end{pmatrix} \begin{pmatrix} u \\ t \\ s \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ -p \end{pmatrix}, \tag{5.47}$$

as discussed in Section 5.7 (note V_0 is replaced by $\widetilde{D}^{-1}$). This has the advantage that only the Neumann data p is present on the right hand side. Hence, the corresponding matrix formulation

$$\begin{pmatrix} D_{k,h} & -\frac{1}{2}M_h^\top + K_{k,h}^\top & \\ \frac{1}{2}M_h - K_{k,h} & V_{k,h} & M_h \\ & -M_h^\top & -i\widetilde{D}_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \\ \underline{s} \end{pmatrix} = \begin{pmatrix} -\underline{p} \\ 0 \\ -\underline{p} \end{pmatrix} \tag{5.48}$$

can be used within an iterative global algorithm, see Section 7.6. In the practical application we prefer the Schur complement formulation

$$\begin{pmatrix} D_{k,h} & -\frac{1}{2}M_h^\top + K_{k,h}^\top \\ \frac{1}{2}M_h - K_{k,h} & V_{k,h} + iM_h\widetilde{D}_h^{-1}M_h^\top \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} -\underline{p} \\ iM_h^\top \widetilde{\widetilde{D}}_h^{-1}\underline{p} \end{pmatrix} \tag{5.49}$$

were the artificial variable $\underline{s}$ is eliminated.

level	Diri. V1	Diri. V2	Diri. V3	Neu. V1	Neu. V2	Neu. V3
0	2.47127	1.24449	1.29537	0.47323	0.11159	0.21507
1	1.10817	0.99370	0.99422	0.29747	0.14349	0.18713
2	0.26375	0.22568	0.22029	0.59564	0.08578	0.11482
3	0.05436	0.04978	0.04968	0.62962	0.02510	0.03477
4	0.01229	0.01173	0.01176	0.60604	0.00793	0.01109
5	0.00292	0.00285	0.00286	0.55294	0.00276	0.00385

Table 5.2: Relative L_2–errors of the Dirichlet and Neumann datum for the non regularized approach V1, the regularized approach V2, and the regularized domain decomposition approach V3.

Numerical example

As a numerical example we consider the Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{in } \Omega^c = \mathbb{R}^3 \setminus [0,1]^3, \\ \gamma_1 U &= p \qquad \text{on } \Gamma. \end{aligned} \tag{5.50}$$

As an exact solution we use the fundamental solution

$$\hat{U}(x) = \frac{e^{ik|x-\hat{x}|}}{|x-\hat{x}|}$$

of the Helmholtz equation, with the pole in $\hat{x} = (0.5, 0.5, 0.5)^\top$. For this example we consider the first Neumann eigen wave number $k = \sqrt{3}\pi \approx 5.4414$ of the unit cube. Although the Neumann boundary value problem (5.50) has a unique solution, this is not true for the non–regularized formulation (5.42). Hence the corresponding discrete problem (5.44) is nearly singular. Therefore we compare it with the regularized formulation (5.46) for the pure exterior problem and with the regularized formulation (5.49) which is used within the domain decomposition approach. The boundary element discretization is done with respect to a globally uniform boundary mesh of N_i plane triangular elements with M_i nodes and by using piecewise constant basis functions ψ_m^i and piecewise linear continuous basis functions ϕ_n^i. The linear systems are solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. The results are given in Table 5.2 and Table 5.3. As discussed in Section 5.7, also the non–regularized approach provides in this example useful results for the Dirichlet datum (which is the point of interest), which are only slightly worse as those of the regularized formulations. The artificial Neumann datum is in this approach completely wrong. The bad conditioning of the non–regularized formulation leads further to higher iteration numbers. The two regularized formulations lead to quite similar results, still the formulation for the pure exterior Neumann boundary value problem gives slightly better results.

level	NP V1	NP V2	NP V3	BP V1	BP V2	BP V3
0	19	19	19	19	19	19
1	55	53	54	50	40	45
2	143	130	136	72	50	57
3	278	219	231	85	53	63
4	468	347	370	92	54	64
5	815	572	614	101	55	66

Table 5.3: Iteration numbers without preconditioner (NP) and with a block diagonal preconditioner (BP) (see Section 5.9) for the non regularized approach V1, the regularized approach V2, and the regularized domain decomposition approach V3.

5.9 Preconditioning strategies

In this section we describe a suitable preconditioner for the local system

$$A = \begin{pmatrix} -V_{k,h} & \frac{1}{2}M_h + K_{k,h} \\ (\frac{1}{2}M_h + K_{k,h})^\top & D_{k,h} \end{pmatrix}. \tag{5.51}$$

Note that we changed the sign of the first line due to an aimed achievement of better iteration numbers. We use a block diagonal preconditioner, which is based on the idea of operators of opposite order, see [45, 103, 126]. This preconditioner consists of the single layer potential and the regularized hypersingular operator of the Laplace operator,

$$C_A = \begin{pmatrix} -M_{0,h}^{-1}\hat{D}_h M_{0,h}^{-1} & \\ & M_{1,h}^{-1}\hat{V}_h M_{1,h}^{-1} \end{pmatrix}.$$

The matrices $M_{0,h}$ and $M_{1,h}$ are the mass matrices of constant and linear basis functions, respectively. The matrix $\hat{V}_h$ is the discretization of the Laplace single layer potential with piecewise linear ansatz functions,

$$\hat{V}_h[i,j] = \int_\Gamma \int_\Gamma \phi_i(x) \frac{1}{4\pi|x-y|} \phi_j(x) \mathrm{d}s_x \mathrm{d}s_y, \qquad \phi_i, \phi_j \in S_h^1(\Gamma).$$

Since $S_h^1(\Gamma)$ is a conforming ansatz space in $H^{-1/2}(\Gamma)$, $\hat{V}_h$ is well defined. A more detailed discussion about the theory and results if preconditioning $D_{k,h}$ by $M_{1,h}^{-1}\hat{V}_h M_{1,h}^{-1}$ can be found in [45, 126].

$\hat{D}_h$ is the discretization of the regularized Laplace hypersingular operator with piecewise constant ansatz functions based on the representation (5.12), i.e.

$$\hat{D}_h[i,j] = \int_\Gamma \int_\Gamma \frac{\mathbf{curl}_\Gamma \psi_i(x) \cdot \mathbf{curl}_\Gamma \psi_j(x)}{4\pi|x-y|} \mathrm{d}s_x \mathrm{d}s_y + \langle 1, \psi_i \rangle_\Gamma \langle 1, \psi_j \rangle_\Gamma, \quad \psi_i, \psi_j \in S_h^0(\Gamma).$$

With the help of Stoke's theorem we can define $\mathbf{curl}_\Gamma \psi_i$ in the weak sense by

$$\begin{aligned}\int_\Gamma \mathbf{curl}_\Gamma \psi_i(x) \cdot \mathbf{v}(x) \mathrm{d}x &= \int_{\partial\tau_i} \mathbf{v}(x) \cdot \mathbf{t}_i(x) \mathrm{d}s_x \\ &= \int_{\tau_i} 1 \cdot \operatorname{curl}_\Gamma \mathbf{v}(x) \mathrm{d}x = \int_\Gamma \psi_i(x) \operatorname{curl}_\Gamma \mathbf{v}(x) \mathrm{d}x\end{aligned}$$

for all test functions $\mathbf{v}$. This leads to a formulation based on line integrals, i.e.

$$\hat{D}_h[i,j] = \int_{\partial\tau_i} \int_{\partial\tau_j} \frac{\mathbf{t}_i \cdot \mathbf{t}_j}{4\pi|x-y|} \mathrm{d}s_x \mathrm{d}s_y + \langle 1, \psi_i \rangle_\Gamma \langle 1, \psi_j \rangle_\Gamma.$$

Hereby $\mathbf{t}_i$ and $\mathbf{t}_j$ are the direction vectors of the edges of the triangles τ_i and τ_j. Note that this is a heuristic approach, since constant ansatz functions are not in $H^{1/2}(\Gamma)$. Therefore, to the best of our knowledge, in contrast to the single layer potential no rigorous theory is available yet. Another drawback is that double integrals on the same line are not well defined, for these integrals heuristic values, which were optimized by testing, were chosen.

Another possible preconditioning strategy, is to use an approximate inverse of $V_{0,h}$ instead of $M_{0,h}^{-1} \hat{D}_h M_{0,h}^{-1}$ and an approximate inverse of $\widetilde{D}_h$ instead of $M_{1,h}^{-1} \hat{V}_h M_{1,h}^{-1}$. Such approximate inverses can be computed quite efficiently by using the $\mathcal{H}$–matrix algebra. Of course efficiency depends on the approximation quality of the inverse, which on the other hand influences the quality of the preconditioner. Therefore, a compromise between efficiency and quality has to be made. For a single application the construction of this preconditioner is probably to expensive, but for multiple applications as in the domain decomposition approach presented in this work, it can be more efficient. For further details see [14].

In this section we presented a particular choice of preconditioners, but there are also many other preconditioners for the single layer potential and the hypersingular operator, which could be used to create such a block diagonal preconditioner. Other possible preconditioners would be the BPX preconditioner [70] or algebraic multigrid preconditioners [94, 113].

Numerical example

To evaluate the performance of the local preconditioners, we use exactly the same problem as we did for the interior scattering problem. Hence, the triangulation and all other parameters are the same. We use a rather high approximation quality of $\varepsilon = 10^{-5}$ for the inverse operators. For a detailed explanation of this parameter see [14]. The results are given in Table 5.4. The iteration numbers seem for both preconditioners to be bounded. As anticipated by the theory, the preconditioner

level	$k = 1.0$			$k = 2.0$			$k = 4.0$			$k = 8.0$		
	OO	AI	non	OO	AI	non	OO	AI	non	OO	AI	non
0	17	15	17	18	15	19	19	15	19	19	15	19
1	27	20	48	28	23	49	39	29	52	58	43	51
2	31	23	116	33	25	120	48	33	135	108	57	143
3	33	25	194	35	27	201	52	33	229	120	59	214
4	34	25	326	36	28	337	53	37	371	122	61	>400
5	35	27	>400	36	29	>400	53	37	>400	122	61	>400

Table 5.4: Iteration numbers for different wave numbers k. 'OO' stands for the preconditioner based on operators of opposite order, 'AI' stands for the preconditioner using the approximate inverse and 'non' for the unpreconditioned system.

level	AI	OO	system
0	0.10	0.02	0.08
1	1.23	0.31	0.82
2	19.54	4.81	12.76
3	370.03	60.83	247.95
4	5504.56	1138.78	2967.56

Table 5.5: Times to construct the preconditioners based on the hierarchical matrices (AI) and operators of opposite order (OO). In addition the construction time of the linear equation system is given (system). All times are given in seconds.

based on the approximate inverses works better, especially for higher wave numbers. This has to be balanced with it's more expensive construction. The times to construct the linear equation system and two preconditioners for the case $k = 1.0$ are given in Table 5.5. To achieve even better iteration numbers, probably a non block–diagonal preconditioner has to be constructed.

6 BOUNDARY INTEGRAL EQUATIONS FOR ELECTROMAGNETIC SCATTERING PROBLEMS

In this chapter we discuss boundary value problems of electromagnetic scattering. First we discuss under which circumstances the problems possess a unique solution. Based on the Stratton–Chu representation formula we then introduce the surface potentials and boundary integral operators for the electromagnetic wave equation. After stating the corresponding boundary integral equations, we discuss some important properties of the boundary integral operators. Thereafter, we explain the solution strategy which fits to the domain decomposition approach. Then we introduce suitable boundary element formulations. Finally we give some numerical examples and describe a possible preconditioning strategy.

6.1 Boundary value problems

The Maxwell boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^3$ is stated by

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} && \text{in } \Omega, \\ \boldsymbol{\gamma}_D\mathbf{U} = \mathbf{n} \times \mathbf{U}_{|\Gamma} \times \mathbf{n} &= \mathbf{g} && \text{on } \Gamma_D, \\ \boldsymbol{\gamma}_N\mathbf{U} = (\mathbf{curl}\,\mathbf{U})_{|\Gamma} \times \mathbf{n} &= \mathbf{p} && \text{on } \Gamma_N, \\ \boldsymbol{\gamma}_N\mathbf{U} + i\eta\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{l} && \text{on } \Gamma_R. \end{aligned} \tag{6.1}$$

As in the Helmholtz case, we call the boundary value problem a Dirichlet, Neumann or Robin boundary value problem if $\Gamma = \Gamma_D$, $\Gamma = \Gamma_N$ or $\Gamma = \Gamma_R$, respectively. If non of these cases applies, we call the boundary value problem a mixed boundary value problem. Again we have to make some assumptions for the operator R.

Assumption 6.1. *We assume that the operator* R *is strictly positive, self–adjoint and that the sesquilinear form* $\langle \mathsf{R}\mathbf{u}, \mathcal{Y}\mathbf{u}\rangle_{\Gamma_R}$ *is real valued up to a compact perturbation. For the definition of* $\mathcal{Y}$ *see Definition 4.40.*

As in the acoustic case, the interior boundary value problems are not for every $k^2 \in \mathbb{C}$ uniquely solvable. For $k = 0$, in contrast to the acoustic case, the boundary value problems possess quite different properties. Therefore we exclude the case $k = 0$ in the upcoming theory.

Theorem 6.2. *The solutions of the Dirichlet eigenvalue problem*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} &= \lambda\mathbf{U} \qquad && \text{in } \Omega, \\ \boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{0} && \text{on } \Gamma \end{aligned} \tag{6.2}$$

have the following properties:

- *Corresponding to the eigenvalue $\lambda_0 = 0$ there is an infinite dimensional family of functions $\mathbf{U} = \boldsymbol{\nabla} P$ for any $P \in H_0^1(\Omega)$.*
- *There is an infinite discrete set of eigenvalues $\lambda_j > 0$, $j = 1, 2, \ldots$ and corresponding eigenfunctions $\mathbf{U}_j \neq \mathbf{0}$ such that equation (6.2) is satisfied and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_j \to \infty$ as $j \to \infty$ and $\mathbf{U}_i$ orthogonal to $\mathbf{U}_j$ in $\mathbf{L}_2$ if $i \neq j$.*

Proof. See [106, p. 97]. □

The eigenvalues of the Neumann eigenvalue problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} &= \lambda\mathbf{U} \qquad && \text{in } \Omega, \\ \boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{0} && \text{on } \Gamma, \end{aligned}$$

have the same properties as the eigenvalues of the Dirichlet eigenvalue problem (6.2), see [143]. Based on the introduced Dirichlet and Neumann eigenvalues of the interior electromagnetic wave equation we can transfer the notation of Dirichlet and Neumann eigen wave numbers directly from the Helmholtz case, cf. Remark 5.4.

Remark 6.3. *In the case of a Dirichlet, Neumann or a mixed Dirichlet/Neumann boundary value problem, i.e. $\Gamma_R = \emptyset$, there exists a unique solution of the boundary value problem (6.1) if $k^2 \in \mathbb{C} \setminus \mathbb{R}$, see [143].*

Theorem 6.4. *The mixed boundary value problem*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} \qquad && \text{in } \Omega, \\ \boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{p} && \text{on } \Gamma_N, \\ \boldsymbol{\gamma}_N\mathbf{U} + i\eta\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{l} && \text{on } \Gamma_R. \end{aligned} \tag{6.3}$$

has at most one solution if the operator R fulfills Assumption 6.1 and if Γ_R is non–trivial.

Proof. The weak formulation of the boundary value problem (6.3) is to find $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$\begin{aligned} \int_\Omega \mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\mathbf{V}(x)\mathrm{d}x - k^2 \int_\Omega \mathbf{U}(x) \cdot \mathbf{V}(x) + i\eta \int_{\Gamma_R} (\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U}(x)) \cdot \boldsymbol{\gamma}_D\mathbf{V}(x)\mathrm{d}s_x \\ = \int_{\Gamma_R} \mathbf{l}(x) \cdot \boldsymbol{\gamma}_D\mathbf{V}(x)\mathrm{d}s_x + \int_{\Gamma_N} \mathbf{p}(x) \cdot \boldsymbol{\gamma}_D\mathbf{V}(x)\mathrm{d}s_x \end{aligned}$$

is satisfied for all $\mathbf{V} \in \mathbf{H}(\mathbf{curl}, \Omega)$. To prove that there is at most one solution of the boundary value problem (6.3), we assume that $\mathbf{U}$ is a non–trivial solution of the homogeneous boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} && \text{in } \Omega, \\ \boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{0} && \text{on } \Gamma_N, \\ \boldsymbol{\gamma}_N\mathbf{U} + i\eta\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{0} && \text{on } \Gamma_R. \end{aligned}$$

Further let us set $\mathbf{V} = \overline{\mathbf{U}}$, this gives us

$$\int_\Omega \left[|\,\mathbf{curl}\,\mathbf{U}(x)|^2 - |\mathbf{U}(x)|^2 \right] \mathrm{d}x + i\eta \int_{\Gamma_R} (\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U})(x) \cdot \overline{\boldsymbol{\gamma}_D\mathbf{U}(x)} \mathrm{d}s_x = 0. \tag{6.4}$$

Since R is a strictly positive operator, $\boldsymbol{\gamma}_D\mathbf{U}(x) = \mathbf{0}$ for $x \in \Gamma_R$ follows. The Robin boundary condition further implies $\boldsymbol{\gamma}_N\mathbf{U}(x) = \mathbf{0}$ for $x \in \Gamma_R$. This finally implies, as in the Helmholtz case (see Lemma 5.7), $\mathbf{U}(x) = \mathbf{0}$ for $x \in \Omega$, see also [106, p. 93]. □

Theorem 6.5. *The mixed boundary value problem* (6.3) *has a solution.*

Proof. The sesquilinear form of the variational formulation corresponding to the Robin boundary value problem (6.3) is given by

$$\begin{aligned} a(\mathbf{U}, \mathcal{Y}\mathbf{U}) := &\int_\Omega \left[\mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\mathcal{Y}\overline{\mathbf{U}}(x) - k^2\mathbf{U}(x) \cdot \mathcal{Y}\overline{\mathbf{U}}(x) \right] \mathrm{d}x \\ &+ i\eta \int_{\Gamma_R} (\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U})(x) \cdot \boldsymbol{\gamma}_D\mathcal{Y}\overline{\mathbf{U}}(x) \mathrm{d}s_x. \end{aligned}$$

Since the imaginary part of the boundary integral is compact, see Assumption 6.1, we have for the real part

$$\mathrm{Re}\,(a(\mathbf{U}, \mathcal{Y}\mathbf{U})) = \int_\Omega \left[\mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\overline{\mathbf{U}}(x) + k^2\mathbf{U}(x) \cdot \overline{\mathbf{U}}(x) \right] \mathrm{d}x + C(\mathbf{U}, \mathbf{U})$$

with a compact sesquilinear form C, see Section 4.3.2. So $a(\cdot, \cdot)$ fulfills the generalized Gårding inequality

$$\mathrm{Re}(a(\mathbf{U}, \mathcal{Y}\mathbf{U}) + C(\mathbf{U}, \mathbf{U})) \geq c\,\|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}\,.$$

Since we have already shown injectivity, the proposition follows. □

In contrast to the interior boundary value problems, the exterior boundary value problems are not haunted by eigen wave numbers if $k \in \mathbb{R} \setminus \{0\}$.

Theorem 6.6. *The exterior Dirichlet scattering problem*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} && \text{in } \Omega^c, \\ \boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{g} && \text{on } \Gamma_D, \\ \lim_{r\to 0}\int_{\partial B_r} |\boldsymbol{\gamma}_N\mathbf{U}(x) - ik\boldsymbol{\gamma}_D\mathbf{U}(x)|^2 \mathrm{d}s_x &= 0 \end{aligned}$$

and the exterior Neumann scattering problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} && \text{in } \Omega^c, \\ \boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{p} && \text{on } \Gamma_N, \\ \lim_{r\to 0}\int_{\partial B_r} |\boldsymbol{\gamma}_N\mathbf{U}(x) - ik\boldsymbol{\gamma}_D\mathbf{U}(x)|^2 \mathrm{d}s_x &= 0 \end{aligned}$$

have for $k \in \mathbb{R}\setminus\{0\}$ a unique solution.

Proof. See [106, 139]. □

Remark 6.7. *In Theorem 6.6 we used the weaker integral radiation condition instead of the radiation condition of Silver–Müller, cf. [107], which is given by*

$$\lim_{r\to\infty} r\,(\mathbf{curl}\,\mathbf{U}(x)\times\mathbf{n} - ik\mathbf{U}(x)) = \mathbf{0}, \qquad \text{with } r = |x|. \tag{6.5}$$

This condition is sufficient to eliminate incoming solutions and to ensure unique solvability of exterior electromagnetic scattering problems. The interchangeability of these two conditions is discussed, e.g., in [106, 110].

6.2 Representation formula and integral operators

In this section we present the Stratton–Chu representation formula for the electromagnetic wave equation, which was first shown in [46]. Afterwards we define potential operators in such a way that this representation formula can be written in a compact form. Based on these potential operators, we introduce the boundary integral operators and state the corresponding boundary integral equations.

Theorem 6.8 (Representation formula)**.** *The solution of the interior electromagnetic wave equation* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} = \mathbf{0}$ *in* Ω *is given by the representation formula*

$$\begin{aligned} \mathbf{U}(x) = \ & \mathbf{grad}_y \int_\Gamma g_k(x,y)\gamma_n\mathbf{U}(y)\mathrm{d}s_y + \int_\Gamma g_k(x,y)\boldsymbol{\gamma}_N\mathbf{U}(y)\mathrm{d}s_y \\ & + \mathbf{curl}_y \int_\Gamma g_k(x,y)(\mathbf{U}(y)\times\mathbf{n})\mathrm{d}s_y \qquad \text{for } x\in\Omega. \end{aligned} \tag{6.6}$$

The solution of the exterior electromagnetic wave equation $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} = \mathbf{0}$ *in* Ω^c *which satisfies the Silver–Müller radiation condition* (6.5) *is given by*

$$\mathbf{U}(x) = -\,\mathbf{grad}_x \int_\Gamma g_\kappa(x,y)\gamma_n\mathbf{U}(y)\mathrm{d}s_y - \int_\Gamma g_\kappa(x,y)\boldsymbol{\gamma}_N\mathbf{U}(y)\mathrm{d}s_y$$
$$-\,\mathbf{curl}_x \int_\Gamma g_\kappa(x,y)(\mathbf{U}(y)\times\mathbf{n})\mathrm{d}s_y \qquad \text{for } x\in\Omega^c.$$

Proof. See [26, 47]. □

Motivated by the Stratton–Chu representation formula we define the following potential operators.

Definition 6.9. *The vector–valued single layer potential is defined by*

$$\boldsymbol{\Psi}_k^A(\boldsymbol{\lambda})(x) := \int_\Gamma g_k(x,y)\boldsymbol{\lambda}(y)\mathrm{d}s_y \quad \text{for } x\notin\Gamma,$$

The Maxwell double layer potential (or vector–valued double layer potential) is given by

$$\boldsymbol{\Psi}_k^M(\boldsymbol{\lambda})(x) := \mathbf{curl}\,\boldsymbol{\Psi}_k^A(\mathbf{R}\boldsymbol{\lambda})(x) \quad \text{for } x\notin\Gamma$$

with $\mathbf{RU} = \mathbf{U}\times\mathbf{n}$.

The vector–valued single layer potential has the mapping property

$$\boldsymbol{\Psi}_k^A : \mathbf{H}_\parallel^{-1/2}(\Gamma) \to \mathbf{H}^1_{loc}(\mathbb{R}^3),$$

see for example [81, Theorem 5.1].

The Stratton–Chu representation formula for bounded domains can now be written as

$$\mathbf{U}(x) = \boldsymbol{\Psi}_k^M(\boldsymbol{\gamma}_D\mathbf{U})(x) + \boldsymbol{\Psi}_k^A(\boldsymbol{\gamma}_N\mathbf{U})(x) + \mathbf{grad}\,\Psi_k^S(\gamma_n\mathbf{U})(x) \qquad \text{for } x\in\Omega.$$

Obviously we have to deal with three instead of two traces. But for $k\neq 0$ we can get rid of this third trace by using Theorem 4.31. For solutions of the differential equation $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} = \mathbf{0}$ in Ω we have the identity

$$\gamma_n\mathbf{U} = \frac{1}{k^2}\,\mathrm{div}_\Gamma(\boldsymbol{\gamma}_N\mathbf{U}).$$

Therefore, $\gamma_n\mathbf{U}$ can be eliminated in the representation formula, i.e.

$$\mathbf{U}(x) = \boldsymbol{\Psi}_k^M(\boldsymbol{\gamma}_D\mathbf{U})(x) + \boldsymbol{\Psi}_k^A(\boldsymbol{\gamma}_N\mathbf{U})(x) + \frac{1}{k^2}\,\mathbf{grad}\,\Psi_k^S\,\mathrm{div}_\Gamma(\boldsymbol{\gamma}_N\mathbf{U})(x) \quad \text{for } x\in\Omega.$$

Further it motivates the following definition of the Maxwell single layer potential.

Definition 6.10. *The Maxwell single layer potential is defined as*

$$\boldsymbol{\Psi}_k^S(\boldsymbol{\mu}) := \boldsymbol{\Psi}_k^A(\boldsymbol{\mu}) + \frac{1}{k^2}\,\mathbf{grad}\,\Psi_k^S(\mathrm{div}_\Gamma\,\boldsymbol{\mu}) \qquad \textit{for } \boldsymbol{\mu} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

The representation formula for the interior boundary value problem can now be written in the compact form

$$\mathbf{U}(x) = \boldsymbol{\Psi}_k^M(\boldsymbol{\gamma}_D\mathbf{U}(x)) + \boldsymbol{\Psi}_k^S(\boldsymbol{\gamma}_N\mathbf{U}(x)) \qquad \text{for } x \in \Omega. \tag{6.7}$$

Accordingly the representation formula for the exterior boundary value problem is given by

$$\mathbf{U}(x) = -\boldsymbol{\Psi}_k^M(\boldsymbol{\gamma}_D^c\mathbf{U}(x)) - \boldsymbol{\Psi}_k^S(\boldsymbol{\gamma}_N^c\mathbf{U}(x)) \qquad \text{for } x \in \Omega^c. \tag{6.8}$$

The proofs of the following theorem and corollary can be found in [84].

Theorem 6.11. *The Maxwell single layer potential* $\boldsymbol{\Psi}_S^k\mathbf{u}$ *and the Maxwell double layer potential* $\boldsymbol{\Psi}_M^k\mathbf{v}$ *are for arbitrary* $\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *radiating solutions of the partial differential equation* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} = \mathbf{0}$ *for* $x \notin \Gamma$.

Corollary 6.12. *The Maxwell single and double layer potential have the mapping properties*

$$\boldsymbol{\Psi}_k^S : \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{loc}(\mathbf{curl}^2, \Omega \cup \Omega^c),$$
$$\boldsymbol{\Psi}_k^M : \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{loc}(\mathbf{curl}^2, \Omega \cup \Omega^c).$$

The potentials corresponding to the electromagnetic scattering equation bear some relations, which are not present in the acoustic case.

Lemma 6.13. *The trace and the potential operators fulfill the following relations:*

$$\boldsymbol{\gamma}_N\boldsymbol{\Psi}_k^S(\mathbf{u}) = (\boldsymbol{\gamma}_D\boldsymbol{\Psi}_k^M(\mathbf{n} \times \mathbf{u})) \times \mathbf{n},$$
$$\boldsymbol{\gamma}_N\boldsymbol{\Psi}_k^M(\mathbf{v}) = k^2(\boldsymbol{\gamma}_D(\boldsymbol{\Psi}_k^A(\mathbf{v} \times \mathbf{n}))) \times \mathbf{n}.$$

Proof. We have

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{\Psi}_k^S(\mathbf{u}) \times \mathbf{n} = \mathbf{curl}\,\boldsymbol{\Psi}_k^A(\mathbf{u}) \times \mathbf{n} &= \boldsymbol{\gamma}_\times\boldsymbol{\Psi}_k^M(\mathbf{n} \times \mathbf{u}) \\ &= (\boldsymbol{\gamma}_D(\boldsymbol{\Psi}_k^M(\mathbf{n} \times \mathbf{u}))) \times \mathbf{n}\end{aligned}$$

and

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_k^A(\mathbf{v} \times \mathbf{n}) \times \mathbf{n} = k^2\boldsymbol{\Psi}_k^A(\mathbf{v} \times \mathbf{n}) \times \mathbf{n} = k^2(\boldsymbol{\gamma}_D(\boldsymbol{\Psi}_k^A(\mathbf{v} \times \mathbf{n}))) \times \mathbf{n}.$$

□

Remark 6.14. *If we take a look at Lemma 6.13, then we see that we have a symmetry between the operators. If we would change the trace operators, i.e.*

$$\widetilde{\boldsymbol{\gamma}}_D := \boldsymbol{\gamma}_\times, \qquad \widetilde{\boldsymbol{\gamma}}_N = \frac{1}{k}\boldsymbol{\gamma}_\times \circ \mathbf{curl},$$

and modify the potential operators correspondingly,

$$\widetilde{\boldsymbol{\Psi}}_k^S(\boldsymbol{\mu}) := k\boldsymbol{\Psi}_k^A(\boldsymbol{\mu}) - \frac{1}{k}\,\mathbf{grad}\,\Psi_k^S \operatorname{div}_\Gamma(\boldsymbol{\mu}),$$
$$\widetilde{\boldsymbol{\Psi}}_k^M(\boldsymbol{\mu}) := \mathbf{curl}\,\boldsymbol{\Psi}_k^A(\boldsymbol{\mu}),$$

then Lemma 6.13 would look like:

$$\widetilde{\boldsymbol{\gamma}}_N\boldsymbol{\Psi}_k^S(\boldsymbol{\mu}) = \widetilde{\boldsymbol{\gamma}}_D\boldsymbol{\Psi}_k^M(\boldsymbol{\mu}), \qquad \widetilde{\boldsymbol{\gamma}}_N\boldsymbol{\Psi}_k^M(\boldsymbol{\mu}) = \widetilde{\boldsymbol{\gamma}}_D\boldsymbol{\Psi}_k^S(\boldsymbol{\mu}).$$

In addition, we would have to change the scalar product to

$$\langle \mathbf{u}, \mathbf{v}\rangle_{\widetilde{\Gamma}} := \int_\Gamma (\mathbf{u}\times\mathbf{n})\cdot\mathbf{v}\mathrm{d}s_x.$$

If we would use this notation, which is quite familiar in literature for electromagnetic scattering, we would only have to deal with two boundary integral operators instead of four. Furthermore, only the space $\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ *would be required. On the other hand, one of the trace operators would depend on the wave number* k *and the corresponding inner product would be non–symmetric. Nevertheless, the differences to the Helmholtz case is interesting, since both single layer potential and hypersingular operator are of the same order.*

Based on the surface potentials we can now define boundary integral operators. This happens by applying either the Dirichlet or the Neumann trace on the Maxwell single or double layer potential.

Theorem 6.15 (Jump conditions)**.** *For the transition from the interior the following representations for* $x\in\Gamma$ *hold almost everywhere:*

$$\boldsymbol{\gamma}_D\boldsymbol{\Psi}_k^A\mathbf{w}(x) = \int_\Gamma \boldsymbol{\gamma}_{D,x}(g_k(x,y)\mathbf{w}(y))\mathrm{d}s_y =: \mathsf{A}_k\mathbf{w}(x),$$
$$\boldsymbol{\gamma}_D\boldsymbol{\Psi}_k^S\mathbf{w}(x) = \int_\Gamma \boldsymbol{\gamma}_{D,x}(g_k(x,y)\mathbf{w}(y))\mathrm{d}s_y + \frac{1}{k^2}\,\boldsymbol{\nabla}_{|\Gamma}\,\Psi_k^S(\operatorname{div}_\Gamma\mathbf{w}(x)) =: \mathsf{S}_k\mathbf{w}(x),$$
$$\boldsymbol{\gamma}_D\boldsymbol{\Psi}_k^M\mathbf{v}(x) = \boldsymbol{\gamma}_{D,x}\,\mathbf{curl}_x\int_\Gamma g_k(x,y)(\mathbf{v}(y)\times\mathbf{n}_y)\mathrm{d}s_y =: \left(\frac{1}{2}\mathsf{I}+\mathsf{C}_k\right)\mathbf{v}(x),$$
$$\boldsymbol{\gamma}_N\boldsymbol{\Psi}_k^S\mathbf{w}(x) = \frac{1}{2}\mathbf{w}(x) + \int_\Gamma \boldsymbol{\gamma}_{N,x}(g_k(x,y)\mathbf{w}(y))\mathrm{d}s_y =: \left(\frac{1}{2}\mathsf{I}+\mathsf{B}_k\right)\mathbf{w}(x),$$
$$\boldsymbol{\gamma}_N\boldsymbol{\Psi}_k^M\mathbf{v}(x) = \boldsymbol{\gamma}_{N,x}\,\mathbf{curl}_x\int_\Gamma (g_k(x,y)\mathbf{v}(y)\times\mathbf{n}_y)\mathrm{d}s_y =: \mathsf{N}_k\mathbf{v}(x).$$

For the transition from the exterior it holds respectively, for $x \in \Gamma$

$$\begin{aligned}
\boldsymbol{\gamma}_D^c \boldsymbol{\Psi}_k^A \mathbf{w}(x) &=: \mathsf{A}_k \mathbf{w}(x), \\
\boldsymbol{\gamma}_D^c \boldsymbol{\Psi}_k^S \mathbf{w}(x) &=: \mathsf{S}_k \mathbf{w}(x), \\
\boldsymbol{\gamma}_D^c \boldsymbol{\Psi}_k^M \mathbf{v}(x) &=: \left(-\frac{1}{2}\mathsf{I} + \mathsf{C}_k \right) \mathbf{v}(x), \\
\boldsymbol{\gamma}_N^c \boldsymbol{\Psi}_k^S \mathbf{w}(x) &=: \left(-\frac{1}{2}\mathsf{I} + \mathsf{B}_k \right) \mathbf{w}(x), \\
\boldsymbol{\gamma}_N^c \boldsymbol{\Psi}_k^M \mathbf{v}(x) &=: \mathsf{N}_k \mathbf{v}(x).
\end{aligned}$$

This gives us the jump conditions

$$\begin{aligned}
[\boldsymbol{\gamma}_D \boldsymbol{\Psi}_k^A]_\Gamma &= \mathbf{0}, & [\boldsymbol{\gamma}_N \boldsymbol{\Psi}_k^A]_\Gamma &= -\mathsf{I}, \\
[\boldsymbol{\gamma}_D \boldsymbol{\Psi}_k^S]_\Gamma &= \mathbf{0}, & [\boldsymbol{\gamma}_N \boldsymbol{\Psi}_k^S]_\Gamma &= -\mathsf{I}, \\
[\boldsymbol{\gamma}_D \boldsymbol{\Psi}_k^M]_\Gamma &= -\mathsf{I}, & [\boldsymbol{\gamma}_N \boldsymbol{\Psi}_k^M]_\Gamma &= \mathbf{0}.
\end{aligned}$$

Proof. See [33, 38]. □

Theorem 6.16. *The boundary integral operators have the following mapping properties:*

$$\begin{aligned}
\mathsf{S}_k &: \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\
\mathsf{A}_k &: \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\
\mathsf{B}_k &: \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \\
\mathsf{C}_k &: \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\
\mathsf{N}_k &: \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma).
\end{aligned}$$

For A_k *we have the additional mapping property*

$$\mathsf{A}_k : \mathbf{H}_\parallel^{-1/2}(\Gamma) \to \mathbf{H}_\parallel^{1/2}(\Gamma).$$

Proof. For the first proposition we combine the mapping properties of the potential operators and the trace operators. The proposition for A_k can be found in [88]. □

As hinted in Remark 6.14, the single layer potential and the hypersingular operator bear a similar representation. The proof of the following theorem can be found in [26].

Theorem 6.17. *For the sesquilinear form of the hypersingular operator* N_k *there holds the representation*

$$\langle \mathsf{N}_k \mathbf{u}, \mathbf{v} \rangle_\Gamma = -k^2 \langle \mathsf{A}_k(\mathbf{R}\mathbf{u}), \mathbf{R}\mathbf{v} \rangle_\Gamma + \langle V_k(\mathrm{curl}_\Gamma \mathbf{u}), \mathrm{curl}_\Gamma \mathbf{v} \rangle_\Gamma$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$.

If we apply the trace operators on the representation formula for the solution of the interior boundary value problem (6.7), we get the boundary integral equations

$$\begin{array}{rcl} \boldsymbol{\gamma}_D\mathbf{E} = & \mathsf{S}_k(\boldsymbol{\gamma}_N\mathbf{E}) & + \quad (\frac{1}{2}\mathsf{I} + \mathsf{C}_k)(\boldsymbol{\gamma}_D\mathbf{E}), \\ \boldsymbol{\gamma}_N\mathbf{E} = & (\frac{1}{2}\mathsf{I} + \mathsf{B}_k)(\boldsymbol{\gamma}_N\mathbf{E}) & + \quad \mathsf{N}_k(\boldsymbol{\gamma}_D\mathbf{E}). \end{array} \tag{6.9}$$

If we do the same for the representation formula for the solution of the exterior boundary value problem (6.8), we get

$$\begin{array}{rcl} \boldsymbol{\gamma}_D^c\mathbf{E} = & -\mathsf{S}_k(\boldsymbol{\gamma}_N^c\mathbf{E}) & + \quad (\frac{1}{2}\mathsf{I} - \mathsf{C}_k)(\boldsymbol{\gamma}_D^c\mathbf{E}), \\ \boldsymbol{\gamma}_N^c\mathbf{E} = & (\frac{1}{2}\mathsf{I} - \mathsf{B}_k)(\boldsymbol{\gamma}_N^c\mathbf{E}) & + \quad -\mathsf{N}_k(\boldsymbol{\gamma}_D^c\mathbf{E}). \end{array} \tag{6.10}$$

Now we can prove the so called Calderon property.

Theorem 6.18. *The interior and the exterior Calderon projector, i.e.*

$$\mathcal{C}_{int} = \begin{pmatrix} \frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k \\ \mathsf{N}_k & \frac{1}{2}\mathsf{I} + \mathsf{B}_k \end{pmatrix}, \qquad \mathcal{C}_{ext} = \begin{pmatrix} \frac{1}{2}\mathsf{I} - \mathsf{C}_k & -\mathsf{S}_k \\ -\mathsf{N}_k & \frac{1}{2}\mathsf{I} - \mathsf{B}_k \end{pmatrix}$$

are indeed projectors, i.e., $\mathcal{C}_{int}^2 = \mathcal{C}_{int}$ *and* $\mathcal{C}_{ext}^2 = \mathcal{C}_{ext}$.

Proof. As shown in Theorem 6.11 the Maxwell single and double layer potential are solutions of the differential equation. Hence, the proof can be done as in the scalar case, see [125]. □

This leads to some important relations among boundary integral operators.

Corollary 6.19. *For the boundary integral operators the relations*

$$\mathsf{S}_k\mathsf{N}_k = \frac{1}{4}\mathsf{I} - \mathsf{C}_k^2, \tag{6.11}$$

$$\mathsf{N}_k\mathsf{S}_k = \frac{1}{4}\mathsf{I} - \mathsf{B}_k^2, \tag{6.12}$$

$$-\mathsf{N}_k\mathsf{C}_k = \mathsf{B}_k\mathsf{N}_k, \tag{6.13}$$

$$-\mathsf{C}_k\mathsf{S}_k = \mathsf{S}_k\mathsf{B}_k. \tag{6.14}$$

hold.

Definition 6.20. *If the operator* S_k *is invertible, i.e. if* k^2 *is not a Neumann eigen wave number, then we can define the Steklov–Poincaré operator by*

$$\mathsf{T}_k := \mathsf{S}_k^{-1}(\frac{1}{2}\mathsf{I} - \mathsf{C}_k) \quad : \quad \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma). \tag{6.15}$$

A symmetric representation is given by

$$\mathsf{T}_k := \mathsf{N}_k + (\frac{1}{2}\mathsf{I} + \mathsf{B}_k)\mathsf{S}_k^{-1}(\frac{1}{2}\mathsf{I} - \mathsf{C}_k). \tag{6.16}$$

The Steklov–Poincaré operator of electromagnetic scattering has similar properties as the one in the Helmholtz case, see Section 5.4, i.e. T_k is not well defined if k^2 is a Dirichlet eigen wave number and is not invertible if k^2 is a Neumann eigen wave number.

6.3 Properties of the boundary integral operators

The boundary integral operators corresponding to the electromagnetic scattering problem bear similar properties as those for the acoustic scattering problem.

Lemma 6.21. *For* $\boldsymbol{\mu} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *there holds*

$$\langle \mathsf{B}_k \boldsymbol{\mu}, \boldsymbol{\lambda} \rangle_\Gamma = -\langle \boldsymbol{\mu}, \mathsf{C}_{-k} \boldsymbol{\lambda} \rangle_\Gamma$$

for all $k \in \mathbb{R}$.

Proof. See [38]. □

As in the Helmholtz case, the boundary integral operators for different wave numbers differ only by a compact part.

Theorem 6.22. *The operators*

$$\begin{aligned}
\mathsf{A}_k - \mathsf{A}_{k'} &: \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\
\mathsf{C}_k - \mathsf{C}_{k'} &: \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\
\mathsf{B}_k - \mathsf{B}_{k'} &: \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)
\end{aligned}$$

are compact for all $k, k' \in \mathbb{C}$.

Proof. See [88]. □

Before we can prove generalized Gårding inequalities for arbitrary wave numbers k, we first have to establish some ellipticity results.

Theorem 6.23. *For* $k = i\kappa$ *and* $\kappa \in \mathbb{R}^+$ *there holds*

$$\langle \mathsf{A}_k \mathbf{u}, \mathbf{u} \rangle_\Gamma \geq c \, \|\mathbf{u}\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\Gamma)}$$

for all $\mathbf{u} \in \mathbf{H}_{\parallel}^{-1/2}(\Gamma)$.

Proof. For A_0 see [33], for A_k, the proof works analogously . □

Theorem 6.24. *If* $k \in \mathbb{C}$ *and* $\mathrm{Im}(k) > 0$*, then the single layer potential* S_k *is* $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$*–elliptic. If* $k = i\kappa$*,* $\kappa \in \mathbb{R}^+$ *then it is in addition self–adjoint. The same holds true for the hypersingular operator* N_k *with respect to* $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$.

Proof. We only consider the case $k = i\kappa$, $\kappa \in \mathbb{R}^+$. The proof for the general case $\mathrm{Im}(k) > 0$ can be found in [26]. For the single layer potential we have

$$\begin{aligned}\langle \mathsf{S}_k \mathbf{u}, \mathbf{u}\rangle_\Gamma &= \langle \mathsf{A}_k \mathbf{u}, \mathbf{u}\rangle_\Gamma + \frac{1}{(ik)^2}\langle V_k \operatorname{div}_\Gamma \mathbf{u}, \operatorname{div}_\Gamma \mathbf{u}\rangle_\Gamma \\ &\geq c_1 \|\mathbf{u}\|^2_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} + \frac{c_2}{(ik)^2}\|\operatorname{div}_\Gamma \mathbf{u}\|^2_{H^{-1/2}(\Gamma)} \geq c\,\|\mathbf{u}\|^2_{\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)},\end{aligned}$$

while for the hypersingular operator we conclude

$$\begin{aligned}\langle \mathsf{N}_k \mathbf{u}, \mathbf{u}\rangle_\Gamma &= (ik)^2\langle \mathsf{A}_k(\mathbf{u}\times\mathbf{n}), \mathbf{u}\times\mathbf{n}\rangle_\Gamma + \langle V_k \operatorname{curl}_\Gamma \mathbf{u}, \operatorname{curl}_\Gamma \mathbf{u}\rangle_\Gamma \\ &\geq (ik)^2 c_1 \|\mathbf{u}\times\mathbf{n}\|^2_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} + c_2\|\operatorname{curl}_\Gamma \mathbf{u}\|^2_{H^{-1/2}(\Gamma)} \geq c\,\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)}\end{aligned}$$

due to $\|\mathbf{u}\times\mathbf{n}\|^2_{\mathbf{H}_\parallel^{-1/2}(\Gamma)} = \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\Gamma)}$. Since S_k and N_k are real–valued and symmetric when assuming $k = i\kappa$ and $\kappa \in \mathbb{R}^+$, they are also self–adjoint. □

Let us define the two auxiliary operators

$$\begin{aligned}\widetilde{\mathsf{S}}_k \mathbf{u} &:= \mathsf{A}_0 \mathbf{u} - \frac{1}{k^2}\,\boldsymbol{\nabla}_\Gamma V_0 \operatorname{div}_\Gamma \mathbf{u},\\ \hat{\mathsf{S}}_k \mathbf{u} &:= \mathsf{A}_0 \mathbf{u} + \frac{1}{k^2}\,\boldsymbol{\nabla}_\Gamma V_0 \operatorname{div}_\Gamma \mathbf{u}\end{aligned}$$

for all $\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$. These two operators are rather similar to the Maxwell single layer potential. An obvious observation is that the operator

$$\hat{\mathsf{S}}_k - \mathsf{S}_k : \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$$

is compact, which will be used in the upcoming proof of Theorem 6.25. The operator $\widetilde{\mathsf{S}}_k$ is $\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$–elliptic for $k > 0$, this can be shown as in Theorem 6.24.

In contrast to the Helmholtz case, for an arbitrary $k \in \mathbb{C}$ it is not possible to prove a Gårding inequality for the single layer potential or for the hypersingular operator. However, we can prove the following generalized Gårding inequalities.

Theorem 6.25. *S_k and N_k fulfill for $k > 0$ the generalized Gårding inequality*

$$\begin{aligned}\mathrm{Re}(\langle \mathsf{S}_k \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma + C^1(\boldsymbol{\mu}, \boldsymbol{\mu})) &\geq c\,\|\boldsymbol{\mu}\|^2_{\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)},\\ \mathrm{Re}(\langle \mathsf{N}_k \boldsymbol{\lambda}, \mathcal{Y}\boldsymbol{\lambda}\rangle_\Gamma + C^2(\boldsymbol{\lambda}, \boldsymbol{\lambda})) &\geq c\,\|\boldsymbol{\lambda}\|^2_{\mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)}\end{aligned}$$

for all $\boldsymbol{\mu} \in \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$, where C^1 and C^2 are compact sesquilinear forms, and $\mathcal{X}$ and $\mathcal{Y}$ are as given in Definition 4.40.

Proof. By using the definition of $\hat{\mathsf{S}}_k$ and the splitting properties of $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, as discussed in Subsection 4.3.2, we get

$$\begin{aligned}
\langle \hat{\mathsf{S}}_k \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma &= \langle \mathsf{A}_0 \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma + \frac{1}{k^2}\langle \boldsymbol{\nabla}_\Gamma V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma \\
&= \langle \mathsf{A}_0 \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma - \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}, \mathrm{div}_\Gamma\, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma \\
&= \langle \mathsf{A}_0 \boldsymbol{\mu}^0, \boldsymbol{\mu}^0\rangle_\Gamma + \langle \mathsf{A}_0 \boldsymbol{\mu}^\perp, \boldsymbol{\mu}^0\rangle_\Gamma - \langle \mathsf{A}_0 \boldsymbol{\mu}^0, \boldsymbol{\mu}^\perp\rangle_\Gamma - \langle \mathsf{A}_0 \boldsymbol{\mu}^\perp, \boldsymbol{\mu}^\perp\rangle_\Gamma \\
&\quad - \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}^0, \mathrm{div}_\Gamma\, \boldsymbol{\mu}^0\rangle_\Gamma - \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp, \mathrm{div}_\Gamma\, \boldsymbol{\mu}^0\rangle_\Gamma \\
&\quad + \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}^0, \mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp\rangle_\Gamma + \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp, \mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp\rangle_\Gamma.
\end{aligned}$$

From $\mathrm{div}_\Gamma\, \boldsymbol{\mu}^0 = 0$ it follows that

$$\begin{aligned}
\langle \hat{\mathsf{S}}_k \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma &= \langle \mathsf{A}_0 \boldsymbol{\mu}^0, \boldsymbol{\mu}^0\rangle_\Gamma + \langle \mathsf{A}_0 \boldsymbol{\mu}^\perp, \boldsymbol{\mu}^0\rangle_\Gamma - \langle \mathsf{A}_0 \boldsymbol{\mu}^0, \boldsymbol{\mu}^\perp\rangle_\Gamma - \langle \mathsf{A}_0 \boldsymbol{\mu}^\perp, \boldsymbol{\mu}^\perp\rangle_\Gamma \\
&\quad + \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp, \mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp\rangle_\Gamma.
\end{aligned}$$

Moreover, since $\langle \mathsf{A}_0 \boldsymbol{\mu}^\perp, \boldsymbol{\mu}^0\rangle_\Gamma$, $\langle \mathsf{A}_0 \boldsymbol{\mu}^0, \boldsymbol{\mu}^\perp\rangle_\Gamma$ and $\langle \mathsf{A}_0 \boldsymbol{\mu}^\perp, \boldsymbol{\mu}^\perp\rangle_\Gamma$ are compact sesquilinear forms (see [88]), it follows that there exists a compact sesquilinear form $C(\mu, \mu)$ such that

$$\begin{aligned}
\mathrm{Re}(\langle \mathsf{S}_k \boldsymbol{\mu}, \mathcal{X}\boldsymbol{\mu}\rangle_\Gamma + C(\boldsymbol{\mu}, \boldsymbol{\mu})) &= \langle \mathsf{A}_0 \boldsymbol{\mu}^0, \boldsymbol{\mu}^0\rangle_\Gamma + \frac{1}{k^2}\langle V_0 \,\mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp, \mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp\rangle_\Gamma \\
&\geq c_1 \left\|\mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp\right\|^2_{H^{-1/2}(\Gamma)} + c_2 \left\|\boldsymbol{\mu}^0\right\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\Gamma)} \\
&\geq c \left\|\boldsymbol{\mu}\right\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)}
\end{aligned}$$

due to $\left\|\boldsymbol{\mu}^\perp\right\|_{\mathbf{H}_{\parallel}^{-1/2}(\Gamma)} \leq C \left\|\mathrm{div}_\Gamma\, \boldsymbol{\mu}^\perp\right\|_{H^{-1/2}(\Gamma)}$, see [88]. The proof for the hypersingular operator N_k works analogously. □

To establish a generalized Gårding inequality for the upcoming local solution approach we need the following theorem.

Theorem 6.26. *The sesquilinear forms*

$$\begin{aligned}
&\langle (\frac{1}{2}\mathsf{I} \pm \mathsf{B}_k)\mathbf{u}^\perp, \mathbf{v}^\perp\rangle_\Gamma, \qquad \langle (\frac{1}{2}\mathsf{I} \pm \mathsf{B}_k)\mathbf{u}^0, \mathbf{v}^0\rangle_\Gamma, \\
&\langle (\frac{1}{2}\mathsf{I} \pm \mathsf{C}_k)\mathbf{v}^\perp, \mathbf{u}^\perp\rangle_\Gamma, \qquad \langle (\frac{1}{2}\mathsf{I} \pm \mathsf{C}_k)\mathbf{v}^0, \mathbf{u}^0\rangle_\Gamma
\end{aligned}$$

are compact.

Proof. See [38, 84]. □

The following lemma can be found in [139].

Lemma 6.27. *For $k > 0$ there holds*

$$\begin{aligned} \mathrm{Im}(\langle \mathsf{S}_k \mathbf{u}, \mathbf{u} \rangle_\Gamma) \geq 0, \qquad & \mathrm{Im}(\langle \mathsf{S}_{-k} \mathbf{u}, \mathbf{u} \rangle_\Gamma) \leq 0, \\ \mathrm{Im}(\langle \mathsf{N}_k \mathbf{v}, \mathbf{v} \rangle_\Gamma) \leq 0, \qquad & \mathrm{Im}(\langle \mathsf{N}_{-k} \mathbf{v}, \mathbf{v} \rangle_\Gamma) \geq 0. \end{aligned}$$

To prove surjectivity of systems of boundary integral equations related to boundary value problems with Robin boundary conditions, the following results about the images of the boundary integral operators are needed.

Lemma 6.28. *For all $\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ we have the image properties*

$$(\frac{1}{2}\mathsf{I} + \mathsf{C}_k)\mathbf{u} \in \mathrm{Imag}(\mathsf{S}_k), \qquad (\frac{1}{2}\mathsf{I} + \mathsf{B}_k)\mathbf{v} \in \mathrm{Imag}(\mathsf{N}_k).$$

Proof. This follows directly from the properties of boundary integral equations related do PDE's in unbounded domains, see (6.10). □

6.4 Robin interface operator R

In contrast to the Helmholtz case we discuss only one possible operator R. This realization is based on the vectorial single layer potential A_0, which is elliptic on $\widetilde{\mathbf{H}}^{-1/2}(\Gamma_R)$. Since the restriction of $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ to Γ_R is only in $\mathbf{H}^{-1/2}(\Gamma_R)$, we have to modify the operator to $\phi'_{\Gamma_R} \circ \mathsf{A}_0 \circ \phi_{\Gamma_R}$ where ϕ_{Γ_R} is a continuous function with

$$\begin{aligned} \phi_{\Gamma_R}(x) &= \min(\mathrm{dist}(x, \Gamma \setminus \Gamma_R), 1) \qquad && \text{for } x \in \Gamma_R, \\ \phi_{\Gamma_R}(x) &= 0 && \text{for } x \in \Gamma \setminus \Gamma_R. \end{aligned}$$

This is a strictly positive and real valued operator which fulfills the mapping properties. Further is the sesquilinear form $\langle \mathsf{R}\mathbf{u}, \mathcal{Y}\mathbf{u} \rangle_\Gamma$ real valued up to a compact perturbation, because the sesquilinear forms $\langle \mathsf{R}\mathbf{u}^0, \mathcal{Y}\mathbf{u}^\perp \rangle_\Gamma$, $\langle \mathsf{R}\mathbf{u}^\perp, \mathcal{Y}\mathbf{u}^0 \rangle_\Gamma$, $\langle \mathsf{R}\mathbf{u}^\perp, \mathcal{Y}\mathbf{u}^\perp \rangle_\Gamma$ are compact, see Section 6.3 and $\langle \mathsf{R}\mathbf{u}^0, \mathcal{Y}\mathbf{u}^0 \rangle_\Gamma$ is real–valued since A_0 is a real–valued and self–adjoint operator.

6.5 Local Neumann/Robin boundary value problems

As in the Helmholtz case, there are various different approaches available to solve the local Neumann or Robin boundary value problems. Since they have, in principle, the same major properties in common, we will skip their detailed discussion in this

section because it would just be a repetition of Subsection 5.6. The boundary value problem we want to solve is given by

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} \qquad \text{in } \Omega, \\ \boldsymbol{\gamma}_N\mathbf{U} + i\eta\mathsf{R}\boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{l} \qquad \text{on } \Gamma \end{aligned} \tag{6.17}$$

Note that if $\eta \equiv 0$ we have a standard Neumann boundary value problem. The formulation of choice to solve the local Neumann/Robin boundary value problem is given by:

Find $\mathbf{u} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{t} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that

$$\begin{pmatrix} \mathsf{N}_k + i\eta\mathsf{R} & \frac{1}{2}\mathsf{I} + \mathsf{B}_k \\ -\frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{l} \\ \mathbf{0} \end{pmatrix}. \tag{6.18}$$

The system (6.18) is uniquely solvable for $\mathrm{Im}(k) > 0$ and $\eta \equiv 0$, or if $k \in \mathbb{R} \setminus \{0\}$ and $\eta \not\equiv 0$. Again this can be proven by showing surjectivity and a generalized Gårding inequality. Surjectivity can be proven as in the Helmholtz case, see Theorem 5.26 and Remark 5.27.

Theorem 6.29. *For $k \neq 0$ and $\eta \in \mathbb{R}$ the generalized Gårding inequality*

$$\begin{aligned} \mathrm{Re}\left(\left\langle \begin{pmatrix} \mathsf{N}_k + i\eta\mathsf{R} & \frac{1}{2}\mathsf{I} + \mathsf{B}_k \\ -\frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}, \begin{pmatrix} \mathcal{Y}\mathbf{u} \\ \mathcal{X}\mathbf{t} \end{pmatrix} \right\rangle_\Gamma + C((\mathbf{u},\mathbf{t}),(\mathbf{u},\mathbf{t})) \right) \\ \geq c \left(\|\mathbf{u}\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} + \|\mathbf{t}\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \right) \end{aligned}$$

is satisfied for all $(\mathbf{u},\mathbf{t}) \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) \times \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and a compact sesquilinear form C.

Proof. For the diagonal parts we have

$$\begin{aligned} \langle \mathsf{N}_k\mathbf{u}, \mathcal{Y}\mathbf{u}\rangle_\Gamma + C^1(\mathbf{u},\mathbf{u}) &\geq c_1 \|\mathbf{u}\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)}, \\ \langle \mathsf{S}_k\mathbf{t}, \mathcal{X}\mathbf{t}\rangle_\Gamma + C^2(\mathbf{t},\mathbf{t}) &\geq c_2 \|\mathbf{t}\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}, \\ \mathrm{Im}(\langle \mathsf{R}\mathbf{u}, \mathcal{X}\mathbf{u}\rangle_\Gamma + C^3(\mathbf{u},\mathbf{u})) &= 0 \end{aligned}$$

with compact sesquilinear forms C^1, C^2 and C^3, see Theorem. By using Theorem 6.26 we obtain

$$\begin{aligned} &\mathrm{Re}\Big(\langle (-\frac{1}{2}\mathsf{I} + \mathsf{C}_k)\mathbf{u}, \mathcal{X}\mathbf{t}\rangle_\Gamma + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_k)\mathbf{t}, \mathcal{Y}\mathbf{u}\rangle_\Gamma \Big) \\ &\quad = \mathrm{Re}\Big(-\langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_0)\mathbf{t}^\perp, \mathbf{u}^0\rangle_\Gamma + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_0)\mathbf{t}^0, \mathbf{u}^\perp\rangle_\Gamma + \langle (-\frac{1}{2}\mathsf{I} + \mathsf{C}_0)\mathbf{u}^\perp, \mathbf{t}^0\rangle_\Gamma \\ &\qquad - \langle (-\frac{1}{2}\mathsf{I} + \mathsf{C}_0)\mathbf{u}^0, \mathbf{t}^\perp\rangle_\Gamma + C^4((\mathbf{u},\mathbf{t}),(\mathbf{u},\mathbf{t})) \Big) = C^5((\mathbf{u},\mathbf{t}),(\mathbf{u},\mathbf{t})) \end{aligned}$$

for the off diagonal parts with compact sesquilinear forms C^4 and C^5. These two statements combined prove the Gårding inequality. □

6.6 The exterior Neumann boundary value problem

For $k \in \mathbb{R}^+$ we consider the exterior Neumann boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} \qquad \text{in } \Omega^c, \\ \boldsymbol{\gamma}_N^c\mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma_N, \\ \lim_{r\to 0}\int_{\partial B_r} |\boldsymbol{\gamma}_N\mathbf{U}(x) - ik\boldsymbol{\gamma}_D\mathbf{U}(x)|^2 \mathrm{d}s_x &= 0. \end{aligned} \tag{6.19}$$

Although the exterior Neumann boundary value problem (6.19) has a unique solution, this is not true for standard boundary integral approaches. The reasons are the same as in the acoustic scattering case, see the discussion in Section 5.7. Therefore, modified approaches were developed. A first approach was given by Panich in [114], which is based on the complex linear combination

$$\mathbf{U}(x) = \boldsymbol{\Psi}_k^S\mathbf{u}(x) + ik\boldsymbol{\Psi}_k^D\mathbf{u}(x)$$

of the single and double layer potential. However, this approach only works for smooth domains and in a L_2–setting. Other approaches, so–called combined field integral equations (CFIE) for Lipschitz domains were developed, e.g. [36,127]. These approaches use an additional operator B, i.e.

$$\mathbf{U}(x) = \boldsymbol{\Psi}_k^S\mathbf{u}(x) + ik\boldsymbol{\Psi}_k^D B\mathbf{u}(x)$$

such that the analysis can be done in the natural trace spaces. In this work we want to give another regularization approach, which is based on the boundary integral equations (6.10) related to the exterior problem. This equations can be rewritten by

$$\begin{pmatrix} \mathsf{N}_k & -\frac{1}{2}\mathsf{I} + \mathsf{B}_k \\ \frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} -\mathbf{p} \\ \mathbf{0} \end{pmatrix} \tag{6.20}$$

with $\mathbf{u} = \boldsymbol{\gamma}_D^c\mathbf{U}$ and $\mathbf{t} = \boldsymbol{\gamma}_N^c\mathbf{U}$. Equation (6.20) is not uniquely solvable if k^2 is an eigen wave number of the interior Dirichlet problem, since the Neumann trace of the Dirichlet eigensolution U_{k^2} is in the kernel of S_k and $\frac{1}{2}\mathsf{I} - \mathsf{B}_k$. To regularize equation (6.20), we add on both sides, as in the Helmholtz case, a term including the known Neumann boundary datum $\mathbf{p} = \mathbf{t}$. This leads to the regularized formulation

$$\begin{pmatrix} \mathsf{N}_k & -\frac{1}{2}\mathsf{I} + \mathsf{B}_k \\ \frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k + i\widetilde{\mathsf{S}}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} -\mathbf{p} \\ i\widetilde{\mathsf{S}}_k\mathbf{p} \end{pmatrix} \tag{6.21}$$

which is uniquely solvable. To utilize Theorem 3.16 we have to show the surjectivity of the operator as given in (6.21) and the following generalized Gårding inequality.

Theorem 6.30. *The operator in* (6.21) *satisfies a generalized Gårding inequality, i.e. there exists a compact sesquilinear form* $C((\cdot,\cdot),(\cdot,\cdot))$ *such that*

$$\begin{aligned} &\mathrm{Re}\left(\left\langle \begin{pmatrix} \mathsf{N}_k & -\frac{1}{2}\mathsf{I} + \mathsf{B}_k \\ \frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k + i\widetilde{\mathsf{S}}_k \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}, \begin{pmatrix} \mathcal{Y}\mathbf{u} \\ \mathcal{X}\mathbf{t} \end{pmatrix} \right\rangle_\Gamma + C((\mathbf{u},\mathbf{t}),(\mathbf{u},\mathbf{t}))\right) \\ &\qquad \geq c\left(\|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} + \|\mathbf{t}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\right) \end{aligned}$$

for all $(\mathbf{u}, \mathbf{t}) \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Proof. Since the imaginary part of $\langle \widetilde{\mathsf{S}}_k \mathbf{t}, \mathcal{Y}\mathbf{t} \rangle_\Gamma$ is compact, the proof works analogically as the proof of Theorem 6.29. □

The surjectivity of the system in (6.21) can be proven by interpreting it as the following interior Robin boundary value problem:

Find $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} \qquad \text{in } \Omega, \\ \boldsymbol{\gamma}_D\mathbf{U} + i\widetilde{\mathsf{S}}_k\boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma. \end{aligned}$$

The proof works as in the Helmholtz case, see Lemma 5.29. Since

$$\mathrm{Im}(\langle \mathsf{S}_k + i\widetilde{\mathsf{S}}_k\mathbf{v}, \mathbf{v}\rangle_\Gamma) \geq c\,\|\mathbf{v}\|^2_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \qquad \text{for all } \mathbf{v} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma),$$

see Lemma 6.27, it is possible to eliminate $\mathbf{t}$ in (6.21). This leads to the combined field integral equation

$$(\mathsf{N}_k + (\frac{1}{2}\mathsf{I} - \mathsf{B}_k)(\mathsf{S}_k + i\widetilde{\mathsf{S}}_k)^{-1}(\frac{1}{2}\mathsf{I} + \mathsf{C}_k))\mathbf{u} = -\mathbf{p} - i\widetilde{\mathsf{S}}_k\mathbf{p}.$$

Note that this equation has a unique solution for all $k > 0$ since the same is true for equation (6.21).

However, this formulation is not suitable for the domain decomposition approach presented in this thesis. A possibility to construct a suitable formulation is to copy the approach from the Helmholtz case. To do so, it would be necessary to find an operator X such that the system

$$\begin{pmatrix} \mathsf{N}_k & -\frac{1}{2}\mathsf{I} + \mathsf{B}_k & -\mathsf{N}_k \\ \frac{1}{2}\mathsf{I} + \mathsf{C}_k & \mathsf{S}_k & \frac{1}{2}\mathsf{I} - \mathsf{C}_k \\ -\mathsf{N}_k & -\frac{1}{2}\mathsf{I} - \mathsf{B}_k & \mathsf{N}_k - i\mathsf{X} \end{pmatrix}$$

is injective/surjective and fulfills a generalized Gårding inequality. However, it is still an open problem to find such an operator X.

6.7 Boundary element methods

In this section we give a short introduction to suitable discrete ansatz spaces for the electromagnetic wave equation. This deduction is mainly based on [26, 105]. Afterwards, we state the discretized formulation of the Neumann and Robin boundary value problems. Finally we give a numerical example for the Neumann problem.

For the discretization of the boundary integral equations (6.10), we use a conforming Galerkin method with trial spaces

$$\mathbf{V}_h \subset \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \qquad \mathbf{W}_h \subset \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

As in the Helmholtz case we assume a regular triangulation of the boundary Γ. The Cauchy datum $\boldsymbol{\gamma}_D\mathbf{E}$ is the trace of the electric field. A suitable mathematical description of an electric field is given by 1–forms, cf. [20]. Hence, it seems to be natural to approximate an electric field by discrete 1–forms and the Cauchy data by traces of discrete 1–forms. 1–forms are given by $\mathbf{H}(\mathbf{curl}, \Omega)$ conforming elements, see [108]. Let us consider a regular finite element triangulation Ω_h which fits to the boundary triangulation Γ_h. The discrete 1–forms are then given by

$$\begin{aligned}\mathcal{N}_h^d(\omega_l) &:= \left\{\mathbf{U} \in \left[\mathcal{P}^d(\omega_l)\right]^3 : \mathbf{U}(x) \cdot x = 0 \text{ for } x \in \omega_l\right\},\\ \mathcal{N}_h^d(\Omega_h) &:= \left\{\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{U}(x)|_{\omega_l} \in \mathcal{N}_h^d \text{ for all } \omega_l \in \Omega_h\right\}.\end{aligned}$$

$\mathcal{P}^d$ is hereby the space of multivariate polynomials of degree d. By applying the traces $\boldsymbol{\gamma}_\times$ and $\boldsymbol{\gamma}_D$ we construct conforming trace spaces, i.e.

$$\begin{aligned}\mathcal{E}_h^d(\Gamma) &:= \boldsymbol{\gamma}_D(\mathcal{N}_h^d(\Omega_h)) \subset \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma),\\ \mathcal{F}_h^d(\Gamma) &:= \boldsymbol{\gamma}_\times(\mathcal{N}_h^d(\Omega_h)) \subset \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma).\end{aligned}$$

In the simplest case $d = 1$, the space $\mathcal{F}_h^1$ is a two dimensional $\mathbf{H}(\mathrm{div}\,, \Omega)$–conforming trial space on Γ_h, see [27]. The Raviart–Thomas elements [119] are given by

$$\mathcal{F}_h^d(\Gamma_h) = \mathcal{RT}_h^d(\Gamma_h) := \left\{\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) : \mathbf{u}|_{\tau_l} \in \mathcal{RT}_h^d(\tau_l) \text{ for all } \tau_l \in \Gamma_h\right\}$$

with

$$\mathcal{RT}_h^d(\tau_l) := \left[\mathcal{P}^{d-1}(\tau_l)\right]^3 \oplus x\mathcal{P}^{d-1}(\tau_l).$$

Since $\boldsymbol{\gamma}_\times = \boldsymbol{\gamma}_D \times \mathbf{n}$, we have an equivalent definition for the space $\mathcal{E}_h^d(\Gamma_h)$

$$\mathcal{E}_h^d(\Gamma_h) = \mathcal{RT}_h^{d,\times}(\Gamma_h) := \left\{\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) : \mathbf{u}|_{\tau_l} \times \mathbf{n} \in \mathcal{RT}_h^d(\tau_l) \text{ for all } \tau_l \in \Gamma_h\right\}.$$

In this way we can define the boundary element spaces without introducing a discretization of the domain Ω. Since we will only use the lowest order elements, i.e. $d = 1$, we will skip this parameter in the notation. We further abandon the explicit notation of the boundary discretization, i.e. we write $\mathcal{E}_h$ instead of $\mathcal{E}_h^d(\Gamma_h)$.

The trial functions of $\mathcal{E}_h$ and $\mathcal{F}_h$ are edge based, hence each edge belongs to one degree of freedom and vice versa. The support of a basis function is the area of the two adjacent elements. A basis function in $\mathcal{F}_h$ corresponding to an edge e_n is explicitly given by

$$\boldsymbol{\lambda}_{lk}^n := \frac{x_k - x}{2\Delta_l}$$

if we restrict it to one adjacent triangle τ_l. x_k is hereby the corner node of the triangle τ_l which is not part of $\overline{e_n}$. For the space $\mathcal{E}_h$ the basis function is given by, see Figure 6.1,

$$\mathbf{u}_{lk}^n := \mathbf{n} \times \left(\frac{x_k - x}{2\Delta_l} \right).$$

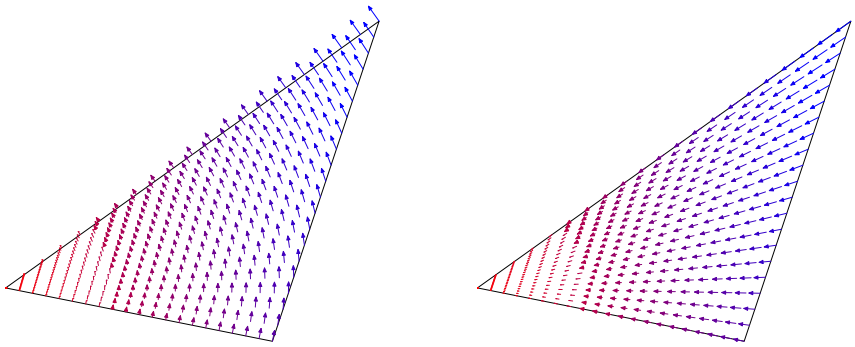


Figure 6.1: Basis functions in $\mathcal{E}_h$ and $\mathcal{F}_h$ restricted to one triangle.

For these conforming ansatz spaces we have the following approximation estimates.

Theorem 6.31. *Let* $\mathbf{t} \in \mathbf{H}_t^s(\Gamma)$, $\operatorname{div}_\Gamma \mathbf{t} \in H^s(\Gamma)$ *for some* $s \in [0, d]$. *Then there exists a* $\mathbf{t}_h \in \mathcal{F}_h^d$ *such that*

$$\begin{aligned} \|\mathbf{t} - \mathbf{t}_h\|_{\mathbf{L}_t^2(\Gamma)} &\leq ch^s(\|\mathbf{t}\|_{\mathbf{H}^s(\Gamma)} + \|\operatorname{div}_\Gamma \mathbf{t}\|_{H^s(\Gamma)}), \\ \|\operatorname{div}_\Gamma(\mathbf{t} - \mathbf{t}_h)\|_{L_2(\Gamma)} &\leq ch^s \|\operatorname{div}_\Gamma \mathbf{t}\|_{H^s(\Gamma)}. \end{aligned}$$

Let $\mathbf{u} \in \mathbf{H}_t^s(\Gamma)$, $\operatorname{curl}_\Gamma \mathbf{u} \in H^s(\Gamma)$ *for some* $s \in [0, d]$. *Then there exists a* $\mathbf{u} \in \mathcal{E}_h^d$ *such that*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}_t^2(\Gamma)} &\leq ch^s(\|\mathbf{u}\|_{\mathbf{H}^s(\Gamma)} + \|\operatorname{curl}_\Gamma \mathbf{u}\|_{H^s(\Gamma)}), \\ \|\operatorname{curl}_\Gamma(\mathbf{u} - \mathbf{u}_h)\|_{L_2(\Gamma)} &\leq ch^s \|\operatorname{curl}_\Gamma \mathbf{u}\|_{H^s(\Gamma)}. \end{aligned}$$

Proof. See [88, 119]. □

Theorem 6.32. *[81] Under the assumptions as given in Theorem 6.31 and for any* $\varepsilon > 0$ *we conclude the error estimates*

$$\begin{aligned} \inf_{\mathbf{t}_h \in \mathcal{F}_h^d} \|\mathbf{t} - \mathbf{t}_h\|_{\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma)} &\leq ch^{\min\left\{\frac{3}{2}-\varepsilon, s+\frac{1}{2}-\varepsilon, 1+s^*, s+s^*\right\}} \left(\|\mathbf{t}\|_{\mathbf{H}^s(\Gamma)} + \|\operatorname{div}_\Gamma \mathbf{t}\|_{H^s(\Gamma)} \right), \\ \inf_{\mathbf{u}_h \in \mathcal{F}_h^d} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)} &\leq ch^{\min\left\{\frac{3}{2}-\varepsilon, s+\frac{1}{2}-\varepsilon, 1+s^*, s+s^*\right\}} \left(\|\mathbf{u}\|_{\mathbf{H}^s(\Gamma)} + \|\operatorname{curl}_\Gamma \mathbf{u}\|_{H^s(\Gamma)} \right), \end{aligned}$$

where s^* *is a domain dependent constant.*

Theorem 6.32 implies that a sequence of Raviart–Thomas element spaces $\mathcal{F}_{h_i}$ is a conforming and approximating sequence for the space $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. The same holds for $\mathcal{E}_{h_i}$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, respectively. But in contrast to the Helmholtz case this is not sufficient to show unique solvability or convergence properties of a discrete formulation. Since we use a generalized Gårding inequality to analyze the continuous formulation, it is necessary that the discrete ansatz spaces satisfy the gap property, see Definition 3.14.

Lemma 6.33. *The approximating sequences $\mathcal{E}_{h_i}$ and $\mathcal{F}_{h_i}$, fulfill the gap property with respect to the splittings of $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ as introduced in Section 4.3.2.*

Proof. See [29, 105]. □

Due to the definition of the gap property, the same holds also for the product spaces $\mathcal{E}_{h_i} \times \mathcal{F}_{h_i}$ with respect to $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

6.7.1 The interior Neumann/Robin scattering problem

The discrete variational formulation of (6.18) is given by:

Find $\mathbf{u}_h \in \mathcal{E}_h$ and $\mathbf{t}_h \in \mathcal{F}_h$ such that

$$\begin{aligned} \langle (\mathsf{N}_k + i\eta \mathsf{R})\mathbf{u}_h, \boldsymbol{\lambda}_h\rangle_\Gamma + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_k)\mathbf{t}_h, \boldsymbol{\lambda}_h\rangle_\Gamma &= \langle \mathbf{l}, \boldsymbol{\lambda}\rangle_\Gamma, \\ \langle (-\frac{1}{2}\mathsf{I} + \mathsf{C}_k)\mathbf{u}_h, \boldsymbol{\mu}_h\rangle_\Gamma + \langle \mathsf{S}_k\mathbf{t}_h, \boldsymbol{\mu}_h\rangle_\Gamma &= 0, \end{aligned} \tag{6.22}$$

for all $\boldsymbol{\lambda}_h \in \mathcal{E}_h$ and $\boldsymbol{\mu}_h \in \mathcal{F}_h$.

Theorem 3.18 ensures that (6.22) has a unique solution if (6.18) has a unique solution and if the mesh width h is fine enough. Further it provides the quasi–optimal error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} + \|\mathbf{t} - \mathbf{t}_h\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \leq c \left(\inf_{\mathbf{x}_h \in \mathcal{E}_h} \|\mathbf{u} - \mathbf{x}_h\| + \inf_{\mathbf{y}_h \in \mathcal{F}_h} \|\mathbf{t} - \mathbf{y}_h\| \right).$$

Numerical example

As a numerical example for an interior scattering problem we consider the Neumann problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} \qquad \text{in } \Omega, \\ \boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma. \end{aligned} \tag{6.23}$$

E_i	It	rel. L_2–error
18	35	0.570153
72	140	0.324963
288	482	0.174921
1152	1595	0.087530
4608	4129	0.058317

(a) $k = 1.0$

E_i	It	rel. L_2–error
18	35	0.657105
72	130	0.356047
288	420	0.184889
1152	1267	0.095502
4608	3132	0.056626

(b) $k = 2.0$

E_i	It	rel. L_2–error
18	35	1.927830
72	130	0.801602
288	402	0.274487
1152	1000	0.131463
4608	1952	0.067883

(c) $k = 4.0$

E_i	It	rel. L_2–error
18	35	1.203550
72	134	1.655500
288	432	0.578021
1152	1390	0.234201
4608	3103	0.109620

(d) $k = 8.0$

Table 6.1: Errors and iteration numbers for the domain $\Omega = (0,1)^3$ with different wave numbers k.

As domain Ω we consider the unit cube $(0,1)^3$ and the unit sphere with the center in the origin. As exact solution we chose

$$\mathbf{U}(x) = \left[\frac{1 + ikr - k^2r^2}{r^3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{3 + 3ikr - k^2r^2}{r^5} (x_1 - \hat{x}_1) \begin{pmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \end{pmatrix} \right] e^{ikr} \quad (6.24)$$

for $x \in \Omega$ and with $r = |\hat{x} - x|$. The source point $\hat{x}$ is given by $(2.0, 1.0, 1.0)^\top$. The boundary element discretization of the variational formulation (6.22) is done with respect to a globally uniform boundary mesh of plane triangular elements with E_i edges and by using first order Raviart–Thomas elements. The corresponding linear system is solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. The results are given in Table 6.1 and Table 6.2. Note that no preconditioner was used to achieve these results, since efficiency was not the aim of these examples. Nevertheless the rather high iteration numbers hint that a local preconditioner has to be used for real–life problems. However, the convergence rate matches with the theoretical predicted convergence rate.

If we chose $k = \sqrt{2} \approx 4.44288$, which is the first eigen wave number of the unit cube, then the Neumann boundary value problem (6.23) has no unique solution. Within the domain decomposition approach we therefore have to consider a Robin boundary value problem. As a numerical example we consider the boundary value problem (6.23) with the same parameters as for the last example. In addition we solve the

E_i	It	rel. L_2–error
183	314	0.157496
732	1009	0.080443
2928	2494	0.042221
11712	6138	0.027018

(a) $k = 1.0$

E_i	It	rel. L_2–error
183	261	0.199227
732	763	0.101210
2928	1958	0.051270
11712	4929	0.027965

(b) $k = 2.0$

E_i	It	rel. L_2–error
183	295	0.852927
732	872	0.301616
2928	1786	0.153936
11712	3534	0.065094

(c) $k = 4.0$

E_i	It	rel. L_2–error
183	299	1.827410
732	910	0.355420
2928	2303	0.164986
11712	4451	0.081153

(d) $k = 8.0$

Table 6.2: Errors and iteration numbers for the unit sphere with different wave numbers k.

E_i	It	rel. L_2–error
18	35	5.555442
72	128	0.929762
288	423	2.075114
1152	1236	7.755683
4608	2852	21.55586

(a) Neumann problem

E_i	It	rel. L_2–error
18	35	4.445549
72	128	0.924866
288	411	0.351006
1152	1061	0.146243
4608	1935	0.071093

(b) Robin problem

Table 6.3: Errors and iteration numbers for the domain $\Omega = (0, 1)^3$ of the Neumann and Robin boundary value problem for the eigen wave number $k = \sqrt{2}\pi \approx 4.44288$.

problem (6.23) with the Robin boundary condition

$$\boldsymbol{\gamma}_N \mathbf{U} + i\mathsf{R}\boldsymbol{\gamma}_D \mathbf{U} = \mathbf{l} \qquad \text{on } \Gamma.$$

As Robin operator R we chose hereby the vectorial single layer potential A_0. The results, which clearly indicate the stability of the boundary element method in the case of Robin boundary conditions, are given in Table 6.3.

6.7.2 Preconditioning strategies

The implementation of a preconditioner for the local problem is nearly unavoidable for practical problems. But standard preconditioners fail since the singular values of

the matrices of the single layer potential and the hypersingular operator accumulate around zero and infinity, see [109]. Suitable preconditioners were given in [2,18], but these preconditioners enforce a complete reimplementation of standard electromagnetic scattering boundary element codes. Another approach is given in [8]. This approach reuses parts of existing boundary element codes but it is quite expensive since the single layer potential has to be computed on a barycentric mesh, which has six times more edges (degrees of freedom) as the original mesh, see [30] and Figure 6.2. Hence there is a strong need in a further investigation of appropriate preconditioning strategies.

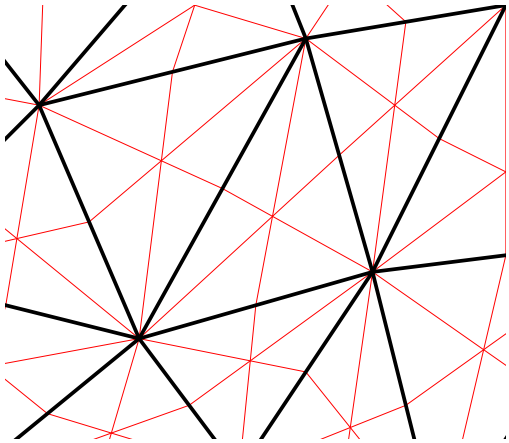


Figure 6.2: Part of an ordinary boundary mesh (black lines) with it's corresponding barycentric mesh (black and red lines).

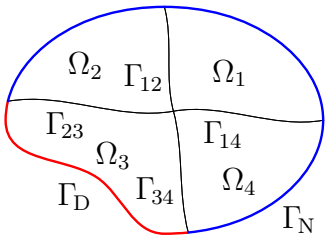
7 DOMAIN DECOMPOSITION METHODS FOR ACOUSTIC SCATTERING

In this chapter we first discuss some geometrical assumptions and definitions for the domain decomposition approach. Afterwards, we formulate a classical Dirichlet domain decomposition method, see for example [101, 117, 124, 136]. In the beginning we assume that all appearing local Dirichlet and Neumann problems are uniquely solvable. Then, step by step, we get rid of these assumptions by introducing artificial Robin interface conditions and by replacing the local Steklov–Poincaré operator by a local system of boundary integral equations. Therefore, we explain how these artificial Robin interface conditions can be chosen and which class of problems is covered by the presented theory. Thereafter, we show that the boundary value problem can be reformulated in such a way that also transmission problems can be treated. Then we introduce suitable trial and test spaces for the variational formulation and give some approximation estimates for the resulting discrete problems. Afterwards, we discuss the idea of the tearing and interconnecting method and introduce an algebraic deduction of this approach. Since one of the motivations for this tearing and interconnecting approach is a feasible efficient parallel implementation, a major concern is to reduce the global iteration numbers, because communication between processors often becomes the most time consuming factor. Therefore we present a preconditioner for the global tearing and interconnecting system which was motivated by Farhat, Macedo and Lesoinne [65]. Finally, we describe some numerical examples.

7.1 Geometric domain decomposition

We assume that the Lipschitz domain Ω is divided into p non overlapping subdomains Ω_i, such that

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega_i}, \qquad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for } i \neq j.$$



By $\Gamma_i := \partial\Omega_i$ we denote the boundary of a subdomain. For neighbouring domains we define the interface $\Gamma_{ij} := \Gamma_i \cap \Gamma_j$. The skeleton of the domain decomposition is defined by

$$\Gamma_S := \bigcup_{i=1}^{p} \Gamma_i = \Gamma \cup \bigcup_{i<j} \overline{\Gamma}_{ij}.$$

The normal vectors $\mathbf{n}_i$ are defined as the outgoing normal vectors for every subdomain. We further assume that the function $k(x)$ is piecewise constant, i.e. $k(x) = k_i$ in Ω_i.

Notation 7.1. *In this chapter we will use a simplified notation for the boundary integral operators. Since we assume a constant wave number k_i with respect to each subdomain, the boundary integral operators will just be indexed with their subdomain number, e.g. the single layer potential with wave number k_i for the domain i will just be donated by V_i.*

7.2 Dirichlet domain decomposition methods

In this section we derive the variational formulation of a domain decomposition approach starting from a global boundary value problem. In principle different boundary conditions are possible, for example Dirichlet, Neumann, Robin or mixed boundary conditions. For simplicity and to reduce technical efforts we will restrict ourselves to Neumann boundary conditions. Parts of the upcoming theory are already published in [129].

As a model problem we consider the Neumann boundary value problem

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U - k^2(x)U &= 0 \qquad \text{in } \Omega, \\ \gamma_1 U &= p \qquad \text{on } \Gamma. \end{aligned} \tag{7.1}$$

We will refer to this problem as the global problem. Let us assume that this problem is uniquely solvable, although we have seen that unique solvability will not hold for all wave numbers k.

The global problem (7.1) can be described by local subproblems which are coupled by interface conditions. These local problems are given by

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U_i - k^2(x)U_i &= 0 \qquad \text{in } \Omega_i, \\ \gamma_{1,i} U_i + \gamma_{1,j} U_j &= 0 \qquad \text{on } \Gamma_{ij}, \\ \gamma_{1,i} U_i &= p \qquad \text{on } \Gamma \cap \Gamma_i. \end{aligned} \tag{7.2}$$

with $U_i = U_{|\Omega_i}$.

A rigorous deduction of the equivalence of (7.2) and (7.1) in the weak sense can be found in [117].

By enforcing the continuity of the Dirichlet traces by an additional equation we don't have to find one global function $U \in H^1(\Omega)$ instead we are now searching for local

functions $U_i \in H^1(\Omega_i)$ for $i = 1, \ldots, p$, which fulfill transmission conditions. This leads to

-0.6cm-

Find $U_i \in H^1(\Omega_i)$ for $i = 1, \ldots, p$ such that

0.6cm

$$\begin{aligned} -\Delta U_i - k^2(x) U_i &= 0 \qquad \text{in } \Omega_i, \\ \gamma_0 U_i - \gamma_0 U_j &= 0 \qquad \text{on } \Gamma_{ij}, \\ \gamma_{1,i} U_i + \gamma_{1,j} U_j &= 0 \qquad \text{on } \Gamma_{ij}, \\ \gamma_{1,i} U_i &= p \qquad \text{on } \Gamma \cap \Gamma_i. \end{aligned}$$

At this point we assume that k_i^2 is not a Dirichlet eigenvalue of the subdomain Ω_i. We do not want to force the differential equation in the domain explicitly, hence we introduce the local Steklov–Poincaré operators

$$S_i = D_i + (\frac{1}{2} I + K_i^{\perp}) V_i^{-1} (\frac{1}{2} I + K_i),$$

see Section 5.4, to get rid of the partial differential equation in the domain. If we substitute the Neumann trace $\gamma_1 U_i$ by $S_i \gamma_0 U_i$ we can implicitly enforce that the partial differential equation is fulfilled in the subdomains. It remains to find local Dirichlet traces $u_i = \gamma_0 U_i \in H^{1/2}(\Gamma_i)$ such that

$$\begin{aligned} u_i - u_j &= 0 \qquad \text{on } \Gamma_{ij}, \\ S_i u_i + S_j u_j &= 0 \qquad \text{on } \Gamma_{ij}, \\ S_i u_i &= p \qquad \text{on } \Gamma \cap \Gamma_i. \end{aligned} \tag{7.3}$$

The coupled transmission problem (7.3) is uniquely solvable if (7.1) is uniquely solvable and if all local Steklov–Poincaré operators S_i are well defined.

By enforcing the continuity of the Dirichlet traces in a strong way we end up with

Find $u = \gamma_0 U \in H^{1/2}(\Gamma_S)$ such that

$$\begin{aligned} S_i u_{|\Gamma_i} + S_j u_{|\Gamma_j} &= 0 \qquad \text{on } \Gamma_{ij}, \\ S_i u_{|\Gamma_i} &= p \qquad \text{on } \Gamma \cap \Gamma_i. \end{aligned} \tag{7.4}$$

The variational formulation of the operator equation (7.4) is

Find $u \in H^{1/2}(\Gamma_S)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} S_i u(x) \overline{v(x)} \, \mathrm{d}s_x = \int_{\Gamma} p(x) \overline{v(x)} \, \mathrm{d}s_x \tag{7.5}$$

for all $v \in H^{1/2}(\Gamma_S)$.

Since all local Steklov–Poincaré operators S_i are well defined, when assuming that k_i^2 is not a Dirichlet eigen wave number related to Ω_i, and satisfy a Gårding inequality, see Theorem 5.22, the sum of all local operators also fulfills such an inequality, i.e. there exists a compact sesquilinear form $C(\cdot,\cdot)$ such that for all $u \in H^{1/2}(\Gamma_S)$

$$\mathrm{Re}\left(\sum_{i=1}^{p}\langle S_i u, u\rangle_\Gamma + C(u,u)\right) \geq c_1 \cdot \sum_{i=1}^{p} \|u\|^2_{H^{1/2}(\Gamma_i)} \geq c_2 \cdot \|u\|^2_{H^{1/2}(\Gamma_S)}$$

holds.

The sesquilinear form as considered in the variational formulation (7.5) is injective when assuming that the global boundary value problem (7.1) has a unique solution. Hence, we can ensure stability and related error estimates for a Galerkin discretization of (7.5) when using a mesh with a sufficient small mesh size h. But since we will use a tearing and interconnecting approach for an iterative solution of the resulting linear system, we need to consider the local Steklov–Poincaré operators for their invertibility which is violated if k_i^2 is a Neumann eigen wave number of Ω_i. In this case we have to modify the interface conditions.

7.3 Robin interface conditions

To achieve a uniquely solvable boundary value problem we will use an idea as given by Farhat and Roux, see [67]. Instead of Neumann boundary conditions we use modified Robin interface conditions. As proven in Chapter 5, these modified Robin boundary value problems always posses a unique solution. Let us assume that appropriate Robin operators are given per interface, i.e. on the interface Γ_{ij} we have the Robin operator R_{ij}.

Let us define

$$(R_i u_{|\Gamma_i})(x) := (R_{ij} u_{|\Gamma_{ij}})(x) \quad \text{for } x \in \Gamma_{ij} \tag{7.6}$$

and

$$\eta_i(x) := \begin{cases} \eta_{ij} & \text{for } x \in \Gamma_{ij},\ i < j, \\ -\eta_{ij} & \text{for } x \in \Gamma_{ij},\ i > j, \\ 0 & \text{for } x \in \Gamma_i \cap \Gamma\,. \end{cases} \tag{7.7}$$

In the case of a wave number $k_i \notin \mathbb{R}^+$ we define $\eta_i(\cdot) \equiv 0$, otherwise we assume $\eta_i(\cdot) \not\equiv 0$ and that η_i does not change its sign on Γ_i. This can be guaranteed either when considering a checker board domain decomposition [65], or when enforcing Robin type boundary conditions only on a part of the local boundary Γ_i, i.e. setting $\eta_{ij} = 0$ on some coupling boundaries Γ_{ij}. The only assumption we have to make is that each bounded scattering subdomain Ω_i with $k_i^2 \in \mathbb{R}^+$ has at least one bounded scattering neighbour, then the Robin interfaces can be chosen such that all subproblems are uniquely solvable. Note that only bounded scattering domains rely

Algorithm 1 Algorithm to define the Robin interfaces

Require: All domains Ω_i are white
  for $i = 1$ to p **do**
    if Ω_i has no black neighbour **then**
      color Ω_i black.
    end if
  end for
  for $i = 2$ to p **do**
    for $j = 1$ to $i - 1$ **do**
      if (Ω_i=white AND Ω_j=black) OR (Ω_i=black AND Ω_j=white) **then**
        set Γ_{ij} to Robin interface.
      end if
    end for
  end for

on a Robin boundary condition to be uniquely solvable, for other domains Robin boundary conditions may not lead to uniquely solvable problems. A possible algorithm describing how to chose the Robin interface conditions is given by Algorithm 1. There we assume that all domains are bounded scattering domains, if this is not the case one has to apply this algorithm to all groups of connected scattering domains. In the white domains we set $\eta_i \geq 0$ and in the black domains $\eta_j \leq 0$. If we have an isolated bounded scattering domain we can not ensure unique solvability for the boundary value problem considered in this subdomain. A possible solution is to divide this scattering subdomain into two scattering subdomains.

Instead of (7.4), we will use the obviously equivalent formulation:

Find $\gamma_0 U \in H^{1/2}(\Gamma_S)$ such that

$$\begin{aligned} (S_i + i\eta_i R_{ij})\gamma_0 U_{|\Gamma_i} + (S_j + i\eta_j R_{ij})\gamma_0 U_{|\Gamma_j} = 0 \qquad & \text{on } \Gamma_{ij}, \\ S_i \gamma_0 U_{|\Gamma_i} = p \qquad & \text{on } \Gamma_N \cap \Gamma_i. \end{aligned} \tag{7.8}$$

Since this is in fact the same system as before, it is of course also uniquely solvable. Differences to the original formulation appear only if applying the tearing and interconnecting approach, i.e. the local problems are changed even though the global problem remains unchanged.

Remark 7.2. *In this thesis we consider the Robin interface conditions only to ensure uniqueness of the local problems. But they can also be used to improve the spectral condition number of the resulting linear system, see, for example, [72].*

Finally, we focus on the Dirichlet eigenvalues. Since the local Steklov–Poincaré S_i operator is not well defined for Dirichlet eigen wave numbers, we exchange the ap-

plication of the local Steklov–Poincaré operator by

$$\begin{aligned}(S_i + i\eta_i R_i)\gamma_0 U_{|\Gamma_i} &= (D_i + i\eta R_i + (\frac{1}{2}I + K_i^{\perp})V_i^{-1}(\frac{1}{2}I + K_i))\gamma_0 U_{|\Gamma_i} \\ &= (D_i + i\eta R_i)\gamma_0 U_{|\Gamma_i} + (\frac{1}{2}I + K_i^{\perp})t_i\end{aligned}$$

where $t_i \in H^{-1/2}(\Gamma_i)$ is a solution the boundary integral equation

$$V_i t_i = (\frac{1}{2}I + K_i)\gamma_0 U_{|\Gamma_i}.$$

This translates (7.8) to

Find $u \in H^{1/2}(\Gamma_S)$ and $t_i \in H^{-1/2}(\Gamma_i)$ for $i = 1, \ldots, p$ such that

$$\begin{aligned}(D_i + i\eta_i R_i)u_{|\Gamma_i} + (\frac{1}{2}I + K_i^{\perp})t_i & \\ +(D_j + i\eta_j R_j)u_{|\Gamma_j} + (\frac{1}{2}I + K_j^{\perp})t_j = 0 \qquad & \text{on } \Gamma_{ij}, \\ D_i u_{|\Gamma_i} + (\frac{1}{2}I + K_i^{\perp})t_i = p \qquad & \text{on } \Gamma \cap \Gamma_i, \\ -(\frac{1}{2} + K_i)u_{|\Gamma_i} + V_i t_i = 0 \qquad & \text{on } \Gamma_i.\end{aligned}$$

The related variational formulation reads:

Find $u \in H^{1/2}(\Gamma_S)$ and $t_i \in H^{-1/2}(\Gamma_i)$ for $i = 1, \ldots, p$ such that

$$\sum_{i=1}^{p}\left[\langle D_i u_{|\Gamma_i}, v_{|\Gamma_i}\rangle_{\Gamma_i} + \langle(\frac{1}{2}I + K_i^{\perp})t_i, v_{|\Gamma_i}\rangle_{\Gamma_i} + \langle i\eta_i R_i u_{|\Gamma_i}, v_{|\Gamma_i}\rangle_{\Gamma_i}\right] = \int_{\Gamma} p(x)v(x)\mathrm{d}s_x \tag{7.9}$$

for all $v \in H^{1/2}(\Gamma)$ and

$$\langle V_i t_i, \tau_i\rangle_{\Gamma_i} - \langle(\frac{1}{2}I + K_i)u_{|\Gamma_i}, \tau_i\rangle_{\Gamma_i} = 0 \tag{7.10}$$

for all $\tau_i \in H^{-1/2}(\Gamma_i), i = 1, \ldots, p$.

Theorem 7.3. *The coupled variational problem* (7.9) *and* (7.10) *admits a unique solution* $u \in H^{1/2}(\Gamma_S)$. *In particular, the associated sesquilinear form is coercive. Moreover, if the Neumann boundary value problem* (7.1) *is uniquely solvable, the associated sesquilinear form is also injective.*

Proof. Coercivity follows from the coercivity of the local operators, i.e.

$$\begin{aligned}\mathrm{Re}\Bigg(\sum_{i=1}^{p}\left\langle\begin{pmatrix} D_i & \frac{1}{2}I+K_i^{\perp} \\ -(\frac{1}{2}I+K_i) & V_i\end{pmatrix}\begin{pmatrix} u_{|\Gamma_i} \\ t_i\end{pmatrix},\begin{pmatrix} u_{|\Gamma_i} \\ t_i\end{pmatrix}\right\rangle_{\Gamma_i}+C((u,\underline{t}),(u,\underline{t}))\Bigg)\\ \geq c\left(\|u\|^2_{H^{1/2}(\Gamma_S)}+\sum_{i=1}^{p}\|t_i\|^2_{H^{-1/2}(\Gamma_i)}\right)\end{aligned}$$

with $\underline{t}=(t_1,\ldots,t_p)^{\top}$ and a compact sesquilinear form C. It remains to prove injectivity. Let $u\in H^{1/2}(\Gamma_S)$ and $t_i\in H^{-1/2}(\Gamma_i)$ for $i=1,\ldots,p$ be any solution of the homogeneous system

$$\sum_{i=1}^{p}\left[\langle D_i u_{|\Gamma_i},v_{|\Gamma_i}\rangle_{\Gamma_i}+\langle(\frac{1}{2}I+K_i^{\perp})t_i,v_{|\Gamma_i}\rangle_{\Gamma_i}+\langle i\eta_i R_i u_{|\Gamma_i},v_{|\Gamma_i}\rangle_{\Gamma_i}\right]=0$$

for all $v\in H^{1/2}(\Gamma_S)$ and

$$\langle V_i t_i,\tau_i\rangle_{\Gamma_i}-\langle(\frac{1}{2}I+K_i)u_{|\Gamma_i},\tau_i\rangle_{\Gamma_i}=0$$

for all $\tau_i\in H^{-1/2}(\Gamma_i), i=1,\ldots,p$. With the definition of R_i and η_i we also have

$$\sum_{i=1}^{p}\left[\langle D_i u_{|\Gamma_i},v_{|\Gamma_i}\rangle_{\Gamma_i}+\langle(\frac{1}{2}I+K_i^{\perp})t_i,v_{|\Gamma_i}\rangle_{\Gamma_i}\right]=0$$

for all $v\in H^{1/2}(\Gamma_S)$. Let us define

$$\phi_i(x)=\int_{\Gamma_i}U_i^*(x,y)t_i(y)\mathrm{d}s_y-\int_{\Gamma_i}\frac{\partial}{\partial\mathbf{n}_y}U_i^*(x,y)u_{|\Gamma_i}(y)\mathrm{d}s_y\quad\text{for } x\in\Omega_i,$$

which satisfies

$$\Delta\phi_i(x)+k^2\phi_i(x)=0\quad\text{for } x\in\Omega_i,$$

and

$$\frac{\partial}{\partial\mathbf{n}_x}\phi_i(x)=(\frac{1}{2}I+K_i^{\perp})t_i(x)+(D_i u_{|\Gamma_i})(x)\quad\text{for } x\in\Gamma_i$$

as well as

$$\phi_i(x)=(V_i t_i)(x)+\frac{1}{2}u_{|\Gamma_i}(x)-(K_i u_{|\Gamma_i})(x)=u_{|\Gamma_i}\quad\text{for } x\in\Gamma_i.$$

Hence we may consider $\phi_i = \phi_{|\Omega_i} \in H^1(\Omega_i)$ as the restriction of a function $\phi \in H^1(\Omega)$. By using Green's first formula, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{p} \left[\langle D_i u_{|\Gamma_i}, V_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K_i^{\perp}) t_i, V_{|\Gamma_i} \rangle_{\Gamma_i} \right] \\ &= \sum_{i=1}^{p} \int_{\Gamma_i} \frac{\partial}{\partial n_y} \phi_i(x) V_{|\Gamma_i}(x) \mathrm{d}s_x \\ &= \sum_{i=1}^{p} \int_{\Omega_i} \Big[\boldsymbol{\nabla} \phi_i(x) \cdot \boldsymbol{\nabla} V_{|\Omega_i}(x) \mathrm{d}x - k_i^2 \phi_i(x) V_{|\Omega_i}(x) \Big] \mathrm{d}x \\ &= \int_{\Omega} \Big[\boldsymbol{\nabla} \phi(x) \cdot \boldsymbol{\nabla} V(x) - k^2(x) \phi(x) V(x) \Big] \mathrm{d}x \end{aligned}$$

for all $V \in H^1(\Omega)$. Since this is the weak formulation of the Neumann boundary value problem

$$\Delta \phi(x) + k^2(x) \phi(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial \mathbf{n}_x} \phi(x) = 0 \quad \text{for } x \in \Gamma,$$

$\phi(x) = 0$ for $x \in \Omega$ follows. Recall that the Neumann boundary value problem (7.1) was assumed to be uniquely solvable. From $\phi_i(x) = 0$ for $x \in \Omega_i$ we conclude $u_{|\Gamma_i}(x) = 0$ for $x \in \Gamma_i$ as well as $\mathbf{n}_i \cdot \boldsymbol{\nabla}_x \phi_i(x) = 0$ for $x \in \Gamma_i$. Therefore we conclude

$$(\frac{1}{2} I + K_i^{\perp}) t_i(x) = 0, \quad (V_i t_i)(x) = 0 \quad \text{for } x \in \Gamma_i.$$

If k_i^2 is not a Dirichlet eigenvalue, then the single layer potential V_i is injective and $t_i = 0$ follows. On the other hand, if $\lambda = k_i^2$ is a Dirichlet eigenvalue, we also have

$$(\frac{1}{2} I - K_i^{\perp}) t_i(x) = 0 \quad \text{for } x \in \Gamma_i.$$

Again, $t_i(x) = 0$ follows. □

7.4 The exterior Neumann boundary value problem

The Neumann boundary value problem in an unbounded domain Ω^c is given by:

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U - k^2(x) U &= 0 \qquad \text{in } \Omega^c, \\ \gamma_1^c U + p &= 0 \qquad \text{on } \Gamma, \\ \lim_{r \to \infty} \int_{B_r} |\gamma_1 U(x) - ik \gamma_0 U(x)|^2 \, \mathrm{d}s_x &= 0. \end{aligned} \tag{7.11}$$

Note that this problem always admits a unique solution. For a domain decomposition approach we assume that Ω^c is divided into $p+1$ subdomains such that

$$\overline{\Omega} = \bigcup_{i=0}^{p} \overline{\Omega_i}, \qquad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for } i \neq j.$$

We further assume that Ω_0 is the complement of a bounded, simply connected Lipschitz domain. All other subdomains are assumed to fulfill the assumptions of Section 7.1. Also the notations of Section 7.1 can be applied directly. To transfer the results of the interior boundary value problem easily to the exterior boundary value problem we define $\gamma_{1,0} := -\gamma_{1,0}^c$ and $S_0 := -S_0^c$. This enables us to state the localized formulation of the Neumann boundary value problem (7.11) in the compact form:

Find $U \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta U_i - k^2(x) U_i &= 0 && \text{in } \Omega_i, \\ \gamma_{1,i} U_i + \gamma_{1,j} U_i &= 0 && \text{on } \Gamma_{ij}, \\ \gamma_{1,i} U_i &= p && \text{on } \Gamma \cap \Gamma_i, \\ \lim_{r\to\infty} \int_{B_r} |\gamma_1 U_0(x) - ik\gamma_0 U_0(x)|^2 \, \mathrm{d}s_x &\to 0 \end{aligned} \tag{7.12}$$

with $U_i = U_{|\Omega_i}$.

Under the assumption that k_i^2 is not an eigen wave number of the domain Ω_i, the formulation (7.12) can, as in the case of the interior boundary value problem, be reformulated to

Find $\gamma_0 U \in H^{1/2}(\Gamma_S)$ such that

$$\begin{aligned} (S_i + i\eta_i R_{ij})\gamma_0 U_{|\Gamma_i} + (S_j + i\eta_j R_{ij})\gamma_0 U_{|\Gamma_j} &= 0 && \text{on } \Gamma_{ij}, \\ S_i \gamma_0 U_{|\Gamma_i} &= p && \text{on } \Gamma_N \cap \Gamma_i. \end{aligned} \tag{7.13}$$

Note that we assume $\eta_0 \equiv 0$, because the exterior Neumann boundary value problem already admits a unique solution, cf. Section 7.3. To get rid of possible eigen wave numbers we replace for $i > 0$ the application of the Dirichlet–to–Robin map $S_i + i\eta_i R_i$ by

$$\begin{aligned} (S_i + i\eta_i R_i)\gamma_0 U_{|\Gamma_i} &= (D_i + i\eta R_i + (\frac{1}{2} I + K_i^{\perp}) V_i^{-1} (\frac{1}{2} I + K_i))\gamma_0 U_{|\Gamma_i} \\ &= (D_i + i\eta R_i)\gamma_0 U_{|\Gamma_i} + (\frac{1}{2} I + K_i^{\perp}) t_i \end{aligned}$$

where $t_i \in H^{-1/2}(\Gamma_i)$ is a solution the boundary integral equation

$$V_i t_i = (\frac{1}{2} I + K_i)\gamma_0 U_{|\Gamma_i}.$$

The application of the Steklov–Poincaré operator S_0 with respect to the unbounded domain Ω_0 is replaced by

$$\begin{aligned} S_0\gamma_0 U_{|\Gamma_0} &= (D_0 + (\frac{1}{2}I - K_0^\perp)V_0^{-1}(\frac{1}{2}I - K_0))\gamma_0 U_{|\Gamma_0} \\ &= D_0\gamma_0 U_{|\Gamma_0} + (\frac{1}{2}I - K_0^\perp)t_0 - D_0 s \end{aligned}$$

where $t_0 \in H^{-1/2}(\Gamma_i)$ and $s \in H^{1/2}(\Gamma_0)$ are solutions of the boundary integral equation system

$$\begin{aligned} V_0 t_0 + (\frac{1}{2}I + K_0)s &= (-\frac{1}{2}I + K_0)\gamma_0 U_{|\Gamma_0}, \\ -(\frac{1}{2}I + K_0^\perp)t_0 + (D_0 - i\widetilde{D}_0)s &= D_0\gamma_0 U_{|\Gamma_0}, \end{aligned}$$

see Section 5.7. Inserting these replacements of the Dirichlet–to–Robin map in formulation (7.13) leads to:

Find $u \in H^{1/2}(\Gamma_S)$, $t_i \in H^{-1/2}(\Gamma_i)$ for $i = 1, \ldots, p$ and $s \in H^{1/2}(\Gamma_0)$ such that

$$\begin{aligned} D_0 u_{|\Gamma_0} + (\frac{1}{2}I - K_0^\perp)t_0 - D_0 s + D_j u_{|\Gamma_j} + (\frac{1}{2}I + K_j^\perp)t_j &= 0 \qquad \text{on } \Gamma_{0j}, \\ (D_i + i\eta_i R_i)u_{|\Gamma_i} + (\frac{1}{2}I + K_i^\perp)t_i & \\ +(D_j + i\eta_j R_j)u_{|\Gamma_j} + (\frac{1}{2}I + K_j^\perp)t_j &= 0 \qquad \text{on } \Gamma_{ij}, \\ D_0 u_{|\Gamma_0} + (\frac{1}{2}I - K_0^\perp)t_0 - D_0 s &= p \qquad \text{on } \Gamma \cap \Gamma_0, \\ D_i u_{|\Gamma_i} + (\frac{1}{2}I + K_i^\perp)t_i &= p \qquad \text{on } \Gamma \cap \Gamma_i, \\ (\frac{1}{2}I - K_0)u_{|\Gamma_i} + V_0 t_0 + (\frac{1}{2}I + K_0) &= 0 \qquad \text{on } \Gamma_0, \\ -(\frac{1}{2}I + K_i)u_{|\Gamma_i} + V_i t_i &= 0 \qquad \text{on } \Gamma_i, \\ -D_0 u_{|\Gamma_0} - (\frac{1}{2}I + K_0^\perp)t_0 + (D_0 - i\widetilde{D}_0)s &= 0 \qquad \text{on } \Gamma_0. \end{aligned}$$

The resulting variational formulation is then given by:

Find $u \in H^{1/2}(\Gamma_S)$, $t_i \in H^{-1/2}(\Gamma_i)$ for $i = 0, \ldots, p$ and $s \in H^{1/2}(\Gamma_0)$ such that

$$\begin{aligned} &\sum_{i=1}^{p}\Big[\langle D_i u_{|\Gamma_i}, v_{|\Gamma_i}\rangle_{\Gamma_i} + \langle(\frac{1}{2}I + K_i^\perp)t_i, v_{|\Gamma_i}\rangle_{\Gamma_i} + \langle i\eta_i R_i u_{|\Gamma_i}, v_{|\Gamma_i}\rangle_{\Gamma_i}\Big] \\ &\quad + \langle D_0 u_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} + \langle(-\frac{1}{2}I + K_0)t_0, v_{|\Gamma_0}\rangle - \langle D_0 s, v_{|\Gamma_0}\rangle_{\Gamma_0} = \int_\Gamma p(x)v(x)\mathrm{d}s_x \end{aligned} \tag{7.14}$$

is satisfied for all $v \in H^{1/2}(\Gamma_S)$,

$$\langle V_i t_i, \tau_i \rangle_{\Gamma_i} - \langle (\frac{1}{2} I + K_i) u_{|\Gamma_i}, \tau_i \rangle_{\Gamma_i} = 0 \tag{7.15}$$

for all $\tau_i \in H^{-1/2}(\Gamma_i), i = 1, \ldots, p$,

$$\langle V_0 t_0, \tau_0 \rangle_{\Gamma_0} + \langle (\frac{1}{2} I - K_0) u_{|\Gamma_0}, \tau_0 \rangle_{\Gamma_0} + \langle (\frac{1}{2} I + K_0) s, \tau_0 \rangle_{\Gamma_0} = 0 \tag{7.16}$$

for all $\tau_0 \in H^{-1/2}(\Gamma_0)$ and

$$\langle (-\frac{1}{2} I - K_0^{\perp}) t_0, \phi \rangle_{\Gamma_0} - \langle D_0 u_{|\Gamma_0}, \phi \rangle_{\Gamma_0} + \langle (-i\widetilde{D} + D_0) s, \phi \rangle_{\Gamma_0} = 0 \tag{7.17}$$

for all $\phi \in H^{1/2}(\Gamma_0)$.

Theorem 7.4. *The variational formulation* (7.14)–(7.17) *admits a unique solution* $u \in H^{1/2}(\Gamma_S)$. *In particular, the associated sesquilinear form is coercive and injective.*

Proof. An associated sesquilinear form of (7.14)–(7.17) is given by

$$\begin{aligned}
a((u, \underline{t}, s), (\lambda, \underline{\tau}, \mu)) := {} & \langle D_0 u_{|\Gamma_0}, \lambda_{|\Gamma_0} \rangle_{\Gamma_0} + \langle (-\frac{1}{2} I + K_0^{\perp}) t_0, \lambda_{|\Gamma_0} \rangle_{\Gamma_0} - \langle D_0 s, \lambda_{|\Gamma_0} \rangle_{\Gamma_0} \\
& + \overline{\langle (\frac{1}{2} I - K_0) u_{|\Gamma_0}, \tau_0 \rangle_{\Gamma_0}} + \overline{\langle V_0 t_0, \tau_0 \rangle_{\Gamma_0}} + \overline{\langle (\frac{1}{2} I + K_0) s, \tau_0 \rangle_{\Gamma_0}} \\
& - \langle D_0 u_{|\Gamma_0}, \mu \rangle_{\Gamma_0} - \langle (\frac{1}{2} I + K_0^{\perp}) t_0, \mu \rangle_{\Gamma_0} + \langle (D_0 - i\widetilde{D}_0) s, \mu \rangle_{\Gamma_0} \\
& + \sum_{i=1}^{p} \Big(\langle D_i u_{|\Gamma_i}, \lambda_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K_i^{\perp}) t_i, \lambda_{|\Gamma_i} \rangle_{\Gamma_i} \\
& \quad - \overline{\langle (\frac{1}{2} I + K_i) u_{|\Gamma_i}, \tau_i \rangle_{\Gamma_i}} + \overline{\langle V_i t_i, \tau_i \rangle_{\Gamma_i}} \Big) .
\end{aligned}$$

Note that the Robin boundary integral operators cancel each other. To prove a Gårding inequality we exchange all appearing hypersingular operators D_i by the local regularized Laplace hypersingular operators $\widetilde{D}_i$, in the case of all other boundary integral operators we exchange the wave numbers k_i by 0, e.g. V_i, which depends on the local wave number k_i, is replaced by the Laplace single layer potential $\widetilde{V}_i$. Since all of these changes are compact perturbations, see Corollary 5.12, they do not influence the proof of a Gårding inequality. The associated shifted sesquilinear form

is then given by

$$\begin{aligned}\widetilde{a}((u,\underline{t},s),(\lambda,\underline{\tau},\mu)) &:= \langle\widetilde{D}_0 u_{|\Gamma_0},\lambda_{|\Gamma_0}\rangle_{\Gamma_0} + \langle(-\frac{1}{2}I+\widetilde{K}_0^{\perp})t_0,\lambda_{|\Gamma_0}\rangle_{\Gamma_0} - \langle\widetilde{D}_0 s,\lambda_{|\Gamma_0}\rangle_{\Gamma_0}\\ &+ \overline{\langle(\frac{1}{2}I-\widetilde{K}_0)u_{|\Gamma_0},\tau_0\rangle_{\Gamma_0}} + \overline{\langle\widetilde{V}_0 t_0,\tau_0\rangle_{\Gamma_0}} + \overline{\langle(\frac{1}{2}I+\widetilde{K}_0)s,\tau_0\rangle_{\Gamma_0}}\\ &- \langle\widetilde{D}_0 u_{|\Gamma_0},\mu\rangle_{\Gamma_0} - \langle(\frac{1}{2}I+\widetilde{K}_0^{\perp})t_0,\mu\rangle_{\Gamma_0} + \langle(\widetilde{D}_0 - i\widetilde{D}_0)s,\mu\rangle_{\Gamma_0}\\ &+ \sum_{i=1}^{p}\Big(\langle\widetilde{D}_i u_{|\Gamma_i},\lambda_{|\Gamma_i}\rangle_{\Gamma_i} + \langle(\frac{1}{2}I+\widetilde{K}_i^{\perp})t_i,\lambda_{|\Gamma_i}\rangle_{\Gamma_i}\\ &\quad - \overline{\langle(\frac{1}{2}I+\widetilde{K}_i)u_{|\Gamma_i},\tau_i\rangle_{\Gamma_i}} + \overline{\langle\widetilde{V}_i t_i,\tau_i\rangle_{\Gamma_i}}\Big).\end{aligned}$$

By identifying test and trial functions we obtain

$$\begin{aligned}\widetilde{a}((u,\underline{t},s),(u,\underline{t},s)) &= \langle\widetilde{D}_0(u_{|\Gamma_0}-s),(u_{|\Gamma_0}-s)\rangle_{\Gamma_0} + \langle\widetilde{V}_0 t_0,t_0\rangle_{\Gamma_0} - i\langle\widetilde{D}_0 s,s\rangle_{\Gamma_0}\\ &\quad + \sum_{i=1}^{p}\Big(\langle\widetilde{D}_i u_{|\Gamma_i},u_{|\Gamma_i}\rangle_{\Gamma_i} + \langle\widetilde{V}_i t_i,t_i\rangle_{\Gamma_i}\Big).\end{aligned}$$

Finally, this gives us the deserved inequality

$$\begin{aligned}|\widetilde{a}((u,\underline{t},s),(u,\underline{t},s))| &\geq c_1\left|\left\|u_{|\Gamma_0}\right\|^2_{H^{1/2}(\Gamma_0)} + i\left\|s\right\|^2_{H^{1/2}(\Gamma_0)}\right|\\ &\quad + c_2\sum_{i=1}^{p}\left\|u_{|\Gamma_i}\right\|^2_{H^{1/2}(\Gamma_i)} + c_3\sum_{i=0}^{p}\left\|t_i\right\|^2_{H^{-1/2}(\Gamma_i)}\\ &\geq \frac{c_1}{2\sqrt{2}}\left(\left\|u_{|\Gamma_0}\right\|^2_{H^{1/2}(\Gamma_0)} + \left\|s\right\|^2_{H^{1/2}(\Gamma_0)}\right)\\ &\quad + c_2\sum_{i=1}^{p}\left\|u_{|\Gamma_i}\right\|^2_{H^{1/2}(\Gamma_i)} + c_3\sum_{i=0}^{p}\left\|t_i\right\|^2_{H^{-1/2}(\Gamma_i)}\\ &\geq c\left(\left\|u\right\|^2_{H^{1/2}(\Gamma_S)} + \sum_{i=0}^{p}\left\|t_i\right\|^2_{H^{-1/2}(\Gamma_i)} + \left\|s\right\|^2_{H^{1/2}(\Gamma_0)}\right)\end{aligned}$$

with positive constants c, c_1, c_2, c_3. It remains to prove injectivity. Let $u \in H^{1/2}(\Gamma_S)$ and $t_i \in H^{-1/2}$ for $i = 1,\ldots,p$. For $s \in H^{1/2}(\Gamma_0)$ we get $s \equiv 0$, since $s \not\equiv 0$ would be a contradiction to the two local equations (7.16) and (7.17), see Remark 5.33. The variational formulation for the homogeneous system is to find $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)$ and $\mathbf{t}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i)$ such that

$$\begin{aligned}&\sum_{i=1}^{p}\left[\langle D_i u_{|\Gamma_i},v_{|\Gamma_i}\rangle_{\Gamma_i} + \langle(\frac{1}{2}I+K_i^{\perp})t_i,v_{|\Gamma_i}\rangle_{\Gamma_i} + \langle i\eta R_i u_{|\Gamma_i},v_{|\Gamma_i}\rangle_{\Gamma_i}\right]\\ &\quad + \langle D_0 u_{|\Gamma_i},v_{|\Gamma_i}\rangle_{\Gamma_0} + \langle(-\frac{1}{2}I+K_0^{\perp})t_0,v_{|\Gamma_0}\rangle_{\Gamma_0} = 0\end{aligned} \tag{7.18}$$

for all $v \in H^{1/2}(\Gamma_S)$,

$$\langle V_i t_i, \tau_i \rangle_{\Gamma_i} - \langle (\frac{1}{2} I + K_i) u_{|\Gamma_i}, \tau_i \rangle_{\Gamma_i} = 0$$

for all $\tau_i \in H^{-1/2}(\Gamma_i), i = 1, \ldots, p$.

$$\langle V_0 t_0, \tau_0 \rangle_{\Gamma_0} + \langle (\frac{1}{2} I - K_0) u_{|\Gamma_0}, \tau_0 \rangle_{\Gamma_0} = 0$$

for all $\tau_0 \in H^{-1/2}(\Gamma_0)$ and

$$\langle (-\frac{1}{2} I - K_0^{\perp}) t_0, \psi \rangle_{\Gamma_0} - \langle D_0 u_{|\Gamma_0}, \psi \rangle_{\Gamma_0} = 0 \tag{7.19}$$

for all $\psi \in H^{1/2}(\Gamma_0)$. By using the definition of R_i and η_i we can rewrite (7.18) as

$$\sum_{i=1}^{p} \left[\langle D_i u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K_i^{\perp}) t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} \right] + \langle D_0 u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_0} + \langle (-\frac{1}{2} I + K_0^{\perp}) t_0, v_{|\Gamma_0} \rangle_{\Gamma_0} = 0$$

for all $v \in H^{1/2}(\Gamma_S)$. By defining

$$\phi_i(x) = \int_{\Gamma_i} U_i^*(x,y) t_i(y) \mathrm{d}s_y - \int_{\Gamma_i} \frac{\partial}{\partial \mathbf{n}_y} U_i^*(x,y) u_{|\Gamma_i}(y) \mathrm{d}s_y \qquad \text{for } x \in \Omega_i,$$

for $i > 0$ and

$$\phi_0(x) = -\int_{\Gamma_0} U_0^*(x,y) t_0(y) \mathrm{d}s_y + \int_{\Gamma_0} \frac{\partial}{\partial \mathbf{n}_y} U_0^*(x,y) u_{|\Gamma_0}(y) \mathrm{d}s_y \qquad \text{for } x \in \Omega_0$$

we once again gain a function ϕ which is a solution of the homogeneous Neumann problem and fulfills $\phi_i(x) = u_{|\Gamma_i} = 0$, see the proof of Theorem 7.3. Showing $t_i = 0$ for $i > 0$ can be done as in the proof of Theorem 7.3. For the unbounded domain we get instead

$$(\frac{1}{2} I - K_0^{\perp}) t_0 = 0 \qquad \text{on } x \in \Gamma_0.$$

In combination with equation (7.19), this again leads to $t_0 = 0$. □

7.5 The transmission problem

Transmission problems are very common problems of practical interest. For the Helmholtz transmission problem there is a lot of literature available, see, e.g., [52,118].

A coupled finite and boundary element approach seems to be natural, since non constant coefficients can be treated in the interior and the exterior can be treated easily by a boundary element method. One incarnation of such an approach, which guarantees uniqueness is given in [86]. An approach using boundary elements also for the interior, is for example, given in [93]. This approach has several similarities to the one we present here. The major differences are that a mortar method is used to couple the different fields, that only the case of two subdomains is treated and that it is not stable for artificial exterior eigenvalues due to a naive boundary integral formulation. A stable approach for the transmission problem, which can handle multiple interior subdomains is given in [85]. So far, in this work we have only treated boundary value problems, but by reformulating the transmission problem we will see that within this framework this kind of problem can also be dealt with. Since we use a boundary integral approach we assume that in the interior we also have to solve a homogeneous Helmholtz equation. We further assume due to simplicity, that only one interior domain with a constant wave number k_1 is present. The extension to several interior domains is straightforward. So the transmission problem is given by

$$\begin{aligned} -\Delta U - k_1^2 U &= 0 && \text{in } \Omega_1, && (7.20)\\ -\Delta U^s - k_0^2 U^s &= 0 && \text{in } \Omega_0, && (7.21)\\ \gamma_0^c U^s - \gamma_0 U &= g && \text{on } \Gamma, && (7.22)\\ \gamma_1^c U^s + \gamma_1 U &= p && \text{on } \Gamma, && (7.23)\\ \lim_{r\to\infty} r(\gamma_1 U^s - ik_0\gamma_0 U^s) &\to 0 && \text{uniformly.} && (7.24) \end{aligned}$$

Hereby is U^s the scattered field, and $U = U^s + U^i$ is the total field, with U^i as the incoming field. The Cauchy data are induced by the incoming field, in other words $-\gamma_0^c U^i = g \in H^{1/2}(\Gamma)$ and $-\gamma_1^c U^i = p \in H^{-1/2}(\Gamma)$. In our approach the primal unknown will be the Dirichlet trace of the total field, i.e. $\gamma_0 U$. To keep the formulations simple let us assume that both, the interior Steklov–Poincaré operator S_1 and the exterior Steklov–Poincaré operator S_0 are well defined, i.e. k_0^2 and k_1^2 are not Dirichlet eigen wave numbers. For the general case we can use the system formulation (5.28) as introduced for the boundary value problems. Hence, equation (7.23) can be reformulated as

$$S_0\gamma_0 U^s + S_1\gamma_0 U = p$$

and since $U^s = U - U^i$ we can rewrite this as

$$S_0\gamma_0 U + S_1\gamma_0 U = p + S_0 g. \tag{7.25}$$

By solving equation (7.25) we get the Dirichlet datum $\gamma_0 U$, which gives us also $\widetilde{f} := \gamma_0^c U^s$. By solving the exterior Dirichlet boundary value problem

$$\begin{aligned} -\Delta U^s - k_0^2 U^s &= 0 && \text{in } \Omega_0, && (7.26)\\ \gamma_0^c U^s &= \widetilde{f} && \text{on } \Gamma, && (7.27)\\ \lim_{r\to\infty} r(\gamma_1 U^s - ik\gamma_0 U^s) &\to 0 && \text{uniformly.} && (7.28) \end{aligned}$$

we obtain the scattered field U^s. This can be used to compute $\gamma_1 U$, see (7.23), which on the other hand can be used, together with $\gamma_0 U$, to compute U in the interior via the representation formula (5.9). The proofs of unique solvability of the continuous and of the discrete formulation can be done in an analogous way as it was done for the interior and exterior boundary value problems. Note that the transmission problem itself has a unique solution in $H_{loc}(\Delta, \mathbb{R}^3)$ if $k_0, k_1 \in \mathbb{R}^+$, see for example [121].

7.6 Boundary element discretizations

For the Galerkin discretization of the coupled variational formulation (7.9) and (7.10) let

$$W_h = \operatorname{span}\{\varphi_k\}_{k=1}^{M_S} \subset H^{1/2}(\Gamma_S)$$

be a boundary element space on the skeleton, e.g., of piecewise linear continuous basis functions φ_k, which are defined with respect to a quasi–uniform boundary mesh with mesh size h_S. We also define local restrictions of W_h onto Γ_i, in particular

$$W_{i,h} = W_{h|\Gamma_i} = \operatorname{span}\{\varphi_k^i\}_{k=1}^{M_i} \subset H^{1/2}(\Gamma_i).$$

By using the isomorphisms

$$\underline{v}_i \in \mathbb{R}^{M_i} \leftrightarrow v_{i,h} = \sum_{k=1}^{M_i} v_{i,k}\varphi_k^i \in W_{i,h}, \quad \underline{v} \in \mathbb{R}^{M_S} \leftrightarrow v_h = \sum_{k=1}^{M_S} v_k \varphi_k \in W_h$$

there exist Boolian connectivity matrices $A_i \in \mathbb{R}^{M_i \times M_S}$ mapping some $\underline{v} \in \mathbb{R}^{M_S}$ of global nodal values onto the vector $\underline{v}_i = A_i \underline{v} \in \mathbb{R}^{M_i}$ of the local subdomain boundary nodal values. In addition, let

$$Z_{i,h} = \operatorname{span}\{\psi_k^i\}_{k=1}^{N_i} \subset H^{-1/2}(\Gamma_i)$$

be another local boundary element space, e.g., of piecewise constant basis functions ψ_k^i, which are defined with respect to a local quasi–uniform boundary mesh with average mesh size h_i. The Galerkin boundary element discretization of the variational formulation (7.9) and (7.10) now reads:

Find $u_h \in W_h$ and $t_{i,h} \in Z_{i,h}$ such that

$$\begin{aligned} \sum_{i=1}^{p} \Big[\langle D_i u_{h|\Gamma_i}, v_{h|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K_i^{\perp}) t_{i,h}, v_{h|\Gamma_i} \rangle_{\Gamma_i} + \langle i\eta R_i u_{h|\Gamma_i}, v_{h|\Gamma_i} \rangle_{\Gamma_i} \Big] \\ = \int_{\Gamma} p(x) v_h(x) \mathrm{d}s_x \end{aligned} \tag{7.29}$$

for all $v_h \in W_h$ and

$$\langle V_i t_{i,h}, \tau_{i,h} \rangle_{\Gamma_i} - \langle (\frac{1}{2} I + K_i) u_{h|\Gamma_i}, \tau_{i,h} \rangle_{\Gamma_i} = 0 \tag{7.30}$$

for all $\tau_{i,h} \in Z_{i,h}$, $i = 1, \ldots, p$.

Since the sesquilinear form of the coupled variational problem (7.9) and (7.10) is coercive and injective, see Theorem 7.3, the stability of the Galerkin variational formulation (7.29) and (7.30) follows for sufficient small mesh widths h_i and h_S, see Lemma 3.12. In particular, there holds the quasi–optimal error estimate

$$\begin{aligned} \|u - u_h\|^2_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} \|t_i - t_{i,h}\|^2_{H^{-1/2}(\Gamma_i)} \\ \leq c \Big\{ \inf_{v_h \in W_h} \|u - v_h\|^2_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} \inf_{\tau_{i,h} \in Z_{i,h}} \|t_i - \tau_{i,h}\|^2_{H^{-1/2}(\Gamma_i)} \Big\}. \end{aligned}$$

When assuming optimal regularity $\gamma_0 U = u \in H^2_{\mathrm{pw}}(\Gamma_S)$, i.e. $U \in H^{5/2}(\Omega)$ and when using the Aubin–Nitsche trick, see, e.g. [125], we finally obtain the error estimate

$$\|u - u_h\|_{L_2(\Gamma_S)} \leq c(u, t_i) h_S^2 + \sum_{i=1}^{p} c(t_i) h_i^2. \tag{7.31}$$

Remark 7.5. *For a Galerkin discretization of the coupled variational formulation* (7.14)–(7.17) *to solve boundary value problems in unbounded domains* Ω^c *we can use the same trial and test spaces. Then we can ensure the same error estimates for the approximate solution.*

7.7 Tearing and interconnecting methods

In this section we introduce the classical tearing and interconnecting approach, as originally introduced by Farhat and Roux [67, 68] using local finite element solvers. This classical approach is nowadays called one–level–FETI method, since other FETI methods like dual–primal FETI (FETI–DP) [64] have appeared. These methods are generally used for an efficient numerical solution of partial differential equations, which induce an elliptic or semi–elliptic symmetric sesquilinear form. Since the deduction of the method is usually based on a reformulation as a minimization problem, it was often thought that the method is only applicable in these cases, see [101, p. 28]. De La Bourdonnaye, Farhat, Macedo and Tezaur showed in [53, 66] that this method is also applicable for the Helmholtz equation, by using a saddle point formulation instead of a minimization problem. This method is called FETI–Helmholtz or in short FETI–H. The partial differential equation still has to be self–adjoint, otherwise the domain decomposition approach can not to be reformulated as a saddle point problem. In our approach we will only use algebraic arguments to deduce the tearing and interconnecting method. This enables us, in principle, to use the method

for arbitrary elliptic partial differential equations of second order. Furthermore we will use a boundary element method instead of a finite element method to solve the local problems. Tearing and interconnecting methods based on boundary elements were introduced by Langer and Steinbach in [95] and are called BETI methods. The same authors later introduced the FETI/BETI method [96], which couples finite and boundary element approaches.

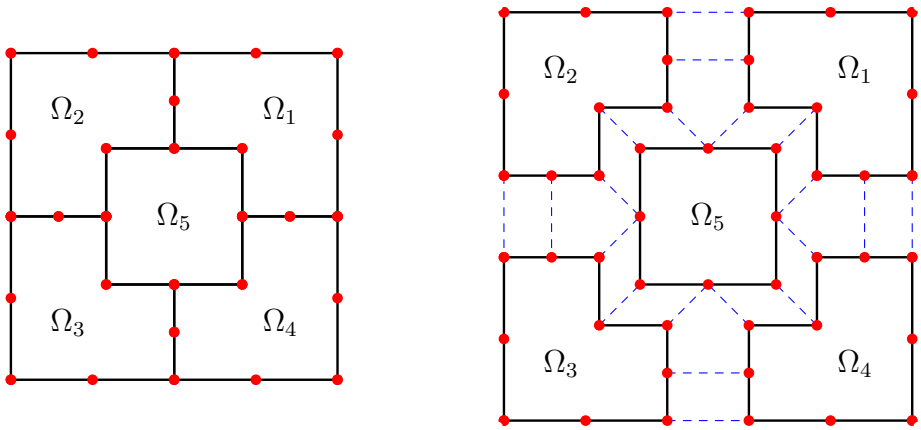


Figure 7.1: In the left figure we see the global degrees of freedom, which are split on the right figure to local degrees of freedom. To ensure continuity of the global solution, these local degrees of freedom are again connected by Lagrangian multipliers (dotted lines).

For simplicity we start by assuming that Ω is a bounded domain. The Galerkin variational formulation (7.29) and (7.30) is equivalent to a linear system of algebraic equations

$$\begin{pmatrix} V_{1,h} & & & -\widetilde{K}_{1,h}A_i \\ & \dots & & \vdots \\ & & V_{p,h} & -\widetilde{K}_{p,h}A_p \\ A_1^\top \widetilde{K}_{1,h}^\top & \dots & A_p^\top \widetilde{K}_{p,h}^\top & \sum_{i=1}^{p} A_i^\top [D_{i,h} + i\eta_i R_{i,h}] A_i \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \sum_{i=1}^{p} A_i^\top \underline{p}_i \end{pmatrix} \tag{7.32}$$

where the block matrices are defined by

$$\begin{aligned} V_{i,h}[\ell,k] &= \langle V_i \psi_k^i, \psi_\ell^i \rangle_{\Gamma_i}, \\ \widetilde{K}_{i,h}[\ell,n] &= \langle (\frac{1}{2}I + K_i)\varphi_n^i, \psi_\ell^i \rangle_{\Gamma_i}, \\ \widetilde{K}_{i,h}^\top[m,k] &= \langle (\frac{1}{2}I + K_i^\perp)\psi_k^i, \varphi_m^i \rangle_{\Gamma_i}, \\ D_{i,h}[m,n] &= \langle D_i \varphi_n^i, \varphi_m^i \rangle_{\Gamma_i}, \\ R_{i,h}[m,n] &= \langle R_i \varphi_n^i, \varphi_m^i \rangle_{\Gamma_i} \end{aligned}$$

for $k, \ell = 1, \ldots, N_i$, $m, n = 1, \ldots, M_i$ and $i = 1, \ldots, p$. In addition,

$$\underline{p}_i[m] = \langle p, \varphi_m^i \rangle_{\Gamma_i \cap \Gamma} \qquad \text{for } m = 1, \ldots, M_i.$$

To tear the global vector $\underline{u} \in \mathbb{R}^{M_S}$ we introduce the local unknowns $\underline{u}_i = A_i \underline{u} \in \mathbb{R}^{M_i}$. To ensure the global continuity, we must additionally require the interconnecting condition in the form

$$\sum_{i=1}^{p} B_i \underline{u}_i = \underline{0}. \tag{7.33}$$

In particular for $x_r \in \Gamma_{ij}$ belonging to two subdomains, the interconnecting condition (7.33) states the continuity condition

$$u_{i,h}(x_r) = u_{j,h}(x_r).$$

For $i < j$ let r_i and r_j denote the local indices of the global index r. Then there exists an index $q(r)$ so that we can define

$$B_i[q(r), r_i] = 1, \qquad B_j[q(r), r_j] = -1, \tag{7.34}$$

to end up with

$$u_{i,r_i} - u_{j,r_j} = 0,$$

where u_{i,r_i} is r_i–th entry of the local vector $\underline{u}_i$. Note that all entries of the matrices B_i which are not explicitly set to 1 or -1 are assumed to be zero. The function $q(\cdot)$ will be discussed later. For degrees of freedom which belong to more than two subdomains, we have several possibilities to enforce continuity. The two most common approaches are fully redundant Lagrange multipliers and non redundant Lagrange multipliers. In Figure 7.2 we can see the two different cases. We will use the non redundant

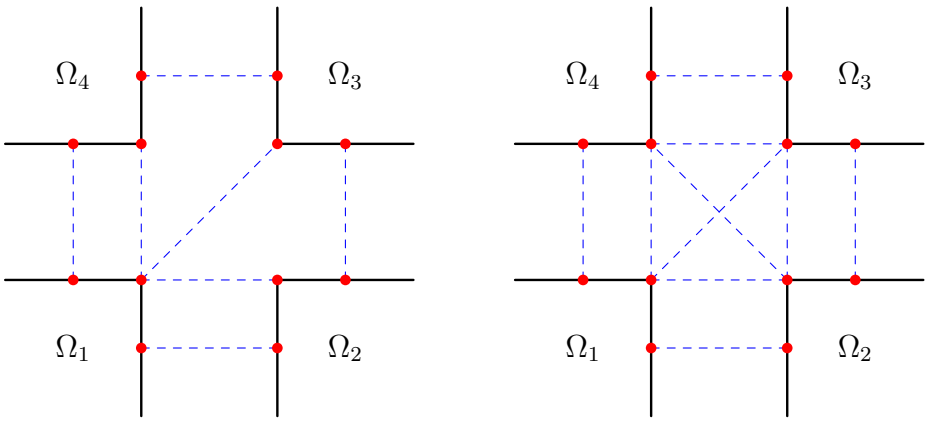


Figure 7.2: Non–redundant and fully redundant Lagrange multipliers.

multipliers. If a global degree of freedom with index r belongs to $l > 2$ subdomains

Ω_i, which have the subdomain numbers $i_1, \ldots, i_l$ with $i_m < i_n$ for $m < n$, we define r_{i_n} as the local index of the global index r in the domain Ω_{i_n}. Further we set

$$\begin{aligned} B_{i_1}[q(r) + n - 2, r_{i_1}] &= 1, \qquad \text{for } n = 2, \ldots, l, \\ B_{i_n}[q(r) + n - 2, r_{i_n}] &= -1, \qquad \text{for } n = 2, \ldots, l. \end{aligned} \tag{7.35}$$

The function $q(\cdot)$ is recursively defined by

$$q(1) := 1, \quad q(r) := q(r-1) + d(r-1) - 1,$$

where $d(r)$ is the number of subdomains adjacent to the global degree of freedom with index r.

The global equation in (7.32) can be written as

$$\sum_{i=1}^{p} A_i^\top \left[(D_{i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}_{i,h}^\top \underline{t}_i - \underline{p}_i \right] = \underline{0}. \tag{7.36}$$

Again we first discuss the case with only two adjacent subdomains. For $x_r \in \Gamma_{ij}$ with local indices r_i and r_j we can rewrite each line of this equation as

$$\left[(D_{i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}_{i,h}^\top \underline{t}_i - \underline{p}_i \right]_{r_i} + \left[(D_{j,h} + i\eta_j R_{j,h})\underline{u}_j + \widetilde{K}_{j,h}^\top \underline{t}_j - \underline{p}_j \right]_{r_j} = 0.$$

Hence, for $i < j$ we may introduce a discrete Lagrange multiplier $\lambda_{q(r)}$ to define

$$\begin{aligned} \left[(D_{i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}_{i,h}^\top \underline{t}_i - \underline{p}_i \right]_{r_i} &= -\lambda_{q(r)}, \\ \left[(D_{j,h} + i\eta_j R_{j,h})\underline{u}_j + \widetilde{K}_{j,h}^\top \underline{t}_j - \underline{p}_j \right]_{r_j} &= \lambda_{q(r)}. \end{aligned}$$

If x_r belongs to $l > 2$ subdomains (again with subdomain numbers i_n, $n = 1, \ldots, l$ and with local indices r_{i_n}), then we can rewrite each line of the global equation (7.36) as

$$\sum_{n=1}^{l} \left[(D_{i_n,h} + i\eta_{i_n} R_{i_n,h})\underline{u}_{i_n} + \widetilde{K}_{i_n,h}^\top \underline{t}_{i_n} - \underline{p}_{i_n} \right]_{r_{i_n}} = \underline{0}.$$

By defining

$$\begin{aligned} \left[(D_{i_n,h} + i\eta_{i_n} R_{i_n,h})\underline{u}_{i_n} + \widetilde{K}_{i_n,h}^\top \underline{t}_{i_n} - \underline{p}_{i_n} \right]_{r_{i_n}} &= \lambda_{q(r)+n-2}, \qquad \text{for } n = 2, .., l \\ \left[(D_{i_1,h} + i\eta_{i_1} R_{i_1,h})\underline{u}_{i_1} + \widetilde{K}_{i_1,h}^\top \underline{t}_{i_1} - \underline{p}_{i_1} \right]_{r_{i_1}} &= -\sum_{n=2}^{l} \lambda_{q(r)+n-2}, \end{aligned}$$

and by using (7.34) and (7.35) we end up with the local systems

$$(D_{i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}_{i,h}^\top \underline{t}_i - \underline{p}_i = B_i^\top \underline{\lambda} \quad \text{for } i = 1, \ldots, p.$$

Hence, the linear system (7.32) is equivalent to

$$\begin{pmatrix} D_{1,h}+i\eta R_{1,h} & \widetilde{K}_{1,h}^\top & & & & -B_1^\top \\ -\widetilde{K}_{1,h} & V_{1,h} & & & & \\ & & \ddots & & & \vdots \\ & & & D_{p,h}+i\eta R_{p,h} & \widetilde{K}_{p,h}^\top & -B_p^\top \\ & & & -\widetilde{K}_{p,h} & V_{p,h} & \\ B_1 & & \dots & B_p & & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{t}_1 \\ \vdots \\ \underline{u}_p \\ \underline{t}_p \\ \underline{\lambda} \end{pmatrix} \begin{pmatrix} \underline{p}_1 \\ \underline{0} \\ \vdots \\ \underline{p}_p \\ \underline{0} \\ \underline{0} \end{pmatrix}. \tag{7.37}$$

Remark 7.6. *Within this thesis we always consider a global Neumann boundary value problem, since this is the technical easiest choice. However, Dirichlet or mixed problems can also be solved by using this approach. For the continuous formulation this is obvious, by using the standard extension approach for the given Dirichlet data and by restricting the ansatz spaces to the Neumann boundary. How to deal with Dirichlet boundary conditions in the tearing and interconnecting is not as obvious. Two similar methods, independently appeared at the same time. These methods deal with Dirichlet boundary conditions in such a way, that the general tearing and interconnecting approach, in the way we presented it, is nearly unaffected. These methods are called all–floating BETI [111,112] and total–FETI [56]. The idea is to not enforce the given Dirichlet data in a strong way, but rather to use Lagrangian multipliers instead to ensure the Dirichlet condition. By using this approach the block diagonal parts of the matrix in* (7.37) *are unaffected, only the right hand side and the matrices B_i are modified.*

Remark 7.7. *Although in this thesis we only use a one–level tearing and interconnecting approach, it is also possible to use this algebraic deduction for dual–primal tearing and interconnecting approaches.*

Let us consider a local system in (7.37),

$$\begin{pmatrix} D_{i,h}+i\eta_i R_{i,h} & \widetilde{K}_{i,h}^\top \\ -\widetilde{K}_{i,h} & V_{i,h} \end{pmatrix} \begin{pmatrix} \underline{u}_i \\ \underline{t}_i \end{pmatrix} = \begin{pmatrix} \underline{p}_i + B_i^\top \underline{\lambda} \\ \underline{0} \end{pmatrix}, \tag{7.38}$$

which corresponds to the Galerkin discretization of the boundary integral equations (5.28). Since the associated sesquilinear form is coercive and injective, stability of the local Galerkin scheme (7.38) follows for a sufficiently small mesh width $h_i < h_0$. Because of this, the Schur complement system of (7.37),

$$\begin{aligned} \sum_{i=1}^{p} \begin{pmatrix} B_i & 0 \end{pmatrix} \begin{pmatrix} D_{i,h}+i\eta_i R_{i,h} & \widetilde{K}_{i,h}^\top \\ -\widetilde{K}_{i,h} & V_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} B_i^\top \underline{\lambda} \\ \underline{0} \end{pmatrix} \\ = -\sum_{i=1}^{p} \begin{pmatrix} B_i & 0 \end{pmatrix} \begin{pmatrix} D_{i,h}+i\eta_i R_{i,h} & \widetilde{K}_{i,h}^\top \\ -\widetilde{K}_{i,h} & V_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{p}_i \\ \underline{0} \end{pmatrix}, \end{aligned}$$

is well defined for $h_i < h_0$ and can be written as

$$F_h \underline{\lambda} = \underline{d}. \tag{7.39}$$

Note that the linear system (7.39) corresponds to the standard dual system in tearing and interconnecting domain decomposition methods.

Since we are using a boundary element approach, the discretization approach is the same for an exterior boundary value problem in an unbounded domain Ω^c. The only difference lies in the additional local equation (7.17) and the additional auxiliary variable s for the unbounded subdomain. Nevertheless, the tearing and interconnecting approach is not influenced by these additional parts. The formulation (7.37) just changes to

$$\begin{pmatrix} D_{0,h} & -\frac{1}{2}M_h^\top + K_{0,h}^\top & -D_{0,h} & & -B_0^\top \\ \frac{1}{2}M_h - K_{0,h} & V_{0,h} & \frac{1}{2}M_h + K_{0,h} & & \\ -D_{0,h} & -\frac{1}{2}M_h^\top - K_{0,h}^\top & -i\widetilde{D}_{0,h} + D_{0,h} & & \\ & & & \ddots & \vdots \\ B_0 & & & \cdots & \end{pmatrix} \begin{pmatrix} \underline{u}_0 + \underline{s} \\ \underline{t}_0 \\ \underline{s} \\ \vdots \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} -\underline{p}_0 \\ \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{pmatrix}.$$

Hence, the block for the unbounded subdomain Ω_0 is a 3×3 block, all other subblocks are still 2×2 blocks as in (7.37). A column and row manipulation leads to

$$\begin{pmatrix} D_{0,h} & -\frac{1}{2}M_h^\top + K_{0,h}^\top & & & -B_0^\top \\ \frac{1}{2}M_h - K_{0,h} & V_{0,h} & M_h & & \\ & M_h^\top & -i\widetilde{D}_{0,h} & & -B_0^\top \\ & & & \ddots & \vdots \\ B_0 & & B_0 & \cdots & \end{pmatrix} \begin{pmatrix} \underline{u}_0 \\ \underline{t}_0 \\ \underline{s} \\ \vdots \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} -\underline{p}_0 \\ \underline{0} \\ -\underline{p}_0 \\ \vdots \\ \underline{0} \end{pmatrix}.$$

The corresponding Schur complement is then given by

$$\begin{aligned} &\begin{pmatrix} B_0 & 0 & B_0 \end{pmatrix} \begin{pmatrix} D_{0,h} & -\frac{1}{2}M_h^\top + K_{0,h}^\top & \\ \frac{1}{2}M_h - K_{0,h} & V_{0,h} & M_h \\ & M_h^\top & -i\widetilde{D}_{0,h} \end{pmatrix}^{-1} \begin{pmatrix} B_i^\top \underline{\lambda} \\ \underline{0} \\ B_i^\top \underline{\lambda} \end{pmatrix} \\ &\quad + \sum_{i=1}^{p} \begin{pmatrix} B_i & 0 \end{pmatrix} \begin{pmatrix} D_{i,h} + i\eta_i R_{i,h} & \widetilde{K}_{i,h}^\top \\ -\widetilde{K}_{i,h} & V_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} B_i^\top \underline{\lambda} \\ \underline{0} \end{pmatrix} \\ &= -\begin{pmatrix} B_0 & 0 & B_0 \end{pmatrix} \begin{pmatrix} D_{0,h} & -\frac{1}{2}M_h^\top + K_{0,h}^\top & \\ \frac{1}{2}M_h - K_{0,h} & V_{0,h} & M_h \\ & M_h^\top & -i\widetilde{D}_{0,h} \end{pmatrix}^{-1} \begin{pmatrix} B_i^\top \underline{p}_0 \\ \underline{0} \\ B_i^\top \underline{p}_0 \end{pmatrix} \\ &\quad - \sum_{i=1}^{p} \begin{pmatrix} B_i & 0 \end{pmatrix} \begin{pmatrix} D_{i,h} + i\eta_i R_{i,h} & \widetilde{K}_{i,h}^\top \\ -\widetilde{K}_{i,h} & V_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{p}_i \\ \underline{0} \end{pmatrix}, \end{aligned}$$

and in short by

$$F_h \underline{\lambda} = \underline{d}.$$

Note that for an implementation we prefer to eliminate the additional unknown $\underline{s}$ in the system for the exterior problem and to use the equivalent regularized 2×2 system instead. For further details see Sections 5.7 and 5.8.

7.8 Preconditioning strategies

The preconditioning of domain decomposition methods is one of the major topics in this field. For tearing and interconnecting approaches various results were obtained. Mandel and Tezaur published the first convergence proof of the one–level FETI method [100] for two–dimensional elliptic problems. Later, Klawonn and Widlund extended these results to three–dimensional problems [91,92]. Similar results for unbounded domains were established by Pechstein [115, 116]. Most of these results provide a preconditioner C_F such that

$$\kappa(C_F F) \leq c \left(1 + \log\left(\frac{H}{h}\right)\right)^p,$$

where H is a measure of the size of the subdomains Ω_i and h describes the mesh width within the local subdomains. However, all of these results rely on the (semi–)ellipticity of the sesquilinear form corresponding to the partial differential equation. In addition, most approaches utilize kernels of local operators, which are known in advance, to construct the preconditioner. The Helmholtz equation does neither lead to an elliptic sesquilinear form, nor possess the local operators in general a kernel. Therefore, most of this established theory can not be used for the acoustic scattering problem. However, Farhat, Macedo and Lesoinne presented a preconditioner for the FETI–H method in [65]. But we do not know any analytical result for an estimate of the condition number of the preconditioned system when using this approach. Because there is no natural coarse space which can be utilized to propagate information between the subdomains, it is necessary to introduce an artificial coarse space.

Let $\underline{r}^k$ be the k–th residual of the global problem within an iterative scheme, i.e.

$$\underline{r}^k = \underline{d} - F\underline{\lambda}^k.$$

Now we modify the algorithm in such a way that the residual $\underline{r}^k$ is orthogonal to a given subspace which is represented by the columns of the matrix Q, e.g.

$$Q^\top \underline{r}^k = Q^\top(\underline{d} - F\underline{\lambda}^k) = 0. \tag{7.40}$$

This restriction is also a solution constraint, since the residual represents the jump of the Dirichlet datum on the interface,

$$\underline{r}^k = \underline{d} - F\underline{\lambda}^k = \sum_{i=1}^{p} B_i \underline{u}_i^k.$$

To enforce the orthogonality, we first introduce a new iterate $\underline{\widetilde{\lambda}}^k$ via

$$\underline{\widetilde{\lambda}}^k = \underline{\lambda}^k + \underline{\mu}^k = \underline{\lambda}^k + Q\underline{\gamma}^k \tag{7.41}$$

where $\underline{\gamma}^k$ is chosen in such a way that the orthogonality constraint holds (this is possible if $Q^\top FQ$ is invertible). By inserting $\underline{\widetilde{\lambda}}^k$ in (7.40) we obtain the linear equation system

$$Q^\top FQ\underline{\gamma}^k = Q^\top(\underline{d} - F\underline{\lambda}^k).$$

By solving this system and inserting $\underline{\gamma}^k$ in (7.41) we get an alternative representation of $\underline{\widetilde{\lambda}}^k$,

$$\underline{\widetilde{\lambda}}^k = P\underline{\lambda}^k + \underline{\lambda}^0,$$

with the projector

$$P = I - Q(Q^\top FQ)^{-1}Q^\top F$$

and

$$\underline{\lambda}^0 = Q(Q^\top FQ)^{-1}Q^\top \underline{d}.$$

By inserting this representation of $\underline{\widetilde{\lambda}} = \lim \underline{\widetilde{\lambda}}^k$ in $F\underline{\lambda} = \underline{d}$ we obtain

$$FP\underline{\lambda} + F\underline{\lambda}^0 = \underline{d}.$$

If we multiply this equation with the transposed projector $P^\top$ we get

$$P^\top FP\underline{\lambda} = P^\top \underline{d},$$

since $P^\top F\underline{\lambda}^0 = FP\underline{\lambda}^0 = 0$. This is the projected system we have to solve. Since

$$\begin{aligned} P^\top FP &= (I - FQ(Q^\top FQ)^{-1}Q^\top)F(I - Q(Q^\top FQ)^{-1}Q^\top F) \\ &= F - FQ(Q^\top FQ)^{-1}Q^\top F \\ &= FP, \end{aligned}$$

we can save one application of P and therefore one application of F in each iteration.

An important question is how to choose the subspace (the columns of Q) to which $\underline{r}^k$ should be orthogonal. As in the paper [65] we have chosen planar waves, which are evaluated at a local level. More precisely we chose for every subdomain Ω_j a set of spatial directions and evaluate the planar wave (which has the wave number k_j) in this direction on every node of the subdomain. So the matrix Q_j for the subdomain Ω_j is given by

$$Q^j[\cdot, l] = e^{ik\theta_l^\top x(\cdot)} \cdot \chi_{\Gamma_j}(x)$$

for all directions θ_l and all subdomains Ω_j, where $\chi_{\Gamma_j} = 1$ if $x \in \Gamma_j$ and 0 otherwise. The global matrix is finally constructed by

$$Q = [Q^1 \cdots Q^i \cdots Q^p].$$

By using this local approach one speeds up the construction of $Q^\top FQ$, since only a few local problems have to be solved for every direction (when computing FQ). Still it is not ensured that $Q^\top FQ$ is invertible at all. To guarantee the invertibility we may have to eliminate certain columns of Q. In our case, this is realized during a LU factorization of $Q^\top FQ$.

Another question is how to chose the directions θ_l. In our numerical examples we got the best results, when we distributed the directions uniformly. We may achieve better results by using a problem dependent choice. The optimal number of directions per subdomain is also not obvious. More directions generally improve the convergence rate, but increase the computational effort to construct the preconditioner. By using too many directions, the matrix $Q^\top FQ$ becomes numerically instable and a lot of directions have to be deleted. In a very few numerical examples we got a better iteration number when using fewer directions, but this happened only for rather small problems. Numerical results for the FETI–H method using this preconditioning approach can be found in [65, 132, 141]. These papers state that this preconditioner is stable with respect to h, H and k, if the number of directions is chosen correctly. Note that all of these statements are based on numerical observations and are not proved in a rigorous way. The correct choice of directions per subdomain is also a difficile problem. Interestingly, the method seems already to be relatively stable with respect to h, when no preconditioner is used at all.

7.9 Numerical examples

In this section we give several numerical examples in order to test the presented theory for the acoustic scattering. Some of these numerical examples have been already published in [128]. In all the upcoming examples we use as Robin operator R the regularized hyper singular operator $\widetilde{D}$ which is discretized only on the internal (with respect to an interface) degrees of freedom, for a more detailed discussion see Section 5.5. In the first example we examine our algorithm on robustness with respect to local eigen wave numbers. In the second example we investigate the behaviour of the method for the case of multiple subdomains. Furthermore, in this example we also study the effectiveness of the proposed preconditioner. Afterwards we apply the method to different geometries, e.g. a sphere and a long bar. Thereafter we give an example with jumping coefficients and test the condition of the system in this case, i.e. we examine the iteration numbers. Finally, we give an example for an unbounded domain Ω^c.

7.9.1 Two bricks

In this example we consider the Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \quad \text{for } x \in \Omega, \\ \gamma_1 U &= p \quad \text{for } x \in \Gamma. \end{aligned}$$

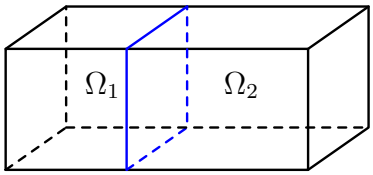


Figure 7.3: Two bricks.

The domain Ω is given by $(-1.0, 1.5) \times (0.0, 1.0) \times (0.0, 1.0)$, which is divided into two subdomains by the yz–plane, see Figure 7.3. As an exact solution we use the fundamental solution $\hat{U}(x)$ with the singularity in $\hat{x} = (2.0, 0.0, 1.5)^\top$, i.e.

$$\hat{U}(x) = \frac{e^{ik|x-\hat{x}|}}{|x-\hat{x}|}.$$

The boundary element discretization of the coupled variational formulation (7.9) and (7.10) is done with respect to a globally uniform boundary mesh of N_i plane triangular elements with M_i nodes per subdomain and by using piecewise constant basis functions ψ_m^i and piecewise linear continuous basis functions ϕ_n^i. The linear system (7.39) is solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. First we consider the global wave number $k = 2.0$, which neither corresponds to a Dirichlet nor to a Neumann eigenvalue of both subproblems. The results, which confirm the error estimate (7.31), are given in Table 7.1. The iteration numbers seem to be bounded or growing logarithmically with respect to h. Such a behaviour was already observed for the FETI–H method in [65].

N_i	M_i	It	rel. L_2–error
12	8	3	0.528489
48	26	8	0.139577
192	98	10	0.032139
768	386	11	0.007769
3072	1538	12	0.001777
12288	6146	13	0.000441

Table 7.1: Numerical results for two subdomains, $k = 2.0$.

In a second example we consider the global wave number $k = \sqrt{3}\pi \approx 5.4414$, which corresponds to the first Dirichlet and Neumann eigenvalue of the unit cube Ω_1. The results given in Table 7.2 confirm the stability of the proposed approach.

N_i	M_i	It	rel. L_2–error
12	8	3	1.517652
48	26	8	12.03050
192	98	14	0.516757
768	386	15	0.080376
3072	1538	15	0.020412
12288	6146	15	0.005689

Table 7.2: Numerical results for two subdomains, $k = \sqrt{3}\pi$.

7.9.2 Multiple subdomains

Again we apply our method to the Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{for } x \in \Omega, \\ \gamma_1 U &= p \qquad \text{for } x \in \Gamma. \end{aligned} \tag{7.42}$$

For this example we choose the domain Ω as the unit cube $(0,1)^3$. We divide this cube uniformly in p^3 subcubes with $p = 2, 3, 4, 5$. As an exact solution we consider the fundamental solution $\hat{U}(x)$ with the center in $\hat{x} = (-0.2, 2.0, 1.0)^\top$. The boundary element discretization of the coupled variational formulation (7.9) and (7.10) is done with respect to a globally uniform boundary mesh of N_i plane triangular elements with M_i nodes per subdomain and by using piecewise constant basis functions ψ_m^i and piecewise linear continuous basis functions ϕ_n^i. The size of the Schur complement system, i.e. the number of Lagrangian multipliers is given by Λ_i. The linear system (7.39) is solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. We further consider a global wave number $k = 2.0$. The numerical results are given in Table 7.3. Note that no global preconditioner is used in this example. Again we observe a quadratic convergence order as foreseen by the theory. It seems that the iteration numbers are stable with respect to h, but increase quite fast with respect to the number of subdomains.

Therefore, we want to test the efficiency of the proposed preconditioner. For this test we consider different wave numbers $k = 1.0, 2.0, 4.0, 8.0$ and different amounts of plane waves per subdomain $\theta_n = 0, 2, 4, 6, 8$ used by the preconditioner. The same triangulation as in the unpreconditioned example is used. The results are given in Table 7.4. Since every preconditioned iteration step is at least as expensive as two non–preconditioned iteration steps, the preconditioner is possibly only useful in the case of many subdomains. By scaling the amount of plane waves per subdomain in a linear way, with respect to the wave number and the number of subdomains, it seems that the iteration numbers remain constant. The observation that the iteration numbers appear to be stable with respect to h also holds for the preconditioned case.

p^3	N_i	M_i	Λ_i	It	rel. L_2–error
8	24	14	49	23	0.3195067
8	96	50	139	29	0.0982673
8	384	194	462	31	0.0252838
8	1536	770	1687	35	0.0060063
8	6144	3074	6439	41	0.0013786
27	24	14	206	54	0.1599704
27	96	50	602	58	0.0450600
27	384	194	2042	59	0.0109017
27	1536	770	7514	62	0.0024980
64	24	14	531	100	0.0962217
64	96	50	1575	105	0.0252739
64	384	194	5391	104	0.0059219
64	1536	770	19935	105	0.0014740
125	24	14	1084	165	0.0634685
125	96	50	3244	156	0.0156664
125	384	194	11164	137	0.0038170
125	1536	770	41404	137	0.0007976

Table 7.3: Numerical results for p^3 subdomains and $k = 2.0$.

7.9.3 Sphere

In this example we again solve the Neumann boundary value problem (7.42). The domain Ω is given by the unit sphere with the origin as center. We divide the unit sphere in 8 and 27 subdomains, see Figure 7.4. The geometrical properties of the boundary mesh are of rather poor quality. This is due to the construction of the meshes, which were originally boundary meshes of the subdivision of the unit cube. These meshes were afterwards projected onto the unit sphere. For this example the wave number $k = 2.0$ is considered. As an exact solution we use the fundamental solution $\hat{U}(x)$ with the singularity in $(-0.9, 0.0, 2.0)^\top$. The linear system is solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. The results which are obtained without the usage of a global preconditioner can be found in Table 7.5.

7.9.4 Long bar

In this example we once again consider the Neumann boundary value problem (7.42). The domain Ω is given by $(0, 100) \times (0, 1) \times (0, 1)$ and it is divided into 100 unit cubes, see Figure 7.5. We use the fundamental solution of the Helmholtz equation $\hat{U}$ with the singularity in $(-0.9, 0.0, 2.0)^\top$ as exact solution. For this experiment, the wave number k is set to 1.0. We test this example once without a preconditioner and once

p^3	k	l	$\theta_n = 0$	$\theta_n = 2$	$\theta_n = 4$	$\theta_n = 6$	$\theta_n = 8$
8	2.0	0	23	15	11	7	8
		1	29	21	19	17	16
		2	31	24	23	21	21
		3	35	29	27	26	24
	4.0	0	27	19	13	6	12
		1	32	23	21	18	16
		2	35	27	24	22	21
		3	40	31	27	25	24
	8.0	0	43	31	16	2	1
		1	49	39	28	21	17
		2	54	41	31	25	22
		3	56	42	35	29	27
27	2.0	0	54	25	17	12	13
		1	58	32	26	24	22
		2	59	34	28	27	25
		3	62	37	31	29	28
	4.0	0	69	36	23	17	22
		1	70	38	33	28	24
		2	68	39	33	29	27
		3	71	44	39	36	31
	8.0	0	88	56	39	25	2
		1	80	53	42	36	31
		2	81	55	44	37	32
		3	88	53	47	41	35
64	2.0	0	100	32	19	13	14
		1	105	39	28	25	24
		2	104	41	32	23	25
		3	105	43	35	31	29
	4.0	0	130	47	28	18	19
		1	128	52	39	33	29
		2	121	50	41	36	29
		3	119	52	45	40	36
	8.0	0	157	93	69	49	24
		1	162	89	70	59	50
		2	145	77	61	53	47
		3	149	70	60	50	42
125	2.0	0	165	37	21	14	14
		1	156	43	31	26	26
		2	137	44	35	30	29
		3	137	47	40	31	30
	4.0	0	215	59	31	18	17
		1	205	60	40	35	33
		2	191	56	44	38	34
		3	181	57	49	43	38
	8.0	0	254	138	88	57	35
		1	267	128	95	74	64
		2	252	124	81	64	57
		3	234	93	71	60	51

Table 7.4: Iteration numbers for the unit cube with p^3 subdomains and different wave numbers using the proposed preconditioner with different amounts of plane waves per subdomain.

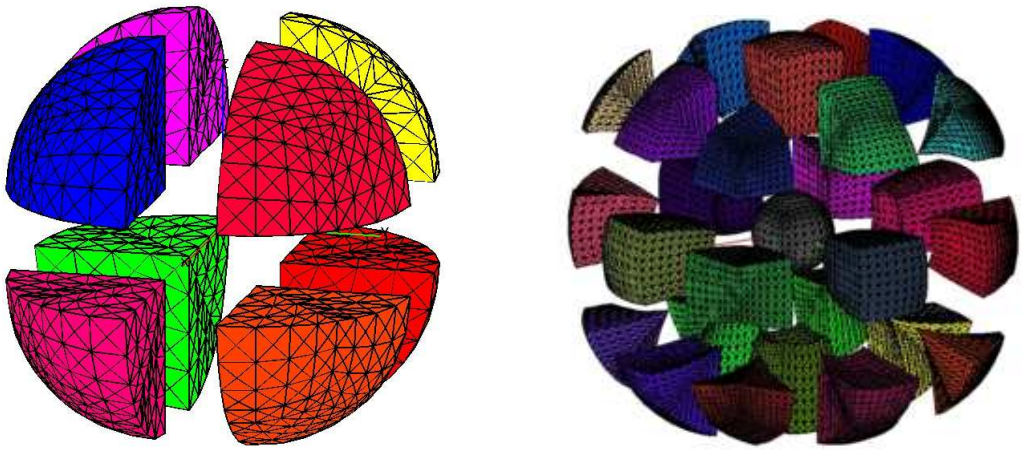


Figure 7.4: The unit sphere divided in 8 and 27 subdomains.

level	N_i	M_i	It	rel. L_2–error
0	24	14	28	0.096700
1	96	50	38	0.030646
2	384	194	42	0.012988
3	1536	770	47	0.004205

(a) 8 subdomains.

level	N_i	M_i	It	rel. L_2–error
0	24	14	72	0.166084
1	96	50	78	0.115071
2	384	194	80	0.078111
3	1536	770	87	0.050307

(b) 27 subdomains.

Table 7.5: Results for the unit sphere with wave number $k = 2.0$.

again with one plane wave per subdomain. We use a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. The results are given in Table 7.6. We can see that for this geometry the preconditioner is already very efficient with only one plane wave per subdomain.

level	N_i	M_i	Λ_i	It1	It2	rel. L_2–error
0	24	14	495	16	121	0.0767380
1	96	50	1287	15	116	0.0199389
2	384	194	4059	17	112	0.0039138
3	1536	770	14355	21	116	0.0011179

Table 7.6: Iteration numbers with (It1) and without (It2) preconditioning for the bar with $k = 1.0$.

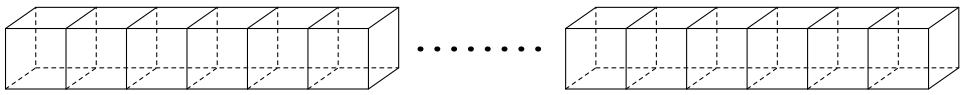


Figure 7.5: Long bar.

7.9.5 Jumping coefficients

Now we solve the Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2(x)U &= 0 \qquad \text{for } x \in \Omega, \\ \gamma_1 U &= p \qquad \text{for } x \in \Gamma, \end{aligned}$$

with a piecewise constant function $k(x)$. Let Ω be the unit cube $(0,1)^3$, which is divided into p^3 subcubes for $p = 2, 3, 4, 5$. For $p = 3, 5$ we assume that the subdomains are colored in a checkerboard way. For $p = 2, 4$ we assume that we have black and white layers, see Figure 7.6. In the white domains we set $k_1 = 1.0$ and in the black domains $k_2 = 4.0$. Since we do not have an exact solution for this example, we only give iteration numbers. Nevertheless, the right hand side is imposed by a fundamental solution with wave number $k = 2.0$ and $\hat{x} = (-0.9, 0.0, 2.0)^\top$. Once more a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$ is used to solve the linear system. We tested the iteration numbers without preconditioning and a second time with 4 plane waves per subdomain. For these plane waves we always used the local wave number k_i. The results are given in Table 7.7. As we can see the iteration numbers hardly differ from those, if we would choose $k = 4.0$ everywhere, see Table 7.4. In the case of a layer–like distribution of the wave numbers, the iteration numbers seem to be slightly higher, on the other hand in the checkerboard case they seem to be slightly lower.

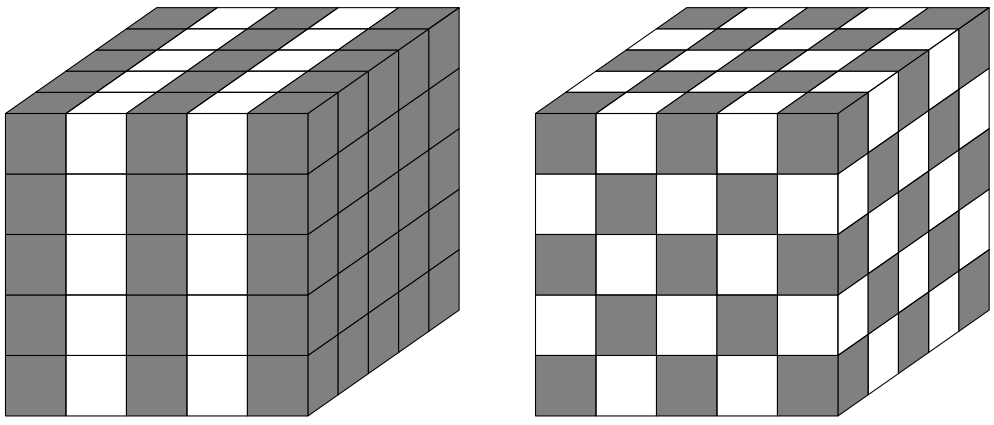


Figure 7.6: Examples for a layer like and checkerboard like distribution of the coefficients k_1 and k_2.

p^3	θ_n	level	It
8	0	0	27
8	0	1	33
8	0	2	35
8	0	3	38
8	4	0	12
8	4	1	21
8	4	2	24
8	4	3	27
64	0	0	122
64	0	1	129
64	0	2	127
64	0	3	125
64	4	0	27
64	4	1	44
64	4	2	47
64	4	3	51

(a) Layer–like domain decomposition.

p^3	θ_n	level	It
27	0	0	62
27	0	1	67
27	0	2	67
27	0	3	71
27	4	0	21
27	4	1	32
27	4	2	35
27	4	3	41
125	0	0	191
125	0	1	193
125	0	2	176
125	0	3	168
125	4	0	28
125	4	1	41
125	4	2	42
125	4	3	45

(b) Checkerboard–like domain decomposition.

Table 7.7: Iteration numbers in the case of jumping coefficients.

7.9.6 Exterior boundary value problem

In this example we solve the Neumann boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \qquad \text{for } x \in \Omega^c, \\ \gamma_1 U &= p \qquad \text{for } x \in \Gamma, \end{aligned} \tag{7.43}$$

for an unbounded domain Ω^c. The geometrical domain decomposition for this example can be explained as follows: we take the unit cube $(0,1)^3$, divide it into $3\times3\times3 = 27$ subcubes and exclude the middle subcube $[1/3, 2/3]^3$, i.e. $\Omega^c = \mathbb{R}^3 \setminus [1/3, 2/3]^3$. So we have 26 bounded subdomains and one unbounded subdomain which is the complement of the unit cube $(0,1)^3$. We use the fundamental solution $\hat{U}(x)$ with the singularity in $(0.5, 0.5, 0.5)^\top$ as an exact solution and the wave number $k = 1.0$ for all domains. We solve the variational formulation (7.14)–(7.17) by using a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-8}$. For the preconditioning we use one plane wave per subdomain. The size of the Schur complement system, i.e. the number of Lagrangian multipliers is given by Λ_i. The results are given in Table 7.8. Once again we observe the estimated convergence rate as given in (7.31). The iteration numbers also seem to be rather stable with respect to h.

level	N_b	M_b	N_u	M_u	Λ_i	It	rel. L_2–error
0	24	14	216	110	302	20	0.0171532
1	96	50	864	434	986	33	0.0031446
2	384	194	3456	1730	3578	37	0.0006652
3	1536	770	13824	6914	13658	43	0.0001682

Table 7.8: Results for the exterior Helmholtz problem.

8 DOMAIN DECOMPOSITION METHODS FOR ELECTROMAGNETIC SCATTERING

Several domain decomposition approaches have been applied to the electromagnetic wave equation, see, e.g., [3, 133]. Mostly the underlying problem is the eddy current problem, which leads to elliptic bilinear forms or operators. For this problem also preconditioners were developed and analyzed, see for example, [4, 90]. The tearing and interconnecting method is also well suited for the eddy current problem, since preconditioners C_F for the tearing and interconnecting system F were developed which, just as in the scalar case, lead to condition number estimates of the form

$$\kappa(C_F F) \leq c \left(1 + \log\left(\frac{H}{h}\right)\right)^p ,$$

see [134, 135]. For the scattering problem, less theory is available. There are some approaches for the transmission problem, see for example [87]. For boundary value problems with multiple subdomains, a FETI–DP approach was applied and tested for the electromagnetic wave equation in [97], the resulting method is called FETI–DPEM. However, only numerical results are given in the above mentioned paper and in the references therein.

In this chapter we will proceed in an analog fashion as in the last chapter, although we will skip some redundant parts. Before we formulate again a classical Dirichlet domain decomposition, we have to carry over the Hodge–splitting of the trace spaces to spaces defined on the skeleton. Thereafter we introduce suitable Robin interface conditions and reformulate the local Steklov–Poincaré operators, such that we get a stable local formulation with respect to local eigen wave numbers. Afterwards we introduce a conforming boundary element formulation based on Raviart–Thomas elements. Since the deduction of the tearing and interconnecting approach does not rely upon the underlying problem, we will skip the rededuction and state the results only. In the end of the chapter we give a numerical example.

In this chapter we will use the same geometrical assumptions, definitions and notations as they were given in Section 7.1. In addition, we make the stronger assumption that all subdomains are Lipschitz polyhedrons and not only Lipschitz domains.

8.1 Spaces on the skeleton

The following construction is close to the one in [29, Section 4.3.1]. The trace space $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma}, \Gamma_S)$ is defined as the Dirichlet trace of $\mathbf{H}(\mathbf{curl}, \Omega)$ on the skeleton. We

define the norm by using local restrictions. For $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ the norm is given by

$$\|\mathbf{u}\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)} := \sum_{i=1}^{p} \left\|\mathbf{u}_{|\Gamma_i}\right\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_i)}.$$

The space $\mathbf{H}_{\|}^{1/2}(\Gamma_S)$ is defined as the Dirichlet trace of $\mathbf{H}^1(\Omega)$ on the skeleton, it's norm is also locally defined as the norm of $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$. Since the surface curl of a function depends on the direction of the normal vector we use $\mathrm{curl}_\Gamma \mathbf{u}$ only on the subdomain level. Note that $(\mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_i})_{|\Gamma_{ij}} = -(\mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_j})_{|\Gamma_{ij}}$. Now let us define a Hodge–type splitting for functions in $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$.

As in Subsection 4.3.2 we consider the local subproblems

$$\begin{aligned} \Delta\Psi_i &= 0 \qquad \text{in } \Omega, \\ \gamma_1\Psi_i &= \omega_i \qquad \text{on } \Gamma \end{aligned}$$

with $\omega_i = \mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_i}$. Again we set $\mathbf{W}_i = \mathbf{grad}\,\Psi_i \in \mathbf{H}(\mathrm{div}\ 0, \Omega_i)$. Now we can define the function $\mathbf{W}$ by $\mathbf{W}_{|\Omega_i} = \mathbf{W}_i$. We have $\mathbf{W} \in \mathbf{H}(\mathrm{div}\ 0, \Omega)$ since $\gamma_n \mathbf{W}_i = -\gamma_n \mathbf{W}_j$ on Γ_{ij}, see [143]. Let us define

$$P^S := \boldsymbol{\gamma}_D \circ J^S \quad : \quad \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S) \to \mathbf{H}_{\|}^{1/2}(\Gamma_S),$$

where $J^S\mathbf{u} := L\mathbf{W}$ and L is defined as in Subsection 4.3.2.

Corollary 8.1. *The operator P^S is a projection and satisfies*

- $\mathrm{curl}_\Gamma(P^S\mathbf{u})_{|\Gamma_i} = \mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_i}$ *for all* $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$.
- $P^S\mathbf{u} = \mathbf{0}$ *for* $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ *and* $\mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_i} = 0$ *for all* i.
- $\left\|P^\Gamma\mathbf{u}\right\|_{\mathbf{H}_{\|}^{1/2}(\Gamma_S)} \le C \sum_{i=1}^{p} \left\|\mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_i}\right\|_{H^{-1/2}(\Gamma_i)}$ *for all* $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$.

Proof. If we utilize Lemma 4.31 we get on the global level

$$\mathrm{curl}_\Gamma P^S\mathbf{u} = \mathrm{curl}_\Gamma \boldsymbol{\gamma}_D J^S\mathbf{u} = \mathrm{curl}_\Gamma \boldsymbol{\gamma}_D L\mathbf{W} = \gamma_n\, \mathbf{curl}\, L\mathbf{W} = \gamma_n \mathbf{W}$$

and locally

$$\gamma_n \mathbf{W}_{|\Gamma_i} = \gamma_1 \Psi_i = \omega_i = \mathrm{curl}_\Gamma \mathbf{u}_{|\Gamma_i}.$$

The other properties follow immediately. □

Now we define

$$\mathbf{X}(\mathrm{curl}_\Gamma, \Gamma_S) := P^S(\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)),$$
$$\mathbf{N}(\mathrm{curl}_\Gamma, \Gamma_S) := \mathrm{Ker}(P^S) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S),$$

and finally we end up with the stable direct splitting

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S) = \mathbf{X}(\mathrm{curl}_\Gamma, \Gamma_S) \oplus \mathbf{N}(\mathrm{curl}_\Gamma, \Gamma_S). \tag{8.1}$$

It is possible to establish some extra regularity for one of the subspaces.

Corollary 8.2. *The embedding* $\mathbf{X}(\mathrm{curl}_\Gamma, \Gamma) \hookrightarrow \mathbf{L}_t^2(\Gamma_S)$ *is compact.*

Proof. This follows immediately from the compact embedding $\mathbf{H}_\parallel^{1/2}(\Gamma_S) \hookrightarrow \mathbf{L}_t^2(\Gamma_S)$. □

Due to the construction, the restriction of the splitting to the subdomains coincides with the local splittings. The sign flip operators $\mathcal{X}$ and $\mathcal{Y}$ for $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ are defined as in Subsection 4.3.2, i.e. $\mathcal{X}$ is defined by

$$(\mathbf{u}, \mathbf{v}) \mapsto (-\mathbf{u}, \mathbf{v}),$$
$$\mathbf{X}(\mathrm{curl}_\Gamma, \Gamma_S) \times \mathbf{N}(\mathrm{curl}_\Gamma, \Gamma_S) \to \mathbf{X}(\mathrm{curl}_\Gamma, \Gamma_S) \times \mathbf{N}(\mathrm{curl}_\Gamma, \Gamma_S),$$

and $\mathcal{Y}$ by $\mathcal{Y} := -\mathcal{X}$.

8.2 Dirichlet domain decomposition methods

In this section we deduce a variational formulation of the domain decomposition approach, starting with the global Neumann boundary value problem. As in the Helmholtz case it would also be possible to treat Dirichlet or mixed boundary value problems, however we once again restrict ourselves to the technically easiest case of a Neumann boundary value problem.

Let us assume that the global Neumann boundary value problem

Find $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2(x)\mathbf{U} &= \mathbf{0} \qquad \text{in } \Omega, \\ \boldsymbol{\gamma}_N \mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma \end{aligned} \tag{8.2}$$

admits a unique solution. The corresponding variational formulation is given by

Find $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$\int_\Omega \left[\mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\overline{\mathbf{V}}(x) - k^2(x)\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\right] \mathrm{d}x - \int_\Gamma \mathbf{p}(x) \cdot \boldsymbol{\gamma}_D \overline{\mathbf{V}}(x) \mathrm{d}s_x = 0$$

for all $\mathbf{V} \in \mathbf{H}(\mathbf{curl}, \Omega)$.

To split this global problem into smaller local ones, we have to determine appropriate interface conditions. From

$$\begin{aligned}
&\int_\Omega \left[\mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\overline{\mathbf{V}}(x) - k^2\mathbf{U}(x) \cdot \overline{\mathbf{V}}\right] \mathrm{d}x - \int_\Gamma \mathbf{p}(x) \cdot \boldsymbol{\gamma}_D \overline{\mathbf{V}}(x) \mathrm{d}s_x \\
&\quad = \int_\Omega \left[\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x) - k^2\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\right] \mathrm{d}x \\
&\quad = \sum_{i=1}^{p} \int_{\Omega_i} \left[\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x) - k^2\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\right] \mathrm{d}x \\
&\quad = \sum_{i=1}^{p} \left[\int_{\Omega_i} \mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\overline{\mathbf{V}}(x) - k^2\mathbf{U}(x) \cdot \overline{\mathbf{V}}(x)\right] \mathrm{d}x \\
&\qquad - \sum_{i,j=1}^{p} \int_{\Gamma_{i,j}} \boldsymbol{\gamma}_{N,i}\mathbf{U}(x) \cdot \boldsymbol{\gamma}_{D,i}\overline{\mathbf{V}}(x) \mathrm{d}s_x - \int_\Gamma \mathbf{p}(x) \cdot \boldsymbol{\gamma}_D \overline{\mathbf{V}}(x) \mathrm{d}s_x = 0
\end{aligned}$$

for all $\mathbf{U}, \mathbf{V} \in \mathbf{H}(\mathbf{curl}, \Omega)$ we conclude that

$$\sum_{i,j=1}^{p} \int_{\Gamma_{i,j}} \boldsymbol{\gamma}_{N_i}\mathbf{U}(x) \cdot \boldsymbol{\gamma}_{D,i}\overline{\mathbf{V}}(x) \mathrm{d}s_x = 0.$$

This provides us the necessary interface conditions. By using a density argument we get

$$\boldsymbol{\gamma}_{N,i}\mathbf{U} + \boldsymbol{\gamma}_{N,j}\mathbf{U} = \mathbf{0} \qquad \text{on } \Gamma_{ij}.$$

The localized formulation of problem (8.2) is then given by:

Find $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$\begin{aligned}
\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_i - k_i^2\mathbf{U}_i &= \mathbf{0} \qquad \text{in } \Omega, \\
\boldsymbol{\gamma}_{N,i}\mathbf{U}_i + \boldsymbol{\gamma}_{N,j}\mathbf{U}_j &= \mathbf{0} \qquad \text{on } \Gamma_{ij}, \\
\boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma,
\end{aligned}$$

with $\mathbf{U}_i = \mathbf{U}_{|\Omega_i}$.

Note that the continuity of the Dirichlet trace $\boldsymbol{\gamma}_D\mathbf{U}_i = \boldsymbol{\gamma}_D\mathbf{U}_j$ is already incorporated in the space $\mathbf{H}(\mathbf{curl}, \Omega)$. By introducing local functions $\mathbf{U}_i$ and explicitly enforcing the continuity of the Dirichlet trace we can reformulate this as:

Find $\mathbf{U}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i)$ for $i = 1, \ldots, p$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_i - k_i^2\mathbf{U}_i &= \mathbf{0} && \text{in } \Omega_i, \\ \boldsymbol{\gamma}_D\mathbf{U}_i - \boldsymbol{\gamma}_D\mathbf{U}_j &= \mathbf{0} && \text{on } \Gamma_{ij}, \\ \boldsymbol{\gamma}_{N,i}\mathbf{U}_i + \boldsymbol{\gamma}_{N,i}\mathbf{U}_j &= \mathbf{0} && \text{on } \Gamma_{ij}, \\ \boldsymbol{\gamma}_{N,i}\mathbf{U} &= \mathbf{p} && \text{on } \Gamma \cap \Gamma_i. \end{aligned}$$

At this point, for simplicity we assume that for all i, k_i is not an eigen wave number of the local Neumann boundary value problem in Ω_i. We soon get rid of this constraint. By using the local Steklov–Poincaré operators T_i, which are well defined due to the previous assumption, we can describe solutions of the partial differential equation in the subdomains. The corresponding global problem then reads:

Find $\mathbf{u}_i = \boldsymbol{\gamma}_D\mathbf{U}_i \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_i)$ for $i = 1, \ldots, p$ such that

$$\begin{aligned} \boldsymbol{\gamma}_D\mathbf{u}_i - \boldsymbol{\gamma}_D\mathbf{u}_j &= \mathbf{0} && \text{on } \Gamma_{ij}, \\ \mathsf{T}_i\mathbf{u}_i + \mathsf{T}_j\mathbf{u}_j &= \mathbf{0} && \text{on } \Gamma_{ij}, \\ \mathsf{T}_i\mathbf{u}_i &= \mathbf{p} && \text{on } \Gamma \cap \Gamma_i. \end{aligned}$$

By enforcing the continuity of the Dirichlet traces in a strong manner we get:

Find $\mathbf{u} = \boldsymbol{\gamma}_D\mathbf{U} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ such that

$$\begin{aligned} \mathsf{T}_i\mathbf{u}_{|\Gamma_i} + \mathsf{T}_j\mathbf{u}_{|\Gamma_j} &= \mathbf{0} && \text{on } \Gamma_{ij}, \\ \mathsf{T}_i\mathbf{u}_{|\Gamma_i} &= \mathbf{p} && \text{on } \Gamma \cap \Gamma_i. \end{aligned} \tag{8.3}$$

The variational form is then given by:

Find $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \mathsf{T}_i\mathbf{u}(x) \cdot \overline{\mathbf{v}}(x) \mathrm{d}s_x = \int_{\Gamma} \mathbf{p}(x) \cdot \overline{\mathbf{v}}(x) \mathrm{d}s_x \tag{8.4}$$

for all $\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$.

If we assume that all local Steklov–Poincaré operators T_i are well defined, we conclude the generalized Gårding inequality

$$\mathrm{Re}\left(\sum_{i=1}^{p} \langle \mathsf{T}_i\mathbf{u}, \mathcal{X}\mathbf{u}\rangle_{\Gamma_i} + C(\mathbf{u}, \mathcal{X}\mathbf{u})\right) \geq c_1 \sum_{i=1}^{p} \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_i)} \geq c_2 \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)}$$

for all $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ where $C(\cdot,\cdot)$ is a compact sesquilinear form. Since the global boundary value problem was assumed to be uniquely solvable, and since all local Steklov–Poincaré operators are assumed to be well defined, a Galerkin discretization of (8.4) is stable for a sufficient small mesh size h and admits related a priori error estimates. However, since we want to use a tearing and interconnecting approach, we have to prove solvability of the local problems for all wave numbers k_i. Therefore, we have to modify the interface conditions.

8.3 Robin interface conditions

As for the Helmholtz equation we use an operator R_i to describe the modified Robin transmission conditions. We also carry over the notation for η_{ij} and η_i, see Section 7.3. An equivalent problem to (8.3) is given by:

Find $\boldsymbol{\gamma}_D\mathbf{U} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ such that

$$\begin{aligned}(\mathsf{T}_i + i\eta_{ij}\mathsf{R}_{ij})\boldsymbol{\gamma}_D\mathbf{U}_{|\Gamma_i} + (\mathsf{T}_j + i\eta_{ij}\mathsf{R}_{ij})\boldsymbol{\gamma}_D\mathbf{U}_{|\Gamma_j} &= \mathbf{0} \qquad \text{on } \Gamma_{ij},\\ \mathsf{T}_i\boldsymbol{\gamma}_D\mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma_i \cap \Gamma.\end{aligned}$$

Since the local Steklov–Poincaré operators T_i are not well defined for eigen wave numbers of the local subproblems, we exchange the local Dirichlet–to–Neumann maps by systems, see Section 6.5. This leads to the formulation:

Find $\mathbf{u} = \boldsymbol{\gamma}_D\mathbf{U} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ and $\mathbf{t}_i = \boldsymbol{\gamma}_N\mathbf{U}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$ such that

$$\begin{aligned}(\mathsf{N}_i + i\eta_i\mathsf{R}_i)\mathbf{u}_{|\Gamma_i} + (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i + (\mathsf{N}_j + i\eta_j\mathsf{R}_j)\mathbf{u}_{|\Gamma_j} + (\frac{1}{2}\mathsf{I} + \mathsf{B}_j)\mathbf{t}_j &= \mathbf{0} \quad \text{on } \Gamma_{ij},\\ (\mathsf{N}_i + i\eta_i\mathsf{R}_i)\mathbf{u}_{|\Gamma_i} + (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i &= \mathbf{p} \quad \text{on } \Gamma_i \cap \Gamma,\\ (-\frac{1}{2}\mathsf{I} + \mathsf{C}_k)\mathbf{u}_{|\Gamma_i} + \mathsf{S}_k\mathbf{t}_i &= \mathbf{0} \quad \text{on } \Gamma_i.\end{aligned}$$

The variational formulation is therefore given by:

Find $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ and $\mathbf{t}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$ for $i = 1, \ldots, p$ such that

$$\sum_{i=1}^{p}\left[\langle \mathsf{N}_i\mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i}\rangle_{\Gamma_i} + \langle(\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i, \mathbf{v}_{|\Gamma_i}\rangle_{\Gamma_i} + \langle i\eta_i\mathsf{R}_i\mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i}\rangle_{\Gamma_i}\right] = \int_\Gamma \mathbf{p}(x)\cdot\mathbf{v}(x)\mathrm{d}s_x \tag{8.5}$$

for all $\mathbf{v} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_S)$ and

$$\langle \mathsf{S}_i\mathbf{t}_i, \boldsymbol{\mu}_i\rangle_{\Gamma_i} + \langle(-\frac{1}{2}\mathsf{I} + \mathsf{C}_i)\mathbf{u}_{|\Gamma_i}, \boldsymbol{\mu}_i\rangle_{\Gamma_i} = 0 \tag{8.6}$$

for all $\boldsymbol{\mu}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i), i = 1, \ldots, p$.

Theorem 8.3. *The coupled variational problem* (8.5) *and* (8.6) *admits a unique solution* $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ *and* $\mathbf{t}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$. *In particular, the associated sesquilinear form fulfills a generalized Gårding inequality. Moreover, if the Neumann boundary value problem* (8.2) *is uniquely solvable, the associated sesquilinear form is also injective.*

Proof. Since the sesquilinear forms of the local Robin systems fulfill a generalized Gårding inequality and because of the splitting properties of the skeleton space $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$, see Section 8.1, the global generalized Gårding inequality

$$\begin{aligned}\mathrm{Re}\Bigg(\sum_{i=1}^{p} \left\langle \begin{pmatrix} \mathsf{N}_i & (\frac{1}{2}\mathsf{I} + \mathsf{B}_i) \\ -(\frac{1}{2}\mathsf{I} - \mathsf{C}_i) & \mathsf{S}_i \end{pmatrix} \begin{pmatrix} \mathbf{u}_{|\Gamma_i} \\ \mathbf{t}_i \end{pmatrix}, \begin{pmatrix} \mathcal{Y}\mathbf{u}_{|\Gamma_i} \\ \mathcal{X}\mathbf{t}_i \end{pmatrix} \right\rangle_{\Gamma_i} + C((\mathbf{u}, \underline{\mathbf{t}}), (\mathbf{u}, \underline{\mathbf{t}})) \Bigg) \\ \geq c \cdot \left(\|\mathbf{u}\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)} + \sum_{i=1}^{p} \|\mathbf{t}_i\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)} \right).\end{aligned}$$

holds for all $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ and $\mathbf{t}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$ with a compact sesquilinear form $C(\cdot, \cdot)$. It remains to prove injectivity. Let $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ and $\mathbf{t}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$ for $i = 1, \ldots, p$ be a non–trivial solution of the homogeneous system

$$\sum_{i=1}^{p} \left[\langle \mathsf{N}_i \mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} + \langle i\eta \mathsf{R}_i \mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} \right] = 0$$

for all $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ and

$$\langle \mathsf{S}_i \mathbf{t}_i, \boldsymbol{\tau}_i \rangle_{\Gamma_i} - \langle (\frac{1}{2}\mathsf{I} - \mathsf{C}_i)\mathbf{u}_{|\Gamma_i}, \boldsymbol{\tau}_i \rangle_{\Gamma_i} = 0$$

for all $\boldsymbol{\tau}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i), i = 1, \ldots, p$. With the definition of R_i and η_i we also have

$$\sum_{i=1}^{p} \left[\langle \mathsf{N}_i \mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} \right] = 0$$

for all $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$. Let us define

$$\boldsymbol{\phi}_i(x) = \boldsymbol{\Psi}_i^S \mathbf{t}_i(x) + \boldsymbol{\Psi}_i^M \mathbf{u}_i(x) \qquad \text{for } x \in \Omega_i,$$

which satisfies

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\phi}_i(x) - k^2 \boldsymbol{\phi}_i(x) = \mathbf{0} \qquad \text{for } x \in \Omega_i$$

and

$$\boldsymbol{\gamma}_N \boldsymbol{\phi}_i(x) = (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i(x) + (\mathsf{N}_i \mathbf{u}_{|\Gamma_i})(x) \qquad \text{for } x \in \Gamma_i$$

as well as

$$\boldsymbol{\phi}_i(x) = (\mathsf{S}_i \mathbf{t}_i)(x) + \frac{1}{2}\mathbf{u}_{|\Gamma_i}(x) - (\mathsf{C}_i \mathbf{u}_{|\Gamma_i})(x) = \mathbf{u}_{|\Gamma_i}(x) \qquad \text{for } x \in \Gamma_i.$$

We consider $\boldsymbol{\phi}_i = \boldsymbol{\phi}_{|\Omega_i} \in \mathbf{H}(\mathbf{curl}, \Omega_i)$ as the restriction of a function $\boldsymbol{\phi} \in \mathbf{H}(\mathbf{curl}, \Omega)$. We then obtain, by using Green's first formula,

$$\begin{aligned}
0 &= \sum_{i=1}^{p} \left[\langle \mathsf{N}_i \mathbf{u}_{|\Gamma_i}, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i, \mathbf{v}_{|\Gamma_i} \rangle_{\Gamma_i} \right] \\
&= \sum_{i=1}^{p} \int_{\Gamma_i} \boldsymbol{\gamma}_{N,y} \boldsymbol{\phi}_i(x) \mathbf{v}_{|\Gamma_i}(x) \mathrm{d}s_x \\
&= \sum_{i=1}^{p} \int_{\Omega_i} \Big[\mathbf{curl}\, \boldsymbol{\phi}_i(x) \cdot \mathbf{curl}\, \mathbf{v}_{|\Omega_i}(x) \mathrm{d}x - k_i^2 \boldsymbol{\phi}_i(x) \mathbf{v}_{|\Omega_i}(x) \Big] \mathrm{d}x \\
&= \int_{\Omega} \Big[\mathbf{curl}\, \boldsymbol{\phi}(x) \cdot \mathbf{curl}\, \mathbf{v}(x) - k^2(x) \boldsymbol{\phi}(x) \mathbf{v}(x) \Big] \mathrm{d}x
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$. Since this is the weak formulation of the Neumann boundary value problem

$$\mathbf{curl}\, \mathbf{curl}\, \boldsymbol{\phi}(x) + k^2(x)\boldsymbol{\phi}(x) = \mathbf{0} \quad \text{for } x \in \Omega, \quad \boldsymbol{\gamma}_N \boldsymbol{\phi}(x) = \mathbf{0} \quad \text{for } x \in \Gamma,$$

$\boldsymbol{\phi}(x) = \mathbf{0}$ for $x \in \Omega$ follows. Recall that the Neumann boundary value problem (8.2) was assumed to be uniquely solvable. From $\boldsymbol{\phi}_i(x) = \mathbf{0}$ for $x \in \Omega_i$ we conclude $\mathbf{u}_{|\Gamma_i}(x) = \mathbf{0}$ for $x \in \Gamma_i$ as well as $\boldsymbol{\gamma}_N \boldsymbol{\phi}_i(x) = 0$ for $x \in \Gamma_i$. Therefore, we conclude

$$(\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_i(x) = \mathbf{0}, \qquad (\mathsf{S}_i \mathbf{t}_i)(x) = \mathbf{0} \qquad \text{for } x \in \Gamma_i.$$

If k_i^2 is not a Dirichlet eigenvalue, the single layer potential S_i is injective and $\mathbf{t}_i(x) = \mathbf{0}$ follows. On the other hand, if $\lambda = k_i^2$ is a Dirichlet eigenvalue, we also have

$$(\frac{1}{2}\mathsf{I} - \mathsf{B}_i)\mathbf{t}_i(x) = \mathbf{0} \qquad \text{for } x \in \Gamma_i.$$

Again, $\mathbf{t}_i(x) = \mathbf{0}$ follows. □

8.4 Boundary element discretizations

For the Galerkin discretization of the coupled variational formulation (8.5) and (8.6) let

$$\mathcal{E}_h := \mathcal{E}_h(\Gamma_S) = \mathrm{span}\{\boldsymbol{\phi}_k\}_{k=1}^{M_S} \subset \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma}, \Gamma_S)$$

be the boundary element space on the skeleton, e.g. of lowest order Raviart–Thomas elements, see Section 6.7. We further assume a quasi–uniform boundary mesh with mesh size h_S.

Theorem 8.4. *The space $\mathcal{E}_h(\Gamma_S)$ fulfills the gap property with respect to the splitting* (8.1) *of $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_S)$ as introduced in Section 8.1.*

Proof. See [29, Section 4.3.2]. □

We also define local restrictions of $\mathcal{E}_h$ onto Γ_i, in particular

$$\mathcal{E}_{i,h} = \mathcal{E}_{h|\Gamma_i} = \mathrm{span}\{\boldsymbol{\phi}_k^i\}_{k=1}^{M_i} \subset \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_i).$$

By using the isomorphisms

$$\underline{v}_i \in \mathbb{R}^{M_i} \leftrightarrow \mathbf{v}_{i,h} = \sum_{k=1}^{M_i} v_{i,k}\boldsymbol{\phi}_k^i \in \mathcal{E}_{i,h}, \quad \underline{v} \in \mathbb{R}^{M_S} \leftrightarrow \mathbf{v}_h = \sum_{k=1}^{M_S} v_k \boldsymbol{\phi}_k \in \mathcal{E}_h$$

there exist Boolian connectivity matrices $A_i \in \mathbb{R}^{M_i \times M_S}$ mapping some $\underline{v} \in \mathbb{R}^{M_S}$ of global edge values onto the vector $\underline{v}_i = A_i \underline{v} \in \mathbb{R}^{M_i}$ of the local subdomain boundary edge values. In addition, let

$$\mathcal{F}_{i,h} = \mathrm{span}\{\boldsymbol{\psi}_k^i\}_{k=1}^{N_i} \subset \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$$

be the local boundary element space of lowest order Raviart–Thomas elements with basis functions $\boldsymbol{\psi}_k^i$, which are defined with respect to a local quasi–uniform boundary mesh with average mesh size h_i. The Galerkin boundary element discretization of the variational formulation (8.5) and (8.6) now reads:

Find $\mathbf{u}_h \in \mathcal{E}_h$ and $\mathbf{t}_{i,h} \in \mathcal{F}_{i,h}$ such that

$$\begin{aligned} \sum_{i=1}^{p} \Big[\langle \mathsf{N}_i \mathbf{u}_{h|\Gamma_i}, \mathbf{v}_{h|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2}\mathsf{I} + \mathsf{B}_i)\mathbf{t}_{i,h}, \mathbf{v}_{h|\Gamma_i} \rangle_{\Gamma_i} + \langle i\eta \mathsf{R}_i \mathbf{u}_{h|\Gamma_i}, \mathbf{v}_{h|\Gamma_i} \rangle_{\Gamma_i} \Big] \\ = \int_\Gamma \mathbf{p}(x) \cdot \mathbf{v}_h(x) \mathrm{d}s_x \end{aligned} \tag{8.7}$$

for all $\mathbf{v}_h \in \mathcal{E}_h$ and

$$\langle \mathsf{S}_i \mathbf{t}_{i,h}, \boldsymbol{\mu}_{i,h} \rangle_{\Gamma_i} + \langle (-\frac{1}{2}\mathsf{I} + \mathsf{C}_i)\mathbf{u}_{h|\Gamma_i}, \boldsymbol{\mu}_{i,h} \rangle_{\Gamma_i} = 0 \tag{8.8}$$

for all $\boldsymbol{\mu}_{i,h} \in \mathcal{F}_{i,h}$, $i = 1, \ldots, p$.

Since the sesquilinear form of the coupled variational problem (8.5) and (8.6) fulfills a generalized Gårding inequality and is injective, see Theorem 8.3, the stability of the Galerkin variational formulation (8.5) and (8.6) follows for sufficient small mesh

widths h_S and h_i, because $\mathcal{E}_h$ and $\mathcal{F}_{h,i}$ fulfill the gap property, see Theorem 3.18. In particular, there holds the quasi–optimal error estimate

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)} + \sum_{i=1}^{p} \|\mathbf{t}_i - \mathbf{t}_{i,h}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i)} \\
&\quad \leq c \left\{ \inf_{\mathbf{v}_h \in \mathcal{E}_h} \|\mathbf{u}-\mathbf{v}_h\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_S)} + \sum_{i=1}^{p} \inf_{\mathbf{z}_{i,h} \in \mathcal{F}_{i,h}} \|\mathbf{t}_i - \mathbf{z}_{i,h}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma_i)} \right\}.
\end{aligned}$$

By using first order Raviart–Thomas elements, we finally obtain the estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}_2(\Gamma_S)} + \sum_{i=1}^{p} \|\mathbf{t}_i - \mathbf{t}_{i,h}\|_{\mathbf{L}_2(\Gamma_i)} \leq c(\mathbf{u},\mathbf{t}_i) h_S + \sum_{i=1}^{p} c(\mathbf{t}_i) h_i \tag{8.9}$$

for $\mathbf{u} \in \mathbf{H}_t^1(\Gamma_S)$, $\mathbf{t}_i \in \mathbf{H}_t^1(\Gamma_i)$, $\mathrm{curl}_\Gamma \mathbf{u} \in H^1(\Gamma_S)$ and $\mathrm{div}_\Gamma \mathbf{t}_i \in H^1(\Gamma_i)$.

8.5 Tearing and interconnecting methods

To derive the tearing and interconnecting approach for the electromagnetic wave equation, the idea is the same as for the Helmholtz case. The only change is that the degrees of freedom are edge based instead of node based, see Figure 8.1. The splitting of the degrees of freedom and the interface equations works therefore as described in detail in Section 7.7.

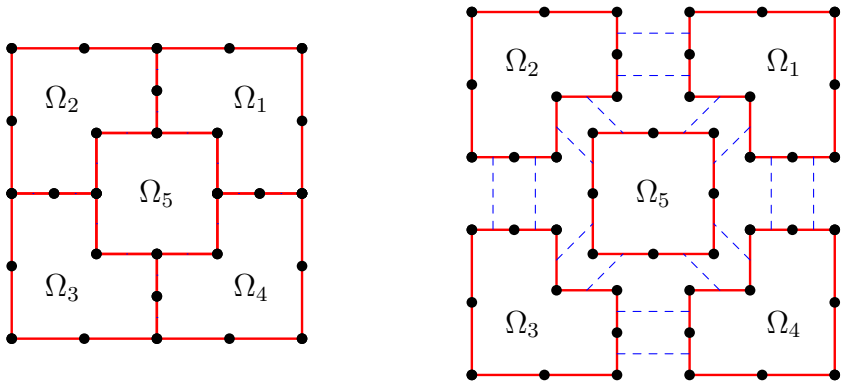


Figure 8.1: The tearing and interconnecting approach for an edge–based discretization.

The Galerkin variational formulation (8.7) and (8.8) is equivalent to the linear system

$$\begin{pmatrix} \mathsf{S}_{1,h} & & & \widetilde{\mathsf{C}}_{1,h}A_i \\ & \dots & & \vdots \\ & & \mathsf{S}_{p,h} & \widetilde{\mathsf{C}}_{p,h}A_p \\ A_1^\top\widetilde{\mathsf{B}}_{1,h} & \dots & A_p^\top\widetilde{\mathsf{B}}_{p,h} & \sum_{i=1}^{p} A_i^\top[\mathsf{N}_{i,h}+i\eta_i\mathsf{R}_{i,h}]A_i \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \sum_{i=1}^{p} A_i^\top \underline{g}_i \end{pmatrix} \tag{8.10}$$

with

$$\begin{aligned} \mathsf{S}_{i,h}[\ell,k] &= \langle \mathsf{S}_i\boldsymbol{\psi}_k^i, \boldsymbol{\psi}_\ell^i\rangle_{\Gamma_i}, \\ \widetilde{\mathsf{C}}_{i,h}[\ell,n] &= \langle (-\frac{1}{2}\mathsf{I}+\mathsf{C}_i)\boldsymbol{\phi}_n^i, \boldsymbol{\psi}_\ell^i\rangle_{\Gamma_i}, \\ \widetilde{\mathsf{B}}_{i,h}[m,k] &= \langle (\frac{1}{2}\mathsf{I}+\mathsf{B}_i)\boldsymbol{\psi}_k^i, \boldsymbol{\phi}_m^i\rangle_{\Gamma_i}, \\ \mathsf{N}_{i,h}[m,n] &= \langle \mathsf{N}_i\boldsymbol{\phi}_n^i, \boldsymbol{\phi}_m^i\rangle_{\Gamma_i}, \\ \mathsf{R}_{i,h}[m,n] &= \langle \mathsf{R}_i\boldsymbol{\phi}_n^i, \boldsymbol{\phi}_m^i\rangle_{\Gamma_i} \end{aligned}$$

for $k,\ell = 1,\dots,N_i$, $m,n = 1,\dots,M_i$, and $i = 1,\dots,p$. In addition,

$$\underline{p}_i[m] = \langle \mathbf{p}, \boldsymbol{\phi}_m^i\rangle_{\Gamma_i\cap\Gamma} \quad \text{for } m = 1,\dots,M_i.$$

By applying the tearing and interconnecting approach as we did in the Helmholtz case, we obtain the linear system

$$\begin{pmatrix} \mathsf{N}_{1,h}+i\eta_1\mathsf{R}_{i,h} & \widetilde{\mathsf{B}}_{1,h} & & & & -B_1^\top \\ \widetilde{\mathsf{C}}_{1,h} & \mathsf{S}_{1,h} & & & & \\ & & \ddots & & & \vdots \\ & & & \mathsf{N}_{p,h}+i\eta_p\mathsf{R}_{p,h} & \widetilde{\mathsf{B}}_{p,h} & -B_p^\top \\ & & & \widetilde{\mathsf{C}}_{p,h} & \mathsf{S}_{p,h} & \\ B_1 & & \dots & B_p & & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{t}_1 \\ \vdots \\ \underline{u}_p \\ \underline{t}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{p}_1 \\ \underline{0} \\ \vdots \\ \underline{p}_p \\ \underline{0} \\ \underline{0} \end{pmatrix}. \tag{8.11}$$

Again we eliminate the primal degrees of freedom and we end up with the Schur complement system

$$\sum_{i=1}^{p} \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} \mathsf{N}_{i,h}+i\eta_i\mathsf{R}_{i,h} & \widetilde{\mathsf{B}}_{i,h} \\ \widetilde{\mathsf{C}}_{i,h} & \mathsf{S}_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} B_i^\top\underline{\lambda} \\ \underline{0} \end{pmatrix} \tag{8.12}$$

$$= -\sum_{i=1}^{p} \begin{pmatrix} B_i & 0 \end{pmatrix} \begin{pmatrix} \mathsf{N}_{i,h}+i\eta_i\mathsf{R}_{i,h} & \widetilde{\mathsf{B}}_{i,h} \\ \widetilde{\mathsf{C}}_{i,h} & \mathsf{S}_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{p}_i \\ \underline{0} \end{pmatrix}. \tag{8.13}$$

As in the Helmholtz case, this can be shortened to

$$F_h\underline{\lambda} = \underline{f}.$$

E_i	Λ_i	It	rel. L_2–error
36	8	5	0.1824189
144	28	17	0.0895037
576	104	49	0.0440296
2304	400	142	0.0234164

(a) $k = 1.0$

E_i	Λ_i	It	rel. L_2–error
36	8	5	0.7042192
144	28	19	0.3055468
576	104	47	0.1472184
2304	400	104	0.0772003

(b) $k = 4.44288$

Table 8.1: Iteration numbers and errors for two bricks.

8.6 Numerical examples

As first example we consider the Neumann boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - k^2\mathbf{U} &= \mathbf{0} \qquad \text{in } \Omega, \\ \boldsymbol{\gamma}_N\mathbf{U} &= \mathbf{p} \qquad \text{on } \Gamma. \end{aligned} \tag{8.14}$$

The domain Ω is given by $(-1.0, 1.5) \times (0.0, 1.0) \times (0.0, 1.0)$ and is divided by the yz–plane, see Figure 7.3. As an analytical solution we use (6.24), with $\hat{x} = (-3.0, 2.1, 1.1)^\top$. The boundary element discretization of the coupled variational formulation (8.7) and (8.8) is done with respect to a globally uniform boundary mesh with E_i edges per subdomain and by using first order Raviart–Thomas elements. As Robin operator we use the discretization of A_0 which is restricted to the degrees of freedom, whose support lies completely within the interface. The number of Lagrangian multipliers is given by Λ_i. The linear system (8.12) is solved by a GMRES method with a relative reduction of the residual norm of $\varepsilon = 10^{-7}$. We consider two different wave numbers, the first is $k = 1.0$ and the second is the first Dirichlet and Neumann eigen wave number of the unit cube, i.e. of Ω_1, $k = \sqrt{2}\pi \approx 4.44288$. The results are given in Table 8.1.

In a second example we solve the Neumann boundary value problem (8.14) in the unit cube, i.e. $\Omega = (0, 1)^3$, which is divided into 8 subcubes. We further use the same analytical solution, variational formulation and boundary elements, as in the last example. In this example we consider different wave numbers, i.e. $k = 1.0, 2.0, 4.0, 8.0$. The results are given in Table 8.2.

Both experiments confirm the error estimate as given in (8.9). Note that the iteration numbers are rapidly increasing without preconditioner. In combination with the effort to solve the local system without a preconditioner, the algorithm is too expensive. Hence, a local and a global preconditioner have to be used to compute larger problems. Probably, a similar preconditioning approach, such as in the acoustic scattering case, is applicable. Another possibility would be to use a dual–primal approach as it was done in [97].

E_i	Λ_i	It	rel. L_2–error
36	90	60	0.1133393
144	324	147	0.0550944
576	1224	476	0.0266769

(a) $k = 1.0$

E_i	Λ_i	It	rel. L_2–error
36	90	60	0.1976629
144	324	154	0.0965666
576	1224	471	0.0465483

(b) $k = 2.0$

E_i	Λ_i	It	rel. L_2–error
36	90	62	0.4037713
144	324	160	0.1874552
576	1224	475	0.0891713

(c) $k = 4.0$

E_i	Λ_i	It	rel. L_2–error
36	90	60	0.9432815
144	324	153	0.3776120
576	1224	397	0.1769975

(d) $k = 8.0$

Table 8.2: Iteration numbers and errors for the unit cube for different wave numbers k.

9 CONCLUSIONS AND OUTLOOK

We have derived a regularized boundary element tearing and interconnecting method for interior acoustic and electromagnetic scattering problems. We also proved that the real valued wave number k can be exchanged by a complex wave number with $\mathrm{Im}(k) > 0$. This corresponds either to damped acoustic waves or to an eddy current problem. A mixture of complex and real valued wave numbers is also possible. The well posedness for real valued wave numbers k was established by introducing Robin boundary interface conditions, which lead to local uniquely solvable subproblems. To avoid ill posed Dirichlet–to–Robin maps we reformulated the local formulation by introducing a local auxiliary Neumann data. A (to the best of our knowledge) new algebraic deduction of the tearing and interconnecting approach was given, which extends in principle the applicability of the tearing and interconnecting method to arbitrary partial differential equations of second order.

We further described another direct regularized combined field integral equation (CFIE) for the exterior acoustic and electromagnetic scattering problem, by interpreting the exterior scattering problem as interior Robin boundary value problem. This approach is different from other CFIE approaches, since it is not based on an artificial combination of single and double layer potentials. Instead it is based on a regularization of the Steklov–Poincaré operator.

For the acoustic scattering problem we modified the regularized exterior problem, such that we could integrate it into the boundary element tearing and interconnecting approach. This allows us to solve exterior scattering problems with jumping coefficients under the assumption that the wave number is constant in the complement of a bounded domain Ω. It further allows us to solve transmission problems with multiple interior subdomains.

For the electromagnetic scattering problem a modification of the regularized exterior field integral equation still has to be found, such that it can be integrated into the boundary element tearing and interconnecting approach. This relies on finding an operator X, which fulfills a generalized Gårding inequality and gives a unique solvable Robin boundary value problem with boundary condition $\boldsymbol{\gamma}_N \mathbf{U} + ik\mathsf{X}^{-1}\boldsymbol{\gamma}_D \mathbf{U}$. The remaining parts of the integration in the boundary element tearing and interconnecting approach should be straightforward.

A big advantage of the tearing and interconnecting approach is the easy coupling of finite and boundary element methods. For the implementation this would be straight

forward and would not differ from other coupled FETI/BETI approaches. For the theory further investigations would be necessary.

For the electromagnetic scattering case, a local preconditioning strategy has to be established and implemented. A hint describing how such a strategy may work, is given in Section 6.7. Furthermore, a global preconditioning approach has to be developed. This could be done by adapting the artificial coarse space idea as used for the acoustic scattering. This may be done by adapting the plane waves in the correct manner. Another possibility could be a dual–primal approach, as it was used in [97].

For larger local electromagnetic scattering subdomains, a fast boundary element technique has to be used. Until now, in this work this is only done for the acoustic scattering case.

The computation of the entries of the Maxwell matrices is very crucial, since accuracy is a major point. Until now the integration is done via a Duffy transformation, see [57], which is quite expensive since a Gauss product approach is used. Optimized integration formulae may lead to better results, see [48, 131].

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