On series $\sum c_k f(kx)$ and Khinchin's conjecture

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Abstract

We prove the optimality of a criterion of Koksma (1953) in Khinchin's conjecture on strong uniform distribution. This verifies a claim of Bourgain (1988) and leads also to a near optimal a.e. convergence condition for series $\sum_{k=1}^{\infty} c_k f(kx)$ with $f \in L^2$. Finally we show that under mild regularity conditions on the Fourier coefficients of f, the Khinchin conjecture is valid assuming only $f \in L^2$.

1 Introduction

Let $\mathbf{T} = \mathbf{R}/\mathbf{Z} \simeq [0,1)$ denote the circle endowed with Lebesgue measure, $e(x) = \exp(2\pi i x)$, $e_n(x) = e(nx)$, $n \in \mathbf{Z}$. Let f be a real function on $\mathbf{T}$ with

$$f \in L^2(\mathbf{T}), \quad \int_{\mathbf{T}} f(x)dx = 0, \quad f(x) \sim \sum_{\ell \in \mathbf{Z}} a_\ell e_\ell, \quad a_0 = 0. \tag{1}$$

Two closely related classical problems of analysis are the almost everywhere convergence of series

$$\sum_{k=1}^{\infty} c_k f(kx) \tag{2}$$

and the a.e. convergence of averages

$$\frac{1}{N} \sum_{k=1}^{N} f(kx). \tag{3}$$

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Khinchin [17] conjectured (assuming only $f \in L^1(\mathbf{T})$ in (1)) that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = 0 \qquad \text{a.e.} \tag{4}$$

This conjecture remained open for nearly 50 years. Koksma [18] proved that (4) holds if the Fourier coefficients of f satisfy

$$\sum_{k=1}^{\infty} |a_k|^2 (\log \log k)^3 < \infty, \tag{5}$$

and in [19] he weakened the condition to

$$\sum_{k=1}^{\infty} |a_k|^2 \sigma_{-1}(k) < \infty, \tag{6}$$

where

$$\sigma_s(k) = \sum_{d|k} d^s. \tag{7}$$

The function $\sigma_{-1}(k)$ is multiplicative and by Gronwall's estimate [14] we have

$$\limsup_{k \to \infty} \frac{\sigma_{-1}(k)}{\log \log k} = e^{\lambda},$$

where λ is Euler's constant. Thus condition (6) is satisfied if

$$\sum_{k=1}^{\infty} |a_k|^2 \log \log k < \infty. \tag{8}$$

Note the difference between (6) and (8): by a theorem of Wintner ([24], p. 180) the function σ_{-1} is bounded in mean, i.e. $\frac{1}{J} \sum_{j=1}^{J} \sigma_{-1}(j) = O(1)$, which implies that for any function $\omega(k) \to \infty$ the inequality $\sigma_{-1}(k) \geq \omega(k)$ can hold only on a set of k's with density 0. Thus (6) is only slightly stronger than $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, showing that relation (4) holds for a very large class of functions $f \in L^2(\mathbf{T})$. However, Marstrand [20] showed that there exist functions $f \in L^2(\mathbf{T})$ (and even bounded functions f) satisfying (1) such that (4) fails, thereby disproving Khinchin's conjecture. In his seminal paper [8], Bourgain used his entropy method to construct a new counterexample and mentioned, concerning conditions (5), (8), that "*... A more detailed analysis of the previous construction shows that Koksma's double logarithmic condition is essentially best possible.*" See the remark on p. 89 of [8]. The purpose of the present paper is to verify Bourgain's claim in a slightly modified form; we will namely prove the following

Theorem 1. *Let $w(n)$ be a nonnegative function of a natural argument, which is sub-multiplicative and bounded in mean. Assume that*

$$w(n) = o(\log \log n).$$

Then there exists a function f satisfying (1) with

$$\sum_{k=1}^{\infty} |a_k|^2 w(k) < \infty$$

such that (4) is not valid.

A nonnegative function of a natural argument w is called sub-multiplicative if $w(nm) \le w(n)w(m)$ for all m, n. As

$$\sigma_{-1}(n) = \prod_{i=1}^{r} \big(1 + \frac{1}{p_i} + \dots \frac{1}{p_i^{\alpha_i}}\big), \qquad n = \prod_{i=1}^{r} p_i^{\alpha_i},$$

it is obvious that σ_{-1} is submultiplicative. Clearly, for any $0 < \varepsilon < 1$ the function $w(n) = \sigma_{-1}(n)^{1-\varepsilon}$ satisfies the assumptions of Theorem 1 and thus it follows that (1) and

$$\sum_{k=1}^{\infty} |a_k|^2 \sigma_{-1}(k)^{1-\varepsilon} < \infty \tag{9}$$

do not generally imply (4). In other words, Koksma's condition (6) is optimal for the a.e. convergence relation (4). Whether Theorem 1 remains valid without the assumption of sub-multiplicativity and bounded means of w remains open.

Theorem 1 shows that $\sigma_{-1}(k)$ is an optimal Weyl factor in the Fourier series of f for the validity of (4), but it does not mean that in the absence of this Weyl factor relation (4) is always false. In fact, we will see that under mild regularity conditions on the Fourier coefficients of f, relation (4) holds under assuming only $f \in L^2$, i.e. $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Also, the norming factor N in (4) can be substantially diminished, see Corollaries 1-3.

The previous results give a fairly satisfactory picture on the validity of the convergence relation (4). Results concerning the convergence of sums (2) are much less complete. Note that, by the Kronecker lemma, the a.e. convergence of (2) with $c_k = 1/k$ implies (4), so the two problems are closely connected. By Carleson's theorem [10], in the case $f(x) = \sin 2\pi x$, $f(x) = \cos 2\pi x$ the series (2) converges a.e. provided $\sum_{k=1}^{\infty} c_k^2 < \infty$. Gaposhkin [13] showed that this remains valid if the Fourier series of f converges absolutely; in particular this holds if f belongs to the Lip (α) class for some $\alpha > 1/2$. However, Nikishin [21] showed that the analogue of Carleson's theorem fails for $f(x) = \operatorname{sgn} \sin 2\pi x$ and fails also for some continuous f. For functions f with bounded variation Aistleitner, Berkes and Seip [2] proved recently that the series (2) converges a.e. provided $\sum_{k=1}^{\infty} c_k^2 (\log\log k)^{\gamma} < \infty$ for $\gamma > 4$, but generally not for $\gamma < 2$. A similar, slightly weaker result holds for the Lip $(1/2)$ class, showing, in particular, that Gaposhkin's result above is sharp. For general $f \in L^2$ the convergence problem has a completely different character and despite a number of sufficient criteria obtained in special cases (see e.g. [1], [4], [5], [6], [9], [11], [12], [23]) no precise a.e. convergence criteria for (2) for L^2 have been found. Our next theorem gives such a criterion, optimal up to a logarithmic factor.

Theorem 2. *Let f satisfy (1) and put*

$$g(r)=\sum_{k=1}^{\infty}|a_{rk}|^2,\qquad G(r)=\sum_{j\le 2r}g(j),\qquad h(n)=\sum_{d|n}(dg(d)+G(d)).\tag{10}$$

Then $\sum_{k=1}^{\infty}c_kf(kx)$ converges a.e. provided

$$\sum_{k=1}^{\infty}c_k^2h(k)(\log k)^2<\infty.\tag{11}$$

On the other hand, for any $\delta>0$ there exists an $f\in L^2$ satisfying (1) and coefficients c_k such that

$$\sum_{k=1}^{\infty}c_k^2h(k)(\log k)^{-\delta}<\infty\tag{12}$$

but $\sum_{k=1}^{\infty}c_kf(kx)$ does not converge a.e.

Theorem 2 is the series analogue of Koksma's theorem and shows that the convergence of $\sum_{k=1}^{\infty}c_kf(kx)$ is intimately connected with the behavior of the function $g(d)=\sum_{k=1}^{\infty}|a_{dk}|^2$ which, in turn, depends on the distribution of the numbers $|a_k|$ on $[0,\infty)$. Under mild regularity conditions on the $|a_k|$, the function $h(n)$ in (10) reduces to classical arithmetic functions like $d(n)$, $\sigma_s(n)$, see Corollaries 1-3 below. For $g(d)\ll d^{-\gamma}$, $0<\gamma<1$ Theorem 2 also implies Theorem 5.19 of Harman [16] in the case $a_n=n$. For general $f\in L^2(\mathbf{T})$, $g(d)$ can behave rather irregularly which, combined with the summation $d|n$ in the definition of $h(n)$, leads to very irregularly changing functions $h(n)$. Clearly, $h(n)$ will be small for numbers n with few prime factors, showing that the a.e. convergence properties of dilated sums $\sum_p c_pf(px)$ extended for primes are considerably better that those of general series (2).

By the Kronecker lemma, Theorem 2 implies for any $f\in L^2(\mathbf{T})$ that

$$\left|\sum_{k=1}^{N}f(kx)\right|\ll\sqrt{N}(\log N)^{3/2+\varepsilon}\hat{h}(N)^{1/2}\qquad\text{a.e.}\tag{13}$$

for any $\varepsilon>0$, where $\hat{h}(n)=\max_{1\le k\le n}h(k)$ is the smallest monotone majorant of the function h. Relation (13) gives much more precise information than (4). In particular, the corollaries below show that under mild monotonicity or regularity conditions on the $|a_k|$, the right hand side of (13) will be $o(N)$, and thus the Khinchin conjecture is valid under such conditions.

Corollary 1. *Let f satisfy (1) where (a_k) satisfies one of the following conditions:*

(a) $|a_k|$ is regularly varying as $k\to\infty$

(b) $k^{-\gamma}|a_k|$ is non-increasing for some $\gamma>0$

(c) There exists a $C>0$ such that for any integer $d\ge 1$ we have $\sum_{k=1}^{\infty}|a_{dk}|^2\le C/d$.

Then the series $\sum_{k=1}^\infty c_k f(kx)$ *converges a.e. provided* $\sum_{k=1}^\infty c_k^2 d(k)(\log k)^2 < \infty$ *and consequently we have*

$$\left|\sum_{k=1}^N f(kx)\right| \ll \sqrt{N}(\log N)^{3/2+\varepsilon}\hat{d}(N)^{1/2} \qquad a.e.$$

for any $\varepsilon > 0$*, where* $d(k) = \sum_{d|k} 1$ *is the divisor function and* $\hat{d}(N) = \max_{1\le k\le N} d(k)$*.*

A sequence (a_n) is called regularly varying if there exists a $\rho \in \mathbb{R}$ such that for any $t > 0$ we have $\lim_{n\to\infty} a_{[nt]}/a_n = t^\rho$; ρ is called the exponent of regularity of (a_n). For basic properties of regularly varying sequences and functions we refer to [7]. Since $d(k) \ll k^\varepsilon$ for any $\varepsilon > 0$ (see e.g. [15]), we get

Corollary 2. *Let* f *satisfy (1). Under any of the regularity conditions (a), (b), (c) we have*

$$\left|\sum_{k=1}^N f(kx)\right| \ll N^{1/2+\varepsilon} \qquad a.e.$$

for any $\varepsilon > 0$.

It is easily seen that in Corollary 1 both (a) and (b) imply (c), so (c) is the weakest condition of the three. It is also quite natural: it requires that all subsums $\sum_{k=1}^\infty |a_{dk}|^2$ carry at most their “fair share” in the sum $\sum_{k=1}^\infty |a_k|^2$. However, in a number of important cases the estimate $\sum_{k=1}^\infty |a_{dk}|^2 \ll d^{-1}$ can be improved, leading to a convergence theorem for $\sum_{k=1}^\infty c_k f(kx)$ with a smaller Weyl factor.

Corollary 3. *Let* f *satisfy (1), where*

$$|a_k| \ll k^{-1/2}\varphi(k) \qquad k = 1, 2, \dots$$

with a non-increasing regularly varying φ *satisfying* $\sum_{k=1}^\infty k^{-1}\varphi^2(k) < \infty$*. Let*

$$\psi(r) = \sum_{k\ge r} k^{-1}\varphi^2(k), \qquad h(N) = \sum_{d|N} \psi(d).$$

Then the series $\sum_{k=1}^\infty c_k f(kx)$ *converges a.e. provided* $\sum_{k=1}^\infty c_k^2 h(k)(\log k)^2 < \infty$ *and consequently (13) holds.*

Note that here, in contrast to Corollary 1, we assume the regularity not for $|a_k|$, but for its majorizing function. For $\varphi(k) = k^{-\gamma}$, $0 < \gamma \le 1/2$ we get $h(k) \ll \sigma_{-2\gamma}(k)$, leading to Theorem 2.8 in Berkes and Weber [6] (cf. also Bremont [9]). For $\varphi(k) = (\log k)^{-\gamma}$, $\gamma > 1/2$ we get

$$h(k) \ll \sum_{d|k} (\log d)^{-(2\gamma-1)}.$$

2 Proof of Theorem 1

Let $w = \{w_n, n \in \mathbf{Z}\}$ be a sequence of positive reals. Let L^2_w be the associated Sobolev space on the circle, namely the subspace of L^2 consisting with functions f such that

$$\|f\|_w^2 := \sum_{n\in\mathbf{Z}} w_n a_n^2(f) < \infty.$$

This is a Hilbert space with scalar product defined by $\langle f, g\rangle = \sum_{n\in\mathbf{Z}} w_n a_n(f)a_n(g)$, $f, g \in L^2_w$.

The proof of Theorem 1 is based on an adaptation to the Sobolev space L^2_w of the method elaborated by Bourgain, and more precisely on a variant of Proposition 1 in [8]. Let $f(x) = \sum_{\ell\in\mathbf{Z}} a_\ell e_\ell$, $a_0 = 0$ and consider the dilation operators $T_j f(x) = f(jx)$. These are positive isometries on L^p, $p \geq 1$, such that $T_j 1 = 1$ for all j; and for all $f \in L^2$

$$\frac{1}{J}\sum_{j\leq J} T_j f \xrightarrow{L^2} \int f d\lambda, \qquad J \to \infty. \tag{14}$$

See [5] p.19 if necessary. To $f \in L^2$ we associate

$$F_{J,f} = \frac{1}{\sqrt{J}} \sum_{1\leq j\leq J} g_j T_j f, \qquad (J \geq 1)$$

where $g_1, g_2, \ldots$ are i.i.d. standard Gaussian random variables.

Proposition. *Let $S_n\colon L^2 \to L^2$, $n = 1, 2, \ldots$ be continuous operators commuting with T_j on L^2, $S_n T_j = T_j S_n$ for all n and j. Assume that the following property is fulfilled:*

$$\lambda\Big\{\sup_{n\geq 1} |S_n(f)| < \infty\Big\} = 1, \qquad \forall f \in L^2_w,$$

then there exists a constant C depending on $\{S_n, n \geq 1\}$ only, such that

$$\sup_{\varepsilon>0} \varepsilon\sqrt{\log N_f(\varepsilon)} \leq C \limsup_{J\to\infty} \big(\mathbf{E}\,\|F_{J,f}\|_w\big)^{1/2}, \qquad \forall f \in L^2_w,$$

where $N_f(\varepsilon)$ is the entropy number associated with the set $C_f = \{S_n f, n \geq 1\}$, namely the minimal number of L^2 open balls of radius ε, centered in C_f and enough to cover C_f.

Proof. By the Banach principle, there exists a non-increasing function $C : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\forall\varepsilon > 0, \forall g \in L^2_w(\mathbf{T}), \qquad \lambda\Big\{\sup_n |S_n(g)| \geq \|g\|_w C(\varepsilon)\Big\} \leq \varepsilon.$$

Let $0 < \varepsilon < 1/4$. Let $f \in L^2(\mathbf{T})$. Taking $g = F_{J,f}$ and using Fubini's theorem, gives

$$\int_{\mathbf{T}} \mathbf{P}\Big\{\sup_{n\geq 1} |S_n(F_{J,f})| \geq C(\varepsilon)\|F_{J,f}\|_w\Big\}\, d\lambda \leq \varepsilon.$$

It follows that

$$\lambda\Big\{x\in\mathbf{T}:\mathbf{P}\Big\{\omega:\sup_{n\ge1}|S_n(F_{J,f}(\omega,.))(x)\ |\ge C(\varepsilon)\|F_{J,f}(\omega,.)\|_w\Big\}\ge\sqrt{\varepsilon}\Big\}\le\sqrt{\varepsilon},$$

which is better rewritten under the following form

$$\lambda\Big\{x\in\mathbf{T}:\mathbf{P}\Big\{\omega:\sup_{n\ge1}|S_n(F_{J,f}(\omega,.))(x)\ |\le C(\varepsilon)\|F_{J,f}(\omega,.)\|_w\Big\}\ge 1-\sqrt{\varepsilon}\Big\}\ge 1-\sqrt{\varepsilon}.$$

By Tchebycheff's inequality, $\mathbf{P}\Big\{\|F_{J,f}\|_w^2>\mathbf{E}\,\|F_{J,f}\|_w^2/\varepsilon\Big\}\le\varepsilon$. We deduce that the set

$$X_{\varepsilon,J,f}=\Big\{x\in\mathbf{T}:\mathbf{P}\Big\{\omega:\sup_{n\ge1}|S_n(F_{J,f}(\omega,.)(x)|\le C(\varepsilon)\big(\mathbf{E}\,\|F_{J,f}\|_w^2/\varepsilon\big)^{1/2}\Big\}\ge 1-2\sqrt{\varepsilon}\Big\}$$

has measure greater than $1-\sqrt{\varepsilon}$. The classical estimate of Gaussian semi-norms implies

$$\forall x\in X_{\varepsilon,J,f},\qquad \mathbf{E}\sup_{n\ge1}|S_n(F_{J,f}(\omega,))(x)|\le\frac{4}{(1-2\sqrt{\varepsilon})}\,\frac{C(\varepsilon)}{\sqrt{\varepsilon}}\Big(\mathbf{E}\,\|F_{J,f}\|_w^2\Big)^{1/2}.$$

Now let I be a finite set of integers such that $\|S_n(f)-S_m(f)\|_2\neq0$, for all distinct elements $m,n\in I$. By the commutation property, $S_n(F_{J,f})=F_{J,S_nf}$; so that

$$\begin{aligned}\mathbf{E}\,\big|S_n(F_{J,f})-S_m(F_{J,f}\big|^2 &= \mathbf{E}\,\big|F_{J,S_nf-S_mf}\big|^2=\frac{1}{J}\sum_{j\le J}(T_j(S_nf-S_mf))^2\\ &= \frac{1}{J}\sum_{j\le J}T_j(S_nf-S_mf)^2\ \to\ \|S_nf-S_mf\|_2^2,\end{aligned}$$

in L^2 as J tends to infinity. We have used (14) and the fact that $(T_jf)^2=T_jf^2$ if $f\in L^2$. By proceeding by extraction, we can find a partial index $\mathcal{J}$ such that the set

$$A(I)=\Big\{\forall J\in\mathcal{J},\ \forall n,m\in I,\ m\neq n,\quad\frac{\big(\mathbf{E}\,|S_n(F_{J,f})-S_m(F_{J,f})|^2\big)^{1/2}}{\|S_n(f)-S_m(f)\|_2}\ge\sqrt{1-\varepsilon}\Big\},$$

has measure greater that $1-\sqrt{\varepsilon}$.

Let $J\in\mathcal{J}$, then $\lambda(A(I)\cap X_{\varepsilon,J,f})\ge1-2\sqrt{\varepsilon}>0$, and for any $x\in A(I)\cap X_{\varepsilon,J,f}$

$$C(\varepsilon)\Big(\mathbf{E}\,\|F_{J,f}\|_w^2\Big)^{1/2}\ge\mathbf{E}\sup_{n\ge1}|S_n(F_{J,f})(x)|\ge\mathbf{E}\sup_{n\in I}S_n(F_{J,f})(x)\ge$$

$$\sqrt{1-\varepsilon}\ \mathbf{E}\sup_{n\in I}Z(S_n(f))\ge(1-2\sqrt{\varepsilon})\ \ \mathbf{E}\sup_{n\in I}Z(S_n(f)).$$

Therefore,

$$\mathbf{E}\sup_{n\in I}Z(S_n(f))\le C\Big(\limsup_{J\to\infty}\mathbf{E}\,\|F_{J,f}\|_w^2\Big)^{1/2}.$$

Sudakov's minoration implies

$$\sup_{\rho>0} \rho\sqrt{\log N_f(\rho)} \le C\Big(\limsup_{J\to\infty} \mathbf{E}\,\|F_{J,f}\|_w^2\Big)^{1/2}.$$

Proof of Theorem 1. We apply the above Proposition with the choice

$$S_n f(x) = \frac{1}{n}\sum_{k=1}^{n} f(kx).$$

The commutation assumption is obviously satisfied. Let $P_1, P_2, \ldots$ denote the sequence of prime numbers. Fix some positive integer s and let d be some other integer such that $2^d \le P_s$. There exists an integer T such that if

$$A_T = \{n = P_1^{\alpha_1}\ldots P_s^{\alpha_s} : 2^T \le n < 2^{T+1},\ \alpha_i \ge 0,\ i = 1,\ldots,s\},$$

then $\sharp(A_{T+d}) \le 2\sharp(A_T)$. Put

$$f = f_T = \frac{1}{\sharp(A_T)^{1/2}}\sum_{n\in A_T} e_n.$$

It follows from Bourgain's proof [8] p. 88-89, (or [22] p. 239-240 for details) that

$$N\Big(\big(S_{4^i}(f), i \le [\frac{d}{2}]\big), \frac{1}{8}\Big) \ge T.$$

So that

$$\sqrt{\log T} \le C\Big(\limsup_{J\to\infty} \mathbf{E}\,\|F_{J,f}\|_w^2\Big)^{1/2}. \tag{15}$$

Now as

$$F_{J,f} = \frac{1}{J^{1/2}}\sum_{j\le J} g_j \frac{1}{\#(A_T)^{1/2}}\sum_{n\in A_T} e_{nj} = \frac{1}{(J\#(A_T))^{1/2}}\sum_{\nu\ge1} e_\nu\Big(\sum_{\substack{1\le j\le J\\ j|\nu\\ \frac{\nu}{j}\in A_T}} g_j\Big)$$

we have

$$\|F_{J,f}\|_w^2 = \frac{1}{J\#(A_T)}\sum_{\nu\ge1} w_\nu\Big(\sum_{\substack{1\le j\le J\\ j|\nu\\ \frac{\nu}{j}\in A_T}} g_j\Big)^2.$$

These sums are finite sums. Further,

$$\begin{aligned}\mathbf{E}\,\|F_{J,f}\|_w^2 &= \frac{1}{J\#(A_T)}\sum_{\nu=1}^{\infty} w_\nu\Big(\sum_{\substack{1\le j\le J\\ j|\nu\\ \frac{\nu}{j}\in A_T}} 1\Big) = \frac{1}{J\#(A_T)}\sum_{j\le J}\sum_{m\in A_T} w_{mj}\\ &\le \Big(\frac{1}{J}\sum_{j\le J} w_j\Big)\Big(\frac{1}{\#(A_T)}\sum_{m\in A_T} w_m\Big)\end{aligned}$$

$$\leq \quad \Big(\frac{1}{J}\sum_{j\leq J} w_j\Big) \max_{m\in A_T} w_m$$

Therefore,

$$\limsup_{J\to\infty} \mathbf{E}\,\|F_{J,f}\|_w^2 \leq \Big(\limsup_{J\to\infty} \frac{1}{J}\sum_{j\leq J} w_j\Big) \max_{m\in A_T} w_m \leq M \max_{m\in A_T} w_m,$$

where $M < \infty$ and further

$$\max_{m\in A_T} w_m = o\big(\max_{m\in A_T} \log\log m\big) = o(\log T),$$

by assumption. Consequently

$$\limsup_{J\to\infty} \big(\mathbf{E}\,\|F_{J,f_T}\|_w\big)^{1/2} = o(\sqrt{\log T}). \tag{16}$$

But this contradicts (15), completing the proof of Theorem 1.

3 Proof of Theorem 2.

We first prove the following lemma.

Lemma. Let f satisfy (1). Then for any $r \geq 1$ and any real coefficients c_j we have

$$\int_0^1 \left(\sum_{\ell=2^r+1}^{2^{r+1}} c_\ell f(\ell x)\right)^2 dx \leq 2 \sum_{\ell=2^r+1}^{2^{r+1}} c_\ell^2 h(\ell), \tag{17}$$

where the arithmetic function h is defined by (10).

Proof. Fix $m, n \geq 1$ and put $m' = m/d$, $n' = n/d$, where $d = (m,n)$. Since f is real, we have $a_{-\ell} = \overline{a}_\ell$ for $\ell \in \mathbb{Z}$ and thus using (1) we get

$$\lambda_{m,n} := \left|\int_0^1 f(mx)f(nx)dx\right| = \left|\sum_{\substack{mk=nl\\k,l\in\mathbb{Z}}} a_k \overline{a}_l\right| \leq \sum_{\substack{mk=nl\\k,l\in\mathbb{Z}}} |a_k||a_l|$$
$$= 2\sum_{\substack{mk=nl\\k,l\geq 1}} |a_k||a_l| = 2\sum_{\substack{m'k=n'l\\k,l\geq 1}} |a_k||a_l|. \tag{18}$$

Since $(m',n') = 1$, the equation $m'k = n'l$ implies that m' is a divisor of l, i.e. $l = m'i$ and consequently $k = n'i$ for some $i \geq 1$. Thus the last expression in (18) equals

$$2\sum_{i=1}^{\infty} |a_{m'i}||a_{n'i}| \leq \sum_{i=1}^{\infty} (|a_{m'i}|^2 + |a_{n'i}|^2) = g(m') + g(n'). \tag{19}$$

Now for any $r \geq 1$ and any coefficients c_ℓ,

$$\int_0^1 \left(\sum_{\ell=2^r+1}^{2^{r+1}} c_\ell f(\ell x) \right)^2 dx$$

$$\leq \sum_{i,j=2^r+1}^{2^{r+1}} \lambda_{i,j} |c_i||c_j| \leq \frac{1}{2} \sum_{i,j=2^r+1}^{2^{r+1}} \lambda_{i,j}(c_i^2 + c_j^2) = \sum_{i,j=2^r+1}^{2^{r+1}} \lambda_{i,j} c_i^2 = \sum_{i=2^r+1}^{2^{r+1}} c_i^2 \rho(i) \quad (20)$$

where

$$\rho(i) = \sum_{j=2^k+1}^{2^{k+1}} \lambda_{i,j} \qquad \text{for } 2^k < i \leq 2^{k+1}. \quad (21)$$

Thus using (18), (19) we get for $2^k < i \leq 2^{k+1}$,

$$\rho(i) = \sum_{j=2^k+1}^{2^{k+1}} \lambda_{i,j} \leq \sum_{j=2^k+1}^{2^{k+1}} (g(i/(i,j)) + g(j/(i,j))). \quad (22)$$

Fix i and $d|i$ and sum here for all j with $(i,j) = i/d$. Then $j = ri/d$ for some $r \leq 2d$ and thus the contribution of these terms is

$$\leq \sum_{r \leq 2d} (g(d) + g(r)) \leq 2dg(d) + G(d).$$

Thus summing now for $d|i$, we get

$$\rho(i) \leq 2 \sum_{d|i} (dg(d) + G(d)) = 2h(i).$$

The lemma now follows from (20).

Proof of Theorem 2. Using the Lemma, the proof of the sufficiency part can be completed by using the method of Rademacher and Mensov, see e.g. [3], pp. 80–81. By the Lemma and (11) we have

$$\sum_{r=1}^{\infty} \int_0^1 r^2 \left[\sum_{j=2^r+1}^{2^{r+1}} c_j f(jx) \right]^2 dx \ll \sum_{r=1}^{\infty} r^2 \sum_{j=2^r+1}^{2^{r+1}} c_j^2 h(j)$$

$$\ll \sum_{r=1}^{\infty} \sum_{j=2^r+1}^{2^{r+1}} c_j^2 (\log j)^2 h(j) < \infty.$$

Thus

$$\sum_{r=1}^{\infty} r^2 \left[\sum_{j=2^r+1}^{2^{r+1}} c_j f(jx) \right]^2 < \infty \qquad \text{a.e.}$$

and the Cauchy-Schwarz inequality yields for any $1 \le M < N$

$$\left| \sum_{j=2^M+1}^{2^N} c_j f(jx) \right|^2 \le \left(\sum_{k=M}^{N-1} \left| \sum_{j=2^k+1}^{2^{k+1}} c_j f(jx) \right| \right)^2$$

$$\le \left(\sum_{k=M}^{N-1} \frac{1}{k^2} \right) \left(\sum_{k=M}^{N-1} k^2 \left| \sum_{j=2^k+1}^{2^{k+1}} c_j f(jx) \right|^2 \right) \le 2 \sum_{k=M}^{\infty} k^2 \left| \sum_{j=2^k+1}^{2^{k+1}} c_j f(jx) \right|^2 \to 0$$

almost everywhere as $M \to \infty$. This implies that $\sum_{j=1}^{2^m} c_j f(jx)$ converges a.e. as $m \to \infty$. Now the Lemma and standard maximal inequalities (see e.g. [22], Lemma 8.3.4) imply that

$$\sum_{k=1}^{\infty} \left\| \max_{2^k+1 \le j \le 2^{k+1}} \Big| \sum_{\ell=2^k+1}^{j} c_\ell f(\ell x) \Big| \right\|^2 \ll \sum_{k=1}^{\infty} (\log 2^k)^2 \left(\sum_{\ell=2^k+1}^{2^{k+1}} c_\ell^2 h(\ell) \right)$$

$$\ll \sum_{\ell=1}^{\infty} c_\ell^2 (\log \ell)^2 h(\ell) < \infty$$

which yields

$$\max_{2^k+1 \le j \le 2^{k+1}} \left| \sum_{\ell=2^k+1}^{j} c_\ell f(\ell x) \right| \to 0 \qquad \text{a.e.} \tag{23}$$

proving the first part of Theorem 2.

To prove the second statement of Theorem 2, let $c_k = 1/k$ and let (ε_k) be a positive non-increasing sequence satisfying $\varepsilon_k/\varepsilon_{2k} = O(1)$. We write $\sum_{k=1}^{\infty} c_k^2 h(k) \varepsilon_k = \sum_1 + \sum_2$, where

$$\sum\nolimits_1 = \sum_{k=1}^{\infty} k^{-2} \varepsilon_k \sum_{d|k} d g(d), \qquad \sum\nolimits_2 = \sum_{k=1}^{\infty} k^{-2} \varepsilon_k \sum_{d|k} G(d).$$

Now

$$\sum\nolimits_1 = \sum_{d=1}^{\infty} d g(d) \sum_{j=1}^{\infty} (dj)^{-2} \varepsilon_{dj} \ll \sum_{d=1}^{\infty} d g(d) d^{-2} \varepsilon_d = \sum_{d=1}^{\infty} \frac{\varepsilon_d}{d} \sum_{k=1}^{\infty} |a_{dk}|^2 \tag{24}$$

$$= \sum_{j=1}^{\infty} |a_j|^2 \sum_{d|j} \frac{\varepsilon_d}{d} = \sum_{j=1}^{\infty} |a_j|^2 \widetilde{\sigma}(j),$$

with

$$\widetilde{\sigma}(k) = \sum_{d|k} \varepsilon_d / d. \tag{25}$$

Similarly,

$$\sum\nolimits_2 = \sum_{d=1}^{\infty} G(d) \sum_{j=1}^{\infty} (dj)^{-2} \varepsilon_{dj} \ll \sum_{d=1}^{\infty} G(d) d^{-2} \varepsilon_d \ll \sum_{d=1}^{\infty} g(d) d^{-1} \varepsilon_d, \tag{26}$$

which is the same bound as the middle expression in (24) and thus continuing, we get the same estimate as for $\sum_1$. To justify the last step in (26), set $G(0) = 0$, $S_d = \sum_{j=d}^{\infty} \varepsilon_j j^{-2}$ and note that

$$\sum_{d=1}^{\infty} G(d)d^{-2}\varepsilon_d = \sum_{d=1}^{\infty} G(d)(S_d - S_{d+1}) = \sum_{\ell=1}^{\infty}(G(\ell) - G(\ell-1))S_\ell$$
$$= \sum_{\ell=1}^{\infty}(g(2\ell) + g(2\ell-1))S_\ell \ll \sum_{\ell=1}^{\infty}(g(2\ell) + g(2\ell-1))\frac{\varepsilon_\ell}{\ell}$$
$$\ll \sum_{\ell=1}^{\infty} g(2\ell)\frac{\varepsilon_{2\ell}}{2\ell} + \sum_{\ell=1}^{\infty} g(2\ell-1)\frac{\varepsilon_{2\ell-1}}{2\ell-1} = \sum_{r=1}^{\infty} g(r)\frac{\varepsilon_r}{r}.$$

Thus we proved

$$\sum_{k=1}^{\infty} c_k^2 h(k)\varepsilon_k \ll \sum_{j=1}^{\infty} a_j^2 \widetilde{\sigma}(j). \tag{27}$$

Now choosing $\varepsilon_k = (\log k)^{-\delta}$ we have

$$\widetilde{\sigma}(k) = \sum_{d|k} \frac{\varepsilon_d}{d} = \sum_{d|k,\, d\le \exp(\sigma_{-1}(k)^\delta)} \frac{\varepsilon_d}{d} + \sum_{d|k,\, d>\exp(\sigma_{-1}(k)^\delta)} \frac{\varepsilon_d}{d} \ll \sum_{d\le \exp(\sigma_{-1}(k)^\delta)} \frac{1}{d}$$
$$+ \varepsilon_{\exp(\sigma_{-1}(k)^\delta)} \sum_{d|k} \frac{1}{d} \ll (\sigma_{-1}(k))^\delta + (\sigma_{-1}(k))^{-\delta^2}\sigma_{-1}(k) \ll (\sigma_{-1}(k))^{1-\delta^2}. \tag{28}$$

By Theorem 1 we can choose a function f satisfying (1) such that $\sum_{j=1}^{\infty} |a_j|^2 \sigma_{-1}(j)^{1-\delta^2}$ converges, but $N^{-1}\sum_{k=1}^{N} f(kx)$ does not converge a.e. But then by relations (27) and (28) we have $\sum_{k=1}^{\infty} c_k^2 h(k)\varepsilon_k < \infty$ for $c_k = 1/k$ and $\varepsilon_k = (\log k)^{-\delta}$ and the series $\sum_{k=1}^{\infty} c_k f(kx)$ cannot converge a.e., since then by the Kronecker lemma we would have $N^{-1}\sum_{k=1}^{N} f(kx) \to 0$ a.e.

Proof of Corollary 3. Let φ be regularly varying with exponent ρ; extend it to all real $t \ge 1$ by $\varphi(t) = \varphi([t])$. Since φ is non-increasing, we have $\rho \le 0$. In view of $\sum_{k=1}^{\infty} k^{-1}\varphi^2(k) < \infty$, Theorem 1.5.11 (ii) of [7] applies with $f(x) = \varphi^2(x)$, $\sigma = -1$ and we get that the ratio

$$\varphi^2(r)/\sum_{k\ge r} k^{-1}\varphi^2(k) \tag{29}$$

converges, as $r \to \infty$, to $-2\rho \ge 0$. For $\rho \ne 0$ this implies that $\psi(r) = \sum_{k\ge r} k^{-1}\varphi^2(k)$ is regularly varying with exponent 2ρ; for $\rho = 0$ the same conclusion follows from [7], Proposition 1.5.9b. Thus in all cases ψ is regularly varying and $\varphi^2(r) \ll \psi(r)$.

Now

$$\begin{aligned} g(r) &= \sum_{k=1}^{\infty} |a_{rk}|^2 \le \sum_{k=1}^{\infty} \frac{1}{rk}\varphi^2(rk) = \frac{1}{r}\varphi^2(r) + \sum_{k=2}^{\infty} \frac{1}{rk}\varphi^2(rk) \\ &\le \frac{1}{r}\varphi^2(r) + \int_1^{\infty} \frac{1}{rx}\varphi^2(rx)\, dx = \frac{1}{r}\varphi^2(r) + \frac{1}{r}\int_r^{\infty} \frac{1}{y}\varphi^2(y)\, dy \\ &\ll \frac{1}{r}\varphi^2(r) + \frac{1}{r}\sum_{k\ge r} \frac{1}{k}\varphi^2(k) \ll \frac{\psi(r)}{r}. \end{aligned}$$

Since ψ is regularly varying and for $2^k < i, j \le 2^{k+1}$ the ratio of the numbers $i/(i,j)$ and $j/(i,j)$ is between $1/2$ and 2, relation (22) in the proof of Theorem 2 yields

$$\rho(i) \ll \sum_{j=2^k+1}^{2^{k+1}} \frac{(i,j)}{i}\psi(i/(i,j)). \tag{30}$$

Fix now $2^k < i \le 2^{k+1}$ and $d|i$ and in (30) sum for all j with $(i,j) = i/d$. Then $j = ri/d$ for some $r \le 2d$ and thus the contribution of these terms in (30) is at most $(\psi(d)/d)2d = 2\psi(d)$. Thus summing for $d|i$, we get

$$\rho(i) \ll \sum_{d|i} \psi(d) = h(i), \tag{31}$$

and hence by (20)

$$\int_0^1 \left(\sum_{\ell=2^r+1}^{2^{r+1}} c_\ell f(\ell x) \right)^2 dx \le \sum_{\ell=2^r+1}^{2^{r+1}} c_\ell^2 h(\ell). \tag{32}$$

The proof can now be completed as in Theorem 2.

Proof of Corollary 1. Assume condition (c) of the Corollary. Then the relation $g(r) \ll \psi(r)/r$ obtained in the proof of Corollary 3 holds with $\psi = 1$ and thus (31) and (32) hold with $h(n) = \sum_{d|n} 1 = d(n)$. Following the proof of Theorem 2, the statement of Corollary 1 follows in the case (c).

Next we show that in Corollary 1 we have (a)$\Longrightarrow$ (c) and (b)$\Longrightarrow$ (c). Assume first that (b) holds, then

$$|a_{n+1}/a_n| \le |(n+1)/n|^\gamma \le 1 + C/n \qquad (n \ge 1)$$

for some constant $C > 0$. Let now $k \ge 1, d \ge 2$ and $0 \le j \le d/2$. Then we get, setting $C_1 = e^C$,

$$\begin{aligned} &|a_{kd}/a_{kd-j}| \\ &= \prod_{r=kd-j}^{kd-1} |a_{r+1}/a_r| \le \prod_{r=kd-j}^{kd-1} (1 + C/r) \le \exp\left(\sum_{r=kd-j}^{kd-1} C/r \right) \le \exp(Cj/(d/2)) \le C_1 \end{aligned}$$

and consequently

$$\sum_{n=1}^{\infty} |a_n|^2 \geq \sum_{k=1}^{\infty} \sum_{j=1}^{[d/2]} |a_{kd-j}|^2 \geq [d/2]\, C_1^{-2} \sum_{k=1}^{\infty} |a_{kd}|^2,$$

proving the validity of condition (c). If condition (a) holds, then by $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ its exponent of regularity is negative and thus by the remark in [7], p. 23 preceding Theorem 1.5.4, there exists a non-increasing sequence $b_n \sim |a_n|$. Clearly, $\sum_{n=1}^{\infty} b_n^2 < \infty$ and condition (b) of Corollary 1 holds for (b_n) as well. But then the previous argument yields that condition (c) holds for (b_n) and consequently for (a_n) as well.

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