Monatsh Math (2017) 182:565–575
DOI 10.1007/s00605-016-0995-9

On Erdős and Sárközy's sequences with Property P

Christian Elsholtz[1] · **Stefan Planitzer**[1]

Received: 27 November 2015 / Accepted: 7 October 2016 / Published online: 18 October 2016

Abstract A sequence A of positive integers having the property that no element $a_i \in A$ divides the sum $a_j + a_k$ of two larger elements is said to have 'Property P'. We construct an infinite set $S \subset \mathbb{N}$ having Property P with counting function $S(x) \gg \frac{\sqrt{x}}{\sqrt{\log x}(\log\log x)^2(\log\log\log x)^2}$. This improves on an example given by Erdős and Sárközy with a lower bound on the counting function of order $\frac{\sqrt{x}}{\log x}$.

Keywords Sequences with Property P · Sums of two squares · Primes in arithmetic progressions · Distribution of integers with given prime factorization

Mathematics Subject Classification 11B83 · 11N13

1 Introduction

Erdős and Sárközy [9] define a monotonically increasing sequence $A = \{a_1 < a_2 < \ldots\}$ of positive integers to have 'Property P' if $a_i \nmid a_j + a_k$ for $i < j \leq k$. They proved

Communicated by J. Schoißengeier.

The authors are supported by the Austrian Science Fund (FWF): W1230, Doctoral Program 'Discrete Mathematics'.

✉ Stefan Planitzer
planitzer@math.tugraz.at

Christian Elsholtz
elsholtz@math.tugraz.at

1 Institute of Analysis and Number Theory, Graz University of Technology, Kopernikusgasse 24/II, 8010 Graz, Austria

that any infinite sequence of integers with Property P has density 0. Schoen [15] showed that if an infinite sequence A has Property P and any two elements in A are coprime then the counting function $A(x) = \sum_{a_i<x} 1$ is bounded from above by $A(x) < 2x^{\frac{2}{3}}$ and Baier [1] improved this to $A(x) < (3+\epsilon)x^{\frac{2}{3}}(\log x)^{-1}$ for any $\epsilon > 0$. Concerning finite sequences with Property P, Erdős and Sárközy [9] get the lower bound $\max A(x) \geq \lfloor \frac{x}{3} \rfloor + 1$ by just taking A to be the set $A = \{x, x-1, \ldots, x - \lfloor \frac{x}{3} \rfloor\}$ for $x \in \mathbb{N}$.

Erdős and Sárközy also thought about large sets with Property P with respect to the size of the counting function (cf. [9, p. 98]). They observed that the set $A = \{q_i^2 : q_i$ the i-th prime with $q_i \equiv 3 \bmod 4\}$ has Property P. This uses the fact that the square of a prime $p \equiv 3 \bmod 4$ has only the trivial representation $p^2 = p^2 + 0^2$ as the sum of two squares. With this set A they get

$$A(x) \sim \frac{\sqrt{x}}{\log x}.$$

Erdős has asked repeatedly to improve this (see e.g. [6, p. 185], [7, p. 535]) and in particular, Erdős [7,8] asked if one can do better than $a_n \sim (2n \log n)^2$. He wanted to know if it is possible to have $a_n < n^2$. We will not quite achieve this but we go a considerable step in this direction. First, we observe that a set of squares of integers consisting of precisely k prime factors $p \equiv 3 \bmod 4$ also has Property P. As for any fixed k this would only lead to a moderate improvement, our next idea is to try to choose k increasing with x. In order to do so, we actually use a union of several sets S_i with Property P. Together, this union will have a good counting function throughout all ranges of x. However, in order to ensure that this union of sets with Property P still has Property P, we employ a third idea, namely to equip all members $a \in S_i$ with a special indicator factor. This seems to be the first improvement going well beyond the example given by Erdős and Sárközy since 1970. Our main result will be the following theorem.

Theorem *The set $S \subset \mathbb{N}$ constructed explicitly below has Property P and counting function*

$$S(x) \gg \frac{\sqrt{x}}{\sqrt{\log x}(\log\log x)^2(\log\log\log x)^2}.$$

We achieve this improvement by not only considering squares of primes $p \equiv 3 \bmod 4$ but products of squares of such primes. More formally we set

$$S = \bigcup_{i=1}^{\infty} S_i. \tag{1}$$

Here the sets S_i are defined by

$$S_i := \left\{ n \in \mathbb{N} : n = q_i^4 \nu^2 \right\}, \tag{2}$$

where ν is the product of exactly i distinct primes $p \equiv 3 \bmod 4$ and we recall that q_i is the i-th prime in the residue class 3 mod 4. The rôle of the q_i is an 'indicator' which uniquely identifies the set S_i a given integer $n \in S$ belongs to. Results from probabilistic number theory like the Theorem of Erdős-Kac suggest that for varying x different sets S_i will yield the main contribution to the counting function $S(x)$. In particular for given $x > 0$ the main contribution comes from the sets S_i with

$$\frac{\log\log\sqrt{x}}{2} - \sqrt{\frac{\log\log\sqrt{x}}{2}} \le i \le \frac{\log\log\sqrt{x}}{2} + \sqrt{\frac{\log\log\sqrt{x}}{2}}.$$

The study of sequences with Property P is closely related to the study of primitive sequences, i.e. sequences where no element divides any other and there is a rich literature on this topic (cf. [10, Chapter V]). Indeed a similar idea as the one described above was used by Martin and Pomerance [13] to construct a large primitive set. While Besicovitch [3] proved that there exist infinite primitive sequences with positive upper density, Erdős [4] showed that the lower density of these sequences is always 0. In his proof Erdős used the fact that for a primitive sequence of positive integers the sum $\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i}$ converges. In more recent work Banks and Martin [2] make some progress towards a conjecture of Erdős which states that in the case of a primitive sequence

$$\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i} \le \sum_{p \in \mathbb{P}} \frac{1}{p \log p}$$

holds. Erdős [5] studied a variant of the Property P problem, also in its multiplicative form.

2 Notation

Before we go into details concerning the proof of the Theorem we need to fix some notation. Throughout this paper $\mathbb{P}$ denotes the set of primes and the letter p (with or without index) will always denote a prime number. We write $\log_k$ for the k-fold iterated logarithm. The functions ω and Ω count, as usual, the prime divisors of a positive integer n without respectively with multiplicity. For two functions $f, g : \mathbb{R} \to \mathbb{R}^+$ the binary relation $f \gg g$ (and analogously $f \ll g$) denotes that there exists a constant $c > 0$ such that for x sufficiently large $f(x) \ge cg(x)$ ($f(x) \le cg(x)$ respectively). Dependence of the implied constant on certain parameters is indicated by subscripts. The same convention is used for the Landau symbol $\mathcal{O}$ where $f = \mathcal{O}(g)$ is equivalent to $f \ll g$. We write $f = o(g)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

3 The set S has Property P

In this section we verify that any union of sets S_i defined in (2) has Property P.

Lemma 1 *Let n_1, n_2 and n_3 be positive integers. If there exists a prime $p \equiv 3 \bmod 4$ with $p|n_1$ and $p \nmid \gcd(n_2, n_3)$, then*

$$n_1^2 \nmid n_2^2 + n_3^2.$$

Proof We prove the Lemma by contradiction. Suppose that $n_1^2 | n_2^2 + n_3^2$. By our assumption there exists a prime $p \equiv 3 \bmod 4$ such that $p|n_1$ and $p \nmid \gcd(n_2, n_3)$. Hence, w.l.o.g. $p \nmid n_2$. We have

$$n_2^2 + n_3^2 \equiv 0 \bmod p$$

and since p does not divide n_2, we get that n_2 is invertible mod p. Hence

$$\left(\frac{n_3}{n_2}\right)^2 \equiv -1 \bmod p$$

a contradiction since -1 is a quadratic non-residue mod p. □

Lemma 2 *Any union of sets S_i defined in (2) has Property P.*

Proof Suppose by contradiction that there exist $a_i \in S_i$, $a_j \in S_j$ and $a_k \in S_k$ with $a_i < a_j \leq a_k$ and $a_i | a_j + a_k$. First suppose that either $S_i \neq S_j$ or $S_i \neq S_k$. Define $l \in \{0, 2\}$ to be the largest exponent such that $q_i^l \,|\, \gcd(a_i, a_j, a_k)$ where we again recall that q_i was defined as the i-th prime in the residue class 3 mod 4. Then

$$\frac{a_i}{q_i^l} \bigg| \frac{a_j}{q_i^l} + \frac{a_k}{q_i^l}.$$

By construction of the sets S_i, S_j and S_k we have that $q_i \big| \frac{a_i}{q_i^l}$ and w.l.o.g. $q_i \nmid \frac{a_j}{q_i^l}$. An application of Lemma 1 finishes this case.

If $S_i = S_j = S_k$ then $\Omega(a_i) = \Omega(a_j) = \Omega(a_k)$. If there is some prime p with $p | \frac{a_i}{q_i^4}$ and ($p \nmid \frac{a_j}{q_i^4}$ or $p \nmid \frac{a_k}{q_i^4}$) we may again use Lemma 1. If no such p exists, then $a_i | a_j$ and $a_i | a_k$ trivially holds. With the restriction on the number of prime factors we get that $a_i = a_j = a_k$. □

4 Products of k distinct primes

In order to establish a lower bound for the counting functions of the sets S_i in (2) we need to count square-free integers containing exactly k distinct prime factors $p \equiv 3 \bmod 4$, but no others, where $k \in \mathbb{N}$ is fixed. For $k \geq 2$ and $\pi_k(x) := \#\{n \leq x : \omega(n) = \Omega(n) = k\}$ Landau [11] proved the following asymptotic formula:

$$\pi_k(x) \sim \frac{x(\log_2 x)^{k-1}}{(k-1)! \log x}.$$

We will need a lower bound of similar asymptotic growth as the formula above for the quantity

$$\pi_k(x;4,3) \;:=\; \#\{n \le x : p|n \Rightarrow p \equiv 3 \bmod 4,\ \omega(n) = \Omega(n) = k\}.$$

Very recently Meng [14] used tools from analytic number theory to prove a generalization of this result to square-free integers having k prime factors in prescribed residue classes. The following is contained as a special case in [14, Lemma 9]:

Lemma A *For any $A > 0$, uniformly for $2 \le k \le A \log\log x$, we have*

$$\pi_k(x;4,3) = \frac{1}{2^k}\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!} \times \left(1+\frac{k-1}{\log\log x}C(3,4)+\frac{2(k-1)(k-2)}{(\log\log x)^2}h''\left(\frac{2(k-3)}{3\log\log x}\right)+\mathcal{O}_A\left(\frac{k^2}{(\log\log x)^3}\right)\right),$$

where $C(3,4) = \gamma + \sum_{p\in\mathbb{P}}\left(\log\left(1-\frac{1}{p}\right)+\frac{2\lambda(p)}{p}\right)$, γ is the Euler-Mascheroni constant, $\lambda(p)$ is the indicator function of primes in the residue class 3 mod 4 *and*

$$h(x) = \frac{1}{\Gamma\left(\frac{x}{2}+1\right)}\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p}\right)^{x/2}\left(1+\frac{x\lambda(p)}{p}\right).$$

We will show that Lemma A with some extra work implies the following Corollary.

Corollary 1 *Uniformly for $\frac{\log\log x}{2} - 1 \le k \le \frac{\log\log x}{2} + \sqrt{\frac{\log\log x}{2}}$ we have*

$$\pi_k(x;4,3) \gg \frac{1}{2^k}\frac{x}{\log x}\frac{(\log_2 x)^{k-1}}{(k-1)!}.$$

Proof In view of Lemma A and with $k \sim \frac{\log\log x}{2}$ we see that it suffices to check that, independent of the choice of k and for sufficiently large x, there exists a constant $c > 0$ such that

$$1+\frac{C(3,4)}{2}+\frac{1}{2}h''\left(\frac{2(k-3)}{3\log\log x}\right) \ge c. \tag{3}$$

Note that the left hand side of the above inequality is exactly the coefficient of the main term $\frac{1}{2^k}\frac{x}{\log x}\frac{(\log_2 x)^{k-1}}{(k-1)!}$ for k in the range given in the Corollary. The constant $C(3,4)$ does not depend on k. Using Mertens' Formula (cf. [16, p. 19: Theorem 1.12]) in the form

$$\sum_{\substack{p\in\mathbb{P}\\ p\le x}}\log\left(1-\frac{1}{p}\right) = -\gamma - \log\log x + o(1)$$

we get

$$C(3,4) = \gamma + \sum_{p\in\mathbb{P}} \left(\log\left(1-\frac{1}{p}\right) + \frac{2\lambda(p)}{p} \right) = 2M(3,4),$$

where $M(3,4)$ is the constant appearing in

$$\sum_{p\in\mathbb{P}} \frac{\lambda(p)}{p} = \frac{\log\log x}{2} + M(3,4) + \mathcal{O}\left(\frac{1}{\log x}\right),$$

which was studied by Languasco and Zaccagnini in [12].[1] The computational results of Languasco and Zaccagnini imply that $0.0482 < M(3,4) < 0.0483$ and hence allow for the following lower bound for $C(3,4)$:

$$C(3,4) = 2M(3,4) > 0.0964. \tag{4}$$

It remains to get a lower bound for $h''\left(\frac{2(k-3)}{3\log\log x}\right)$, where the function h is defined as in Lemma A. A straight forward calculation yields that

$$h' = \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p}\right)^{x/2} \left(1+\frac{x\lambda(p)}{p}\right) \times \frac{\Gamma\left(\frac{x}{2}+1\right)\left(\sum_{p\in\mathbb{P}} \frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p+x\lambda(p)}\right) - \frac{1}{2}\Gamma'\left(\frac{x}{2}+1\right)}{\Gamma\left(\frac{x}{2}+1\right)^2}$$

and

$$h''(x) = f(x) \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p}\right)^{x/2} \left(1+\frac{x\lambda(p)}{p}\right),$$

where

$$f(x) = \frac{\left(\sum_{p\in\mathbb{P}} \frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p+x\lambda(p)}\right)^2}{\Gamma\left(\frac{x}{2}+1\right)} - \frac{\Gamma''\left(\frac{x}{2}+1\right)}{4\Gamma\left(\frac{x}{2}+1\right)^2} - \frac{\sum_{p\in\mathbb{P}} \frac{\lambda(p)}{(p+\lambda(p)x)^2}}{\Gamma\left(\frac{x}{2}+1\right)} - \frac{\Gamma'\left(\frac{x}{2}+1\right)\left(\sum_{p\in\mathbb{P}} \frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p+x\lambda(p)}\right)}{\Gamma\left(\frac{x}{2}+1\right)^2} + \frac{\Gamma'\left(\frac{x}{2}+1\right)^2}{2\Gamma\left(\frac{x}{2}+1\right)^3}.$$

[1] Note that our constant $M(3,4)$ corresponds to the constant $M(4,3)$ in the work of Languasco and Zaccagnini.

Note that for $x \to \infty$ and $\frac{\log\log x}{2} - 1 \le k \le \frac{\log\log x}{2} + \sqrt{\frac{\log\log x}{2}}$ the term $\frac{2(k-3)}{3\log\log x}$ gets arbitrarily close to $\frac{1}{3}$. Hence we may suppose that $\frac{99}{300} \le \frac{2(k-3)}{3\log\log x} \le \frac{101}{300}$ and it suffices to find a lower bound for $h''(x)$ where $\frac{99}{300} \le x \le \frac{101}{300}$. For x in this range Mathematica provides the following bounds on the Gamma function and its derivatives

$$0.9271 \le \Gamma\left(\frac{x}{2}+1\right) \le 0.9283, \ -0.3104 \le \Gamma'\left(\frac{x}{2}+1\right) \le -0.3058,$$
$$1.3209 \le \Gamma''\left(\frac{x}{2}+1\right) \le 1.3302.$$

Furthermore we have

$$\sum_{p\in\mathbb{P}} \frac{\lambda(p)}{(p+x)^2} < \sum_{p\in\mathbb{P}} \frac{\lambda(p)}{p^2} < \sum_{\substack{p\in\mathbb{P}\\ p\le 10^4}} \frac{\lambda(p)}{p^2} + \sum_{n>10^4} \frac{1}{n^2}$$
$$< 0.1485 + \int_{x=10^4}^{\infty} \frac{\mathrm{d}x}{x^2} = 0.1486.$$

Later we will use that

$$\sum_{p\in\mathbb{P}} \left(\frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p+x}\right) = \sum_{p\in\mathbb{P}} \left(\frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p}\right) - x\sum_{p\in\mathbb{P}} \frac{\lambda(p)}{p^2+px}$$
$$> \sum_{p\in\mathbb{P}} \left(\frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p}\right) - x\sum_{p\in\mathbb{P}} \frac{\lambda(p)}{p^2}$$
$$= -\frac{\gamma}{2} + M(3,4) - x\sum_{p\in\mathbb{P}} \frac{\lambda(p)}{p^2} > -0.2905,$$

and

$$\sum_{p\in\mathbb{P}} \left(\frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p+x}\right) < \sum_{p\in\mathbb{P}} \left(\frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p}\right)$$
$$= -\frac{\gamma}{2} + M(3,4) < -0.2403.$$

Finally, using $\log(1+\frac{x}{p}) \le \frac{x}{p}$, we get

$$0 \le \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p}\right)^{x/2} \left(1+\frac{x\lambda(p)}{p}\right) \le \exp\left(x\left(\sum_{p\in\mathbb{P}} \left(\frac{1}{2}\log\left(1-\frac{1}{p}\right) + \frac{\lambda(p)}{p}\right)\right)\right)$$
$$= \exp\left(x\left(-\frac{\gamma}{2} + M(3,4)\right)\right) < \exp\left(-\frac{99}{300}\cdot 0.2403\right) < 0.9238.$$

Applying the explicit bounds calculated above, for $\frac{99}{300} \le x \le \frac{101}{300}$ we obtain:

$$f(x) \ge \frac{0.2403^2}{0.9283} - \frac{1.3302}{4 \cdot 0.9271^2} - \frac{0.1486}{0.9271} - \frac{0.3104 \cdot 0.2905}{0.9271^2} + \frac{0.3058^2}{2 \cdot 0.9283^3} > -0.5315.$$

This implies for sufficiently large x:

$$h''\left(\frac{2(k-3)}{3\log\log x}\right) > -0.492.$$

Together with (4) this leads to an admissible choice of $c = 0.802$ in (3). □

5 The counting function $S(x)$

Proof of Theorem As in (1) we set

$$S = \bigcup_{i=1}^{\infty} S_i$$

where the sets S_i are defined as in (2). The set S has Property P by Lemma 2 and it remains to work out a lower bound for the size of the counting function $S(x)$. For sufficiently large x there exists a uniquely determined integer $k \in \mathbb{N}$ such that $e^{2e^{2k}} \le x < e^{2e^{2(k+1)}}$ hence

$$k \le \frac{\log_2 \sqrt{x}}{2} < k + 1. \tag{5}$$

It depends on the size of x, which S_i makes the largest contribution. For a given x we take several sets $S_{k+2}, S_{k+3}, \ldots, S_{k+l}$, $l = \lfloor \sqrt{\frac{\log_2 \sqrt{x}}{2}} \rfloor$, as the number of prime factors $p \equiv 3 \bmod 4$ of a typical integer less than x is in

$$\left[\frac{\log_2 x}{2} - \sqrt{\frac{\log_2 x}{2}}, \frac{\log_2 x}{2} + \sqrt{\frac{\log_2 x}{2}}\right].$$

Using Corollary 1 as well as the fact that the i-th prime in the residue class 3 mod 4 is asymptotically of size $2i \log i$ for given $2 \le j \le l$ we get

$$S_{k+j}(x) \gg \underbrace{\frac{\sqrt{\frac{x}{16(k+j)^4 \log^4(k+j)}}}{\log\left(\sqrt{\frac{x}{16(k+j)^4 \log^4(k+j)}}\right)}}_{\mathrm{F}_1} \cdot \underbrace{\frac{\left(\log_2 \sqrt{\frac{x}{16(k+j)^4 \log^4(k+j)}}\right)^{k+j-1}}{2^{k+j}(k+j-1)!}}_{\mathrm{F}_2}. \tag{6}$$

We deal with the fractions F_1 and F_2 on the right hand side of (6) separately. With the given range of j and (5) we have that

$$\mathrm{F}_1 \gg \frac{\sqrt{x}}{\log x(\log_2 x)^2(\log_3 x)^2}.$$

It remains to deal with F_2. Using the given range of k and j we have that $k + j \le \log_2 \sqrt{x}$ and, again for sufficiently large x, for the numerator of F_2 we get

$$\begin{aligned}
\log_2^{k+j-1} \sqrt{\frac{x}{16(k+j)^4 \log^4(k+j)}} &\gg (\log(\log \sqrt{x} - \log 4 - 2\log_3 \sqrt{x} - 2\log_4 \sqrt{x}))^{k+j-1} \\
&\gg (\log(\log \sqrt{x} - 5\log_3 \sqrt{x}))^{k+j-1} \\
&= \left(\log_2 \sqrt{x} + \log\left(1 - \frac{5\log_3 \sqrt{x}}{\log \sqrt{x}}\right)\right)^{k+j-1} \\
&\gg \left(\log_2 \sqrt{x} - \frac{10\log_3 \sqrt{x}}{\log \sqrt{x}}\right)^{k+j-1} \\
&\gg \left(1 - \frac{10\log_3 \sqrt{x}}{\log \sqrt{x}\log_2 \sqrt{x}}\right)^{\frac{\log_2 \sqrt{x}}{2} + \sqrt{\frac{\log_2 \sqrt{x}}{2}} - 1} \log_2^{k+j-1} \sqrt{x} \\
&\gg \log_2^{k+j-1} \sqrt{x}.
\end{aligned}$$

Here we used that

$$\lim_{x\to\infty} \left(1 - \frac{10\log_3 \sqrt{x}}{\log \sqrt{x}\log_2 \sqrt{x}}\right)^{\frac{\log_2 \sqrt{x}}{2} + \sqrt{\frac{\log_2 \sqrt{x}}{2}} - 1} = 1$$

and that for $0 \le y \le \frac{1}{2}$ we certainly have that $\log(1-y) \ge -2y$. To deal with the denominator of F_2 we apply Stirling's Formula and get

$$\begin{aligned}
(k+j-1)! &\ll \left(\frac{k+j-1}{e}\right)^{k+j-1} \sqrt{k+j-1} \\
&\ll \left(\frac{\log_2 \sqrt{x} + 2(j-1)}{2e}\right)^{k+j-1} \sqrt{\log_2 x} \\
&\ll (\log_2 \sqrt{x} + 2(j-1))^{k+j-1} \frac{\sqrt{\log_2 x}}{2^{k+j-1} e^{\frac{\log_2 \sqrt{x}}{2} + j - 2}} \\
&\ll (\log_2 \sqrt{x} + 2(j-1))^{k+j-1} \frac{\sqrt{\log_2 x}}{2^{k+j-1} e^{j-2} \sqrt{\log x}}.
\end{aligned}$$

Altogether we get

$$\begin{aligned} \mathrm{F}_2 &\gg \frac{\sqrt{\log x}}{\sqrt{\log_2 x}} e^{j-2} \left(\frac{\log_2 \sqrt{x}}{\log_2 \sqrt{x} + 2(j-1)} \right)^{k+j-1} \\ &\gg \frac{\sqrt{\log x}}{\sqrt{\log_2 x}} e^{j-2} \left(\frac{\log_2 \sqrt{x}}{\log_2 \sqrt{x} + 2(j-1)} \right)^{\frac{\log_2 \sqrt{x}}{2}+j-1}. \end{aligned} \tag{7}$$

Since

$$\left(\frac{\log_2 \sqrt{x}}{\log_2 \sqrt{x} + 2(j-1)} \right)^{\frac{\log_2 \sqrt{x}}{2}} \sim \frac{1}{e^{j-1}}$$

it suffices to check that for any $x > 0$ and for our choices of j there exists a fixed constant $c > 0$ such that

$$\left(1 + \frac{2(j-1)}{\log_2 \sqrt{x}} \right)^{1-j} \geq c. \tag{8}$$

For $j \geq 2$ we have that $\left(1 + \frac{2(j-1)}{\log_2 \sqrt{x}}\right)^{1-j}$ is monotonically decreasing in j and get

$$\left(1 + \frac{2(j-1)}{\log_2 \sqrt{x}} \right)^{1-j} \geq \left(1 + \frac{2\sqrt{\frac{\log_2 \sqrt{x}}{2}}}{\log_2 \sqrt{x}} \right)^{-\sqrt{\frac{\log_2 \sqrt{x}}{2}}} = \left(1 + \frac{1}{\sqrt{\frac{\log_2 \sqrt{x}}{2}}} \right)^{-\sqrt{\frac{\log_2 \sqrt{x}}{2}}} \geq \frac{1}{e}.$$

Therefore for $j \geq 2$ the constant c in (8) may be chosen as $c = \frac{1}{e}$ for sufficiently large x. Together with (7) this implies

$$F_2 \gg \frac{\sqrt{\log x}}{\sqrt{\log_2 x}}.$$

Altogether for the counting function of any of the sets S_i with $\lfloor \frac{\log_2 \sqrt{x}}{2} \rfloor + 2 \leq i \leq \lfloor \frac{\log_2 \sqrt{x}}{2} \rfloor + \lfloor \sqrt{\frac{\log_2 \sqrt{x}}{2}} \rfloor$ we have

$$S_i(x) \gg \frac{\sqrt{x}}{\sqrt{\log x} (\log_2 x)^{\frac{5}{2}} (\log_3 x)^2}.$$

Summing these contributions up we finally get

$$S(x) \gg \frac{\sqrt{x}}{\sqrt{\log x} (\log_2 x)^2 (\log_3 x)^2}.$$

□

Acknowledgements Open access funding provided by Graz University of Technology. Parts of this research work were done when the first author was visiting the FIM at ETH Zürich, and the second author was visiting the Institut Élie Cartan de Lorraine of the University of Lorraine. The authors thank these institutions for their hospitality. The authors are also grateful to the referee for suggestions on the manuscript and would like to thank Xianchang Meng for some discussion on his recent paper [14].

References

1. Baier, S.: A note on P-sets. Integers **4**(A13), 6 (2004)
2. Banks, W. D., Martin, G.: Optimal primitive sets with restricted primes. Integers **13**, A69, 10 (2013)
3. Besicovitch, A.S.: On the density of certain sequences of integers. Math. Ann. **110**(1), 336–341 (1935)
4. Erdős, P.: Note on Sequences of integers no one of which is divisible by any other. J. London Math. Soc. **10**(1), 126–128 (1935)
5. Erdős, P.: On sequences of integers no one of which divides the product of two others and on some related problems. Mitt. Forsch.-Inst. Math. Mech. Univ. Tomsk **2**, 74–82 (1938)
6. Erdős, P.: Some old and new problems on additive and combinatorial number theory. In: Combinatorial Mathematics: Proceedings of the Third International Conference (New York, 1985), *Ann. New York Acad. Sci.*, vol. 555, pp. 181–186. New York Acad. Sci., New York (1989)
7. Erdős, P.: Some of my favourite unsolved problems. Math. Japon. **46**(3), 527–538 (1997)
8. Erdős, P.: Some of my new and almost new problems and results in combinatorial number theory. In: Number theory (Eger, 1996), pp. 169–180. de Gruyter, Berlin (1998)
9. Erdős, P., Sárközi, A.: On the divisibility properties of sequences of integers. Proc. London Math. Soc. **21**, 97–101 (1970)
10. Halberstam, H., Roth, K.F.: Sequences, 2nd edn. Springer-Verlag, New York-Berlin (1983)
11. Landau, E.: Sur quelques problèmes relatifs à la distribution des nombres premiers. Bull. Soc. Math. France **28**, 25–38 (1900)
12. Languasco, A., Zaccagnini, A.: Computing the Mertens and Meissel-Mertens constants for sums over arithmetic progressions. Experiment. Math. **19**(3), 279–284 (2010). With an appendix by Karl K. Norton, computational results available online: http://www.math.unipd.it/~languasc/Mertenscomput/Mqa/Msumfinalresults.pdf (URL last checked: 08.08.2016)
13. Martin, G., Pomerance, C.: Primitive sets with large counting functions. Publ. Math. Debrecen **79**(3–4), 521–530 (2011)
14. Meng, X.: Large bias for integers with prime factors in arithmetic progressions. ArXiv e-prints, available at 1607.01882 (2016)
15. Schoen, T.: On a problem of Erdős and Sárközy. J. Combin. Theory Ser. A **94**(1), 191–195 (2001)
16. Tenenbaum, G.: Introduction to analytic and probabilistic number theory, *Graduate Studies in Mathematics*, vol. 163, third edn. American Mathematical Society, Providence, RI (2015)