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Guaranteeing Disturbance Rejection and Control Signal Continuity for the Saturated Super-Twisting Algorithm

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***Abstract*—The saturated super-twisting algorithm is a second order sliding-mode control law for robust control in the presence of a bounded control input. Its implementation is based on a switching logic and the resulting control signal typically exhibits a single jump discontinuity. This contribution presents novel stability conditions that allow for tuning the algorithm such that perturbation amplitudes up to the control input bound are rejected with a continuous control signal. A simplified control law is furthermore proposed, which is equivalent to the original algorithm while being easier to implement.**

***Index Terms*—Variable-structure/sliding-mode control, constrained control, Lyapunov methods.**

I. Introduction

MITIGATING or eliminating the influence of disturbances is one of the main goals of feedback control. The field of sliding mode control offers several techniques for dealing with large classes of disturbances, such as bounded and Lipschitz continuous signals. One commonly used second order sliding mode control law in that regard is the classical super-twisting algorithm originally proposed in [1]. Closed-loop stability conditions and performance indicators, such as the finite convergence time and chattering amplitudes, have extensively been studied in literature, see, e.g., [2], [3], [4] and [5], [6], [7], [8], [9], respectively.

In practice, control input constraints have to be taken into account. The design of sliding mode algorithms for saturated actuators or constrained systems in general has thus repeatedly been studied, see, e.g., [10], [11], [12]. For the super-twisting algorithm in particular, a number of modifications to deal with bounded control inputs have recently been proposed and analyzed in terms of stability, see, e.g., [13], [14], [15]. One promising approach is the so-called saturated super-twisting algorithm [13]. It consists in applying a relay controller initially, until some switching condition is met, and using the super-twisting control law afterwards. By means of a Lyapunov based stability proof it is shown in [13] that the control input stays within the saturation limits and that all trajectories converge to the origin. The switch leads to a discontinuity in the control signal, however, and the proof in principle can not guarantee convergence for perturbations that exceed half the control input bound in magnitude.

This contribution presents a new stability proof that remedies these facts. In particular, it yields conditions for the rejection of *perturbations with any amplitude* less than the control input bound, and permits to tune the switching condition such that a *continuous control signal* is obtained. For the case of a continuous control signal, an equivalent version of the control law is furthermore proposed that is considerably easier to implement.

This letter is structured as follows: Section II presents the considered plant and discusses some notational conventions and definitions. The saturated super-twisting algorithm and its existing stability proof are then briefly discussed in Section III. Its limitations regarding the continuity of the control signal and the permissible perturbation bounds are illustrated. Section IV presents the letter's main results, a stability condition and a simplified equivalent control law that guarantee disturbance rejection by means of a continuous control signal. The corresponding stability proof is then discussed in Section V; in addition to the main results, it provides additional conditions for the case of a discontinuous control signal that extend the results obtained in [13]. Section VI compares these conditions to results from literature and illustrates the algorithm's performance in a simulation. Section VII concludes this letter. Some of the more technical proofs of some lemmas and propositions are given in an Appendix.

II. Preliminaries

As in [13], a plant

$$\dot{x} = u + w \tag{1a}$$

with the state x, the control input u, and a perturbation w is considered. The perturbation is assumed to be bounded and

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composed of two components w_1 and w_2 that are Hölder continuous in the state and Lipschitz continuous in time, respectively, i.e., $w = w_1 + w_2$ with

$$|w| \leq W, \qquad |w_1| \leq K\sqrt{|x|}, \qquad |\dot{w}_2| \leq L, \tag{1b}$$

where W, K, and L are non-negative constants.

The control task is to steer the state x to zero despite the perturbation w by using a control input u that is bounded by

$$|u| \leq U. \tag{2}$$

Therein, U is a positive control input bound. Clearly, this bound has to exceed the maximum perturbation amplitude W for the task to be feasible; thus, $U > W$ is assumed. For this purpose, the following definition is introduced.

Definition 1: A control law is said to globally stabilize the plant (1) in finite time, if for every initial condition $x(0)$ there exists a positive time instant τ such that for any admissible perturbation w one has $x(t) = 0$ for all $t \geq \tau$.

The abbreviation $\lfloor y \rceil^p := |y|^p \operatorname{sign}(y)$ is commonly used throughout this letter, and $\lfloor y \rceil^0$ in particular is written for the sign of y. The saturation function sat_M with positive parameter M is defined as

$$\operatorname{sat}_M(y) := \begin{cases} y & |y| \leq M \\ M\lfloor y \rceil^0 & |y| > M. \end{cases} \tag{3}$$

All solutions of differential equations with discontinuous right-hand side are understood in the sense of Filippov [16].

III. Control Algorithm

This section discusses the saturated super-twisting algorithm and briefly sketches the original stability proof given in [13].

A. Control Law

The saturated super-twisting algorithm consists in applying a relay controller up to a time instant T, when the state x for the first time satisfies $|x(T)| \leq \gamma^2$. Therein, γ denotes a non-negative switching level parameter. For $t \geq T$, a super-twisting control law is used.

The saturated super-twisting control law is thus given by

$$u = \begin{cases} -U\lfloor x \rceil^0 & t < T \\ -k_1\lfloor x \rceil^{\frac{1}{2}} + v & t \geq T, \end{cases} \tag{4a}$$

$$\dot{v} = \begin{cases} 0 & t < T \\ -k_2\lfloor x \rceil^0 & t \geq T, \end{cases} \tag{4b}$$

$$T = \inf\{t : |x(t)| \leq \gamma^2\}, \tag{4c}$$

with initial value $v(0) = 0$. At the switching time instant, one has $v(T) = 0$ and therefore

$$u(T) = -k_1\lfloor x(T) \rceil^{\frac{1}{2}} + v(T) = -k_1\gamma\lfloor x(T) \rceil^0. \tag{5}$$

Thus, one can see that $k_1\gamma \leq U$ is required for $u(T)$ to satisfy the control input bound, and that a continuous control signal is obtained if and only if

$$\gamma = k_1^{-1}U \tag{6}$$

holds.

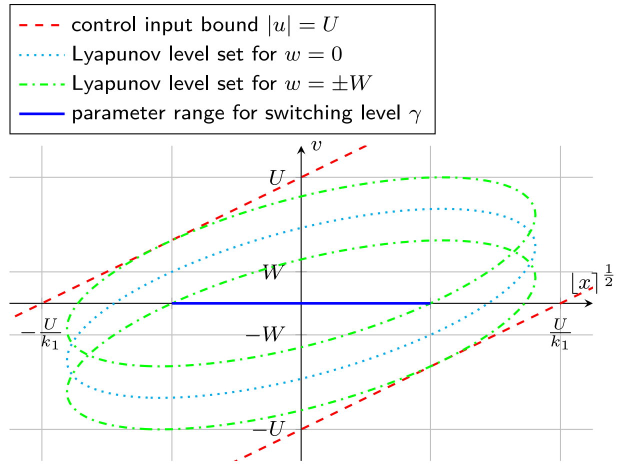

Fig. 1. Illustration of the original Lyapunov-based stability proof from [13] in the $\lfloor x \rceil^{1/2}$-v-plane: a Lyapunov level set, where the control input bound is satisfied, is guaranteed to be entered at the switching time instant T if the switching level γ is chosen within the depicted range.

B. Original Stability Proof

With the proof in [13], the perturbation amplitude W must not exceed half the control input bound U, and obtaining a continuous control signal is only possible if W is zero. To see this, the original proof is briefly discussed in the following.

The proof is based on a Lyapunov function, originally proposed in [2], which is a quadratic form with respect to the variables $\lfloor x \rceil^{\frac{1}{2}}$ and $v + w$. For this Lyapunov function, a level set is constructed, inside which the control input bound is satisfied. This is illustrated in Fig. 1 in the $\lfloor x \rceil^{\frac{1}{2}}$-$v$-plane, where the level set has the shape of an ellipse. As one can see, the location of the ellipse's center depends on the value of the disturbance w. If the trajectory is guaranteed to be inside this level set for $t = T$, it stays inside and satisfies the control input bound for all $t \geq T$. Since v is zero at the switching time instant, the permitted range of the switching level parameter γ is thus given by the part of the $\lfloor x \rceil^{\frac{1}{2}}$-axis that is contained in the level set for any value of w.

One can see geometrically in Fig. 1 that it is impossible to obtain a range for γ that includes the choice (6) required for a continuous control signal, regardless of the ellipse's shape, for a nonzero perturbation bound W. Thus, guaranteeing disturbance rejection in the presence of a continuous control signal is impossible with the original proof. Furthermore, one can see that the maximum perturbation amplitude is constrained to values $W \leq 0.5U$, because the intersection of the level set for $w = W$ and for $w = -W$, and thus also the set of permitted values for γ, is empty otherwise.

IV. Main Result

In the following, the main results of this letter are stated: conditions that permit rejecting perturbations with any amplitude $W < U$ while maintaining a continuous control signal, and a control law that is equivalent to (4) but is easier to implement.

Theorem 1: For given positive parameters k_1, k_2, non-negative perturbation bounds W, K, L, and a positive control input bound U, consider the control law (4) with the switching level $\gamma = k_1^{-1} U$ and initial condition $v(0) = 0$. If the perturbation bounds satisfy the conditions $L < k_2$ and

$$K < k_1 - \sqrt{2L + 2k_2}, \tag{7a}$$

$$W < U \frac{k_1^2 - K k_1 - 2k_2}{k_1^2 - K k_1 + 2k_2}, \tag{7b}$$

then this control law globally stabilizes the plant (1) in finite time with a control input u that is continuous with respect to time, satisfies $|u| \leq U$, and is equivalently obtained via the control law

$$u = \operatorname{sat}_U(-k_1 \lfloor x \rceil^{\frac{1}{2}} + v), \tag{8a}$$

$$\dot{v} = \begin{cases} 0 & |u| = U \\ -k_2 \lfloor x \rceil^0 & |u| < U \end{cases} \tag{8b}$$

with initial condition $v(0) = 0$.

Proof: A proof of Theorem 1 is given in Section V. ■

Remark 1: The simplified control law (8) essentially uses conditional integration to prevent integrator wind-up. A similar modification is proposed in [10] for the so-called "sub-optimal" second order sliding mode controller, see [17]. The result also fits nicely into the context of classical anti-windup schemes for controllers with integral part, with additional stability guarantees in the presence of disturbances.

Remark 2: Solving (7) for k_1 shows that for $W < U$ the theorem's conditions are equivalent to $k_2 > L$ and

$$k_1 > K + \sqrt{2L + 2k_2}, \tag{9a}$$

$$k_1 > \frac{K}{2} + \sqrt{\frac{K^2}{4} + 2k_2 \frac{U + W}{U - W}}. \tag{9b}$$

This form of the conditions is useful for tuning purposes, because it permits to select parameters k_1, k_2 for any given value of K, L, and $W < U$.

Remark 3: One can see from (9) that $k_1 \geq \sqrt{2k_2}$ is a necessary condition for the theorem's applicability. Several parameter settings have been proposed in literature that satisfy this condition. It is shown in [9], for example, that choosing $k_1 \approx 2\sqrt{k_2}$ minimizes the chattering amplitude in the presence of a second order actuator. In [18] it is furthermore shown that having $k_1 \geq 2\sqrt{k_2}$ is desirable if one seeks to minimize the impact of brief violations of the assumed disturbance bounds in practice.

Figs. 2 and 3 show block diagrams of the original control law (4) and the simplified control law (8). One can see that compared to the former the complexity of the latter is considerably reduced.

V. Stability Proof

Similar to [13], the proposed stability proof is based on the construction of an invariant set. To construct this set, the behavior of the control input u at the saturation limit, i.e., for $|u| = U$, is first considered. Computing the time derivative of $|u|$ for $|u| < U$ along the closed-loop trajectories yields

$$\begin{aligned} \frac{\mathrm{d}|u|}{\mathrm{d}t} &= -\frac{k_1}{2}|x|^{-\frac{1}{2}} \dot{x} \lfloor u \rceil^0 + \dot{v} \lfloor u \rceil^0 \\ &= -\frac{k_1}{2}|x|^{-\frac{1}{2}} (|u| + w \lfloor u \rceil^0) - k_2 \lfloor x \rceil^0 \lfloor u \rceil^0. \end{aligned} \tag{10}$$

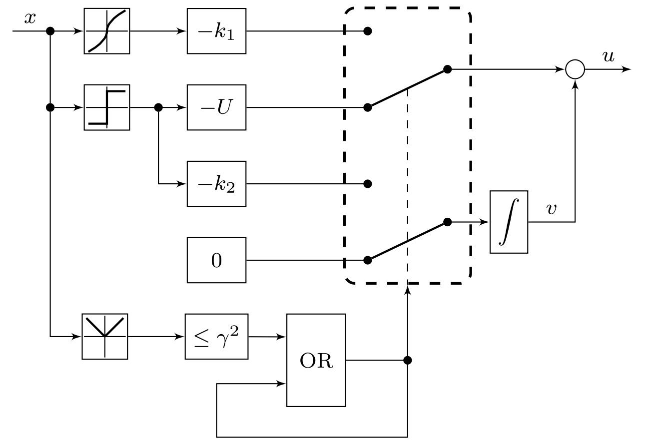

Fig. 2. Block diagram of the saturated super-twisting algorithm (4). The comparison block's output is one when its input does not exceed γ^2 and zero otherwise, the or-gate's output is initialized with zero, and the switches are in the upper position when the input of the switchbox is non-zero.

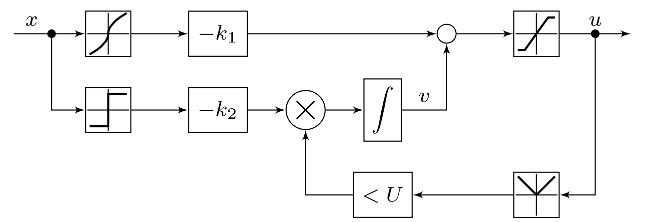

Fig. 3. Block diagram of the simplified control law (8) for the saturated super-twisting algorithm with continuous control input. The comparison block's output is one when its input is less than U and zero otherwise.

In the limit, as $|u|$ tends to U, an upper bound is given by

$$\frac{\mathrm{d}|u|}{\mathrm{d}t} \leq -\frac{k_1}{2}|x|^{-\frac{1}{2}}(U - W) + k_2. \tag{11}$$

This bound is negative, and thus u can neither exceed nor slide along the saturation limit $|u| = U$, if

$$|x|^{\frac{1}{2}} < \frac{k_1}{2k_2}(U - W) \tag{12}$$

holds. In the following, a Lyapunov level set is designed, within which x satisfies this bound.

The region, where (12) holds, corresponds to a vertical strip in the $\lfloor x \rceil^{\frac{1}{2}}$-$v$-plane. The elliptic level sets of the quadratic Lyapunov function shown in Fig. 1 are therefore not the best choice for this purpose. A new Lyapunov function V, whose level sets are tailored to the task at hand, is thus proposed. For this purpose, the state variables $x_1 := x$, $x_2 := v + w_2$ and new perturbations $\delta_1 := w_1 |x_1|^{-\frac{1}{2}}$, $\delta_2 := \dot{w}_2$ are introduced. For $t \geq T$, the closed-loop dynamics are then governed by

$$\dot{x}_1 = -k_1 \lfloor x_1 \rceil^{\frac{1}{2}} + x_2 + \delta_1 |x_1|^{\frac{1}{2}} \tag{13a}$$

$$\dot{x}_2 = -k_2 \lfloor x_1 \rceil^0 + \delta_2 \tag{13b}$$

and the perturbations δ_1, δ_2 are bounded by

$$|\delta_1| \leq K, \quad |\delta_2| \leq L. \tag{13c}$$

With the state vector $\mathbf{x} := \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$, the functions

$$V_1(x_1) := (k_1 - K)^2 |x_1|, \quad V_2(x_2) := x_2^2, \tag{14}$$

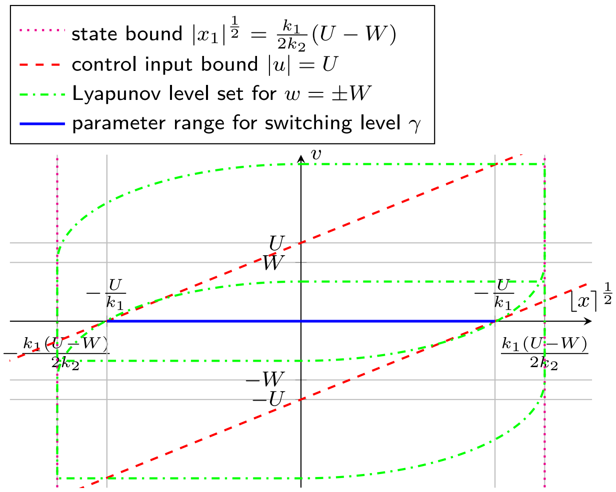

Fig. 4. Illustration of the proposed Lyapunov-based stability proof in the $\lfloor x \rceil^{1/2}$-v-plane: the Lyapunov level set is constructed, inside which the state bound (12) holds; this guarantees that the control input amplitude can not exceed U.

and the state space regions

$$\mathcal{M}_1 := \{\mathbf{x} : 0 < x_2 \lfloor x_1 \rceil^{\frac{1}{2}} \le (k_1 - K)|x_1|\}, \tag{15a}$$
$$\mathcal{M}_2 := \{\mathbf{x} : (k_1 - K)|x_1| < x_2 \lfloor x_1 \rceil^{\frac{1}{2}}\}, \tag{15b}$$
$$\mathcal{M}_3 := \{\mathbf{x} : x_2 \lfloor x_1 \rceil^{\frac{1}{2}} \le 0\}, \tag{15c}$$

the proposed function $V(\mathbf{x})$ is defined as

$$V(\mathbf{x}) = \begin{cases} V_1(x_1) & \mathbf{x} \in \mathcal{M}_1 \\ V_2(x_2) & \mathbf{x} \in \mathcal{M}_2 \\ V_1(x_1) + V_2(x_2) & \mathbf{x} \in \mathcal{M}_3. \end{cases} \tag{16}$$

The conditions, for which it is a Lyapunov function for system (13), are given in the following lemma.

Lemma 1: Suppose that $k_1 > K$ holds. Then, the function $V(\mathbf{x})$ defined in (16) is continuous and positive definite. Furthermore, if the conditions

$$k_2 > L, \qquad k_1 \ge K + \sqrt{2(k_2 + L)} \tag{17}$$

hold, then V is non-increasing along the trajectories of system (13), i.e., it is a (weak) Lyapunov function for that system.

Proof: Given in the Appendix. ■

Using the proposed Lyapunov function, conditions guaranteeing that u satisfies the control input bound (2) after the switching time instant T are derived. In Fig. 4, this procedure is illustrated in the $\lfloor x \rceil^{\frac{1}{2}}$-$v$-plane: The largest intersection of a Lyapunov level set, wherein (12) holds, with the $\lfloor x \rceil^{\frac{1}{2}}$-axis is determined. When γ is contained in the interval thus obtained, (12) and thus $|u| \le U$ are guaranteed for all time. The resulting conditions and their actual computation are given in the following proposition and its proof, respectively.

Proposition 1: Consider the closed loop formed by the plant (1) and the control law (4) with positive parameters k_1, k_2, a positive control input bound U, and the non-negative switching level parameter $\gamma \le k_1^{-1}U$. If the conditions of Lemma 1 are fulfilled and the perturbation bounds K and W additionally satisfy

$$(k_1 - K)^2\gamma^2 + (W + K\gamma)^2 < (k_1 - K)^2 \frac{k_1^2(U-W)^2}{4k_2^2}, \tag{18}$$

then $|u| \le U$ holds for all times and $|u| = U$ only holds at isolated time instants.

Proof: Given in the Appendix. ■

Condition (18) is the best condition that may be obtained with the proposed invariant set, but it is quite involved. To obtain a simpler, though slightly more conservative condition, one may use the fact that

$$\begin{aligned}(k_1 - K)^2\gamma^2 + (W + K\gamma)^2 &< \left[(k_1 - K)\gamma + (W + K\gamma)\right]^2 \\ &= (k_1\gamma + W)^2 \end{aligned} \tag{19}$$

holds, to see that the inequality

$$k_1\gamma + W < (k_1 - K)\frac{k_1(U-W)}{2k_2} \tag{20}$$

implies (18). Solving this inequality (20) for W yields the following corollary to Proposition 1.

Corollary 1: Consider the closed loop formed by the plant (1) and the control law (4) with positive parameters k_1, k_2, a positive control input bound U, and the non-negative switching level parameter $\gamma \le k_1^{-1}U$. If the conditions of Lemma 1 are fulfilled and the maximum perturbation amplitude W is bounded by

$$W < \frac{(k_1^2 - Kk_1)U - 2k_1k_2\gamma}{k_1^2 - Kk_1 + 2k_2}, \tag{21}$$

then the control input bound $|u| \le U$ holds for all times and $|u| = U$ only holds at isolated time instants.

Proof: The inequality (21) is equivalent to (20), which implies condition (18) of Proposition 1. ■

To actually prove Theorem 1, convergence of trajectories to the origin for $t \ge T$, i.e., finite time stability of system (13), remains to be shown. Since the proposed Lyapunov function's time derivative is not negative definite, and proving finite time stability would thus require more extensive arguments, the following result presented in [19] is used in the following.

Lemma 2 [19]: If the parameters k_1, k_2 and perturbation bounds K, L satisfy

$$k_2 > L, \qquad k_1 > K + \sqrt{k_2 + L}, \tag{22}$$

then system (13) is finite time stable.

Proof: Given in [19, Sec. 3]. ■

Theorem 1 may now be proven. As shown in Section III-A, the control signal is continuous for $\gamma = k_1^{-1}U$. Furthermore, conditions (7b) and (21) are equivalent for this value of γ. Thus, $|u| \le U$ holds for all times due to Corollary 1, and the closed-loop dynamics are given by system (13) after the switching time instant. Since (7a) and $k_2 > L$ imply (22), Lemma 2 guarantees finite time stability of that system. To see equivalence of control laws (4) and (8), note that for $\gamma = k_1^{-1}U$ the switching time instant T defined in (4c) can be rewritten as $T = \inf \mathcal{N}$ with the set $\mathcal{N}$ given by

$$\mathcal{N} = \left\{t : \left|-k_1 \lfloor x(t) \rceil^{\frac{1}{2}} + v(t)\right| \le U\right\}, \tag{23}$$

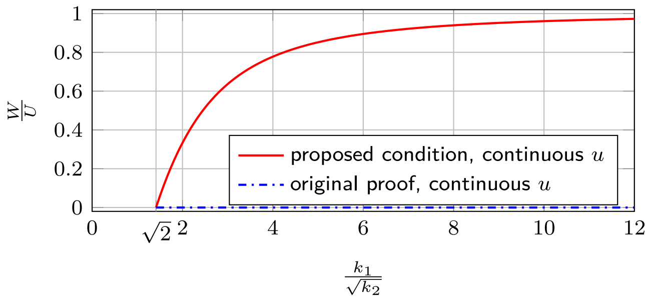

Fig. 5. Upper bounds for the perturbation amplitude W for $K = L = 0$ permitted by the proposed conditions and the stability proof in [13] in case of a continuous control signal.

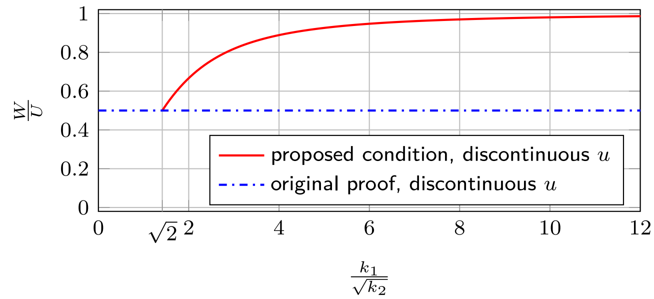

Fig. 6. Upper bounds for the perturbation amplitude W for $K = L = 0$ permitted by the proposed conditions and the stability proof in [13] in case of a discontinuous control signal with switching level parameter $\gamma = 0$.

because $v(t) = 0$ for $t \leq T$. Since $|u| < U$ holds for almost all $t \geq T$, one may replace the condition $t \geq T$ by $|u| < U$. This yields the control law (8), which leads to a continuous $u(t)$, because the saturation function and involved arguments are continuous. ■

VI. Comparisons and Simulation Results

This section compares the proposed stability conditions to those imposed in [13], and illustrates the closed-loop performance by means of a simulation.

A. Comparison of Perturbation Amplitudes

For $\gamma = 0$ or $W = 0$, the conditions in [13] are given by

$$W \leq \frac{U}{1 + \sqrt{1 + \frac{(p_{12}+k_1 p_{22})^2}{p_{11}p_{22}-p_{12}^2}}} \quad \text{or} \quad \gamma \leq \frac{U}{\sqrt{k_1^2 + \frac{(p_{11}+k_1 p_{12})^2}{p_{11}p_{22}-p_{12}^2}}}, \tag{24}$$

respectively. Therein, $p_{11}, p_{22} > 0$ and p_{12} are parameters that satisfy $p_{11}p_{22} > p_{12}^2$ and have to be chosen such that

$$\tilde{V}(\mathbf{x}) = p_{11}|x_1| + 2p_{12}\lfloor x_1 \rceil^{\frac{1}{2}} x_2 + p_{22}|x_2|^2 \tag{25}$$

is a Lyapunov function for system (13). Both conditions in (24) become more restrictive if either γ or W are nonzero. Thus, $W \leq 0.5U$ has to hold in any case. Furthermore, choosing $\gamma = k_1^{-1}U$ to obtain a continuous control signal is only possible, if $W = 0$ and $p_{11} + k_1 p_{12} = 0$ hold. This requires the inequality $k_1 \geq \sqrt{2k_2}$ to be fulfilled. To see this, evaluate the time derivative of $\tilde{V}$ along the trajectories of (13) without perturbations for $x_2 = 0$; this yields

$$\begin{aligned} \dot{\tilde{V}}\Big|_{x_2=0} &= -(k_1 p_{11} + 2k_2 p_{12})|x_1|^{\frac{1}{2}} \\ &= -(k_1 - \frac{2k_2}{k_1})p_{11}|x_1|^{\frac{1}{2}}. \end{aligned} \tag{26}$$

One can thus see that $k_1 \geq \sqrt{2k_2}$ is a necessary condition for $\tilde{V}$ to be a Lyapunov function in this case.

Fig. 5 shows upper bounds imposed on W by Theorem 1 for the case of a continuous control signal in comparison to those imposed by the original proof. One can see that in both cases $k_1 \geq \sqrt{2k_2}$ has to hold, but only the proposed approach permits nonzero perturbation amplitudes W.

Fig. 6 depicts the same comparison for the case of a discontinuous control signal using the bound obtained from Corollary 1 for $\gamma = 0$. One can see that in this case the original proof is the only one applicable for $k_1 < \sqrt{2k_2}$, while the proposed approach imposes a less restrictive condition otherwise.

B. Numerical Simulation

Fig. 7 compares the saturated super-twisting algorithm's performance with a continuous and with a discontinuous control signal. For demonstration purposes, a constant perturbation $w = 0.5U = W$ with $U = 5$ is applied. Therefore, $K = L = 0$ holds, and $k_1 = \sqrt{6k_2}$ with $k_2 = 1$ is chosen in order to satisfy (9). The simulation was performed using forward Euler discretization with a step size of 10^{-4}. One can see that for $\gamma = 0$ the closed loop takes longer to converge, in addition to the discontinuity that is present in the control signal. While the speed of convergence depends on the perturbation, of course, the depicted behavior is representative for the obtained performance on average.

VII. Conclusion

In this contribution, a novel stability condition for the saturated super-twisting algorithm was presented. It guarantees disturbance rejection using a continuous control signal and permits to significantly simplify the control law's implementation. It is furthermore straightforward to use for the purpose of parameter selection. The presented results thus make it considerably easier to implement and tune the saturated super-twisting algorithm in practice.

Appendix

Proof of Lemma 1: The function V defined in (16) is continuous, because $V_1 = 0$ for $x_1 = 0$, $V_2 = 0$ for $x_2 = 0$, and $V_1 = V_2$ for $x_2 = (k_1 - K)\lfloor x_1 \rceil^{\frac{1}{2}}$. It is furthermore positive definite, because $V_1 + V_2$ is positive definite and having $V_1 = 0$ in case $\mathbf{x} \in \mathcal{M}_1$ or $V_2 = 0$ in case $\mathbf{x} \in \mathcal{M}_2$ leads to the contradictions

$$0 < x_2 \lfloor x_1 \rceil^{\frac{1}{2}} = 0 \quad \text{or} \quad 0 \leq (k_1 - K)|x_1| < 0, \tag{27}$$

respectively.

Computing the time derivative of V along the trajectories of (13) yields for $\mathbf{x} \in \mathcal{M}_1$

$$\begin{aligned} \dot{V}_1 &= (k_1 - K)^2(-k_1|x_1|^{\frac{1}{2}} + \lfloor x_1 \rceil^0 x_2 + \delta_1 \lfloor x_1 \rceil^{\frac{1}{2}}) \\ &\leq (k_1 - K)^2(-K + \delta_1 \lfloor x_1 \rceil^0)|x_1|^{\frac{1}{2}} \leq 0, \end{aligned} \tag{28}$$

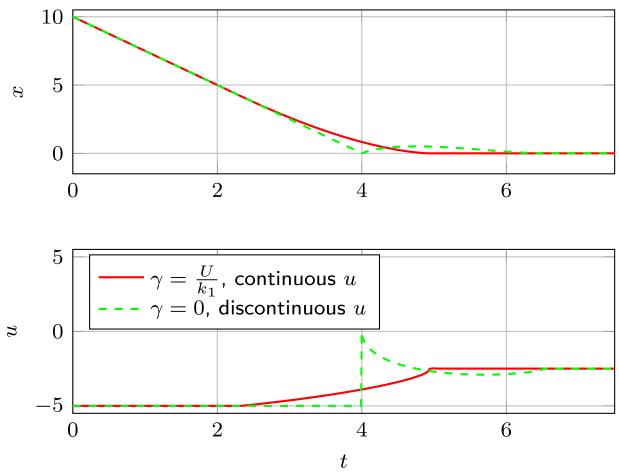

Fig. 7. Plant state x and control input u obtained in a simulation with parameters $k_2 = 1$, $k_1 = \sqrt{6}$, control input bound $U = 5$, and constant perturbation $w = 2.5$ using the saturated super-twisting control law with a continuous and a discontinuous control signal.

because $x_2\lfloor x_1\rceil^0 \le (k_1 - K)|x_1|^{\frac{1}{2}}$, and for $\mathbf{x} \in \mathcal{M}_2$

$$\dot{V}_2 = 2x_2\lfloor x_1\rceil^0(-k_2 + \delta_2\lfloor x_1\rceil^0) \le 0, \tag{29}$$

because $x_2\lfloor x_1\rceil^0 \ge 0$ and $k_2 > L$. In case $\mathbf{x} \in \mathcal{M}_3$, i.e., if $x_2\lfloor x_1\rceil^0 \le 0$ holds, one has

$$\begin{aligned}\dot{V}_1 + \dot{V}_2 &= x_2\lfloor x_1\rceil^0\big[(k_1 - K)^2 - 2k_2 + 2\delta_2\lfloor x_1\rceil^0\big]\\ &\quad + (k_1 - K)^2(-k_1 + \delta_1\lfloor x_1\rceil^0)|x_1|^{\frac{1}{2}}\\ &\le x_2\lfloor x_1\rceil^0\big[(k_1 - K)^2 - 2(k_2 + L)\big] \le 0, \end{aligned}\tag{30}$$

because k_1 satisfies (17). Thus, one has $\dot{V} \le 0$, i.e., V is non-increasing along the trajectories of system (13). ∎

Proof of Proposition 1: Consider the largest level set of the Lyapunov function V defined in (16), inside which $x_1 = x$ is bounded by (12). It is characterized by $V(\mathbf{x}) < c_1$ with

$$c_1 = \min\left\{V(\mathbf{x}) : |x_1| = \frac{k_1^2(U - W)^2}{4k_2^2}\right\}. \tag{31}$$

With this x_1, the minimum is attained for $\mathbf{x} \in \mathcal{M}_2$ as

$$\begin{aligned}c_1 &= \inf\left\{V_2(x_2) : |x_2| > (k_1 - K)|x_1|^{\frac{1}{2}}\right\}\\ &= V_1(x_1) = (k_1 - K)^2\frac{k_1^2(U - W)^2}{4k_2^2},\end{aligned}\tag{32}$$

because V is bounded by this value from below also in the two other cases $\mathbf{x} \in \mathcal{M}_1$ and $\mathbf{x} \in \mathcal{M}_3$.

The largest value of V at the switching time instant T, denoted by c_2 in the following, is now determined. At this time instant one has $|x_1| = \gamma^2$ and $v = 0$, and thus

$$x_2 = v + w_2 = v + w - \delta_1|x_1|^{\frac{1}{2}} = w - \delta_1\gamma. \tag{33}$$

Restricting considerations to positive values of x_1 due to symmetry, one finds that c_2 is given by

$$c_2 = \max_{|w|\le W}\max_{|\delta_1|\le K} V\Big(\big[\gamma^2 \quad w - \delta_1\gamma\big]^{\mathrm{T}}\Big). \tag{34}$$

Looking at (16), one can see that this maximum is obtained in case $\mathbf{x} \in \mathcal{M}_3$, i.e., for $x_2 \le 0$, because V is smaller in either of the two other cases. Thus, one finds

$$\begin{aligned}c_2 &= V_1(\gamma^2) + V_2(-W - K\gamma)\\ &= (k_1 - K)^2\gamma^2 + (W + K\gamma)^2.\end{aligned}\tag{35}$$

If $c_2 < c_1$, i.e., (18), holds, then $V < c_1$ holds at the switching time instant T. Due to Lemma 1, this inequality and thus the state bound (12) are satisfied for all $t \ge T$. Since the relation $|u(T)| = k_1\gamma \le U$ is fulfilled, and $\frac{\mathrm{d}|u|}{\mathrm{d}t} < 0$ holds as $|u|$ tends to U as a consequence of (11), the bound (2) is satisfied for $t \ge T$ and no sliding along $|u| = U$ occurs. ∎

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