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Optimal Lyapunov-Based Reaching Time Bounds for the Super-Twisting Algorithm

Richard Seeber*, Martin Horn

***Abstract*—The super-twisting algorithm is a second order sliding mode control law commonly used for robust control and observation. One of its key properties is the finite time it takes to reach the sliding surface. Using Lyapunov theory, upper bounds for this reaching time may be found. This contribution considers the problem of finding the best bound that may be obtained using a family of quadratic Lyapunov functions. An optimization problem for finding this bound is derived, whose solution may be obtained using semidefinite programming. It is shown that the restrictions imposed on the perturbations and the conservativeness of the obtained bound are significantly reduced compared to existing results from literature.**

***Index Terms*—Variable-structure/sliding-mode control; Convergence Time; Lyapunov Functions; Optimization; LMIs**

I. Introduction

CONTROL in the presence of disturbances or parasitic nonlinearities acting on a plant is an important topic in control engineering. One field that offers several techniques for dealing with this problem is sliding mode control. A common sliding mode control technique is the so-called super-twisting algorithm [1], which may be used as a robust control law, see, e.g., [2], [3], and as an observer, such as, e.g., in [4], [5]. When used as a controller, it can steer the plant towards a sliding surface in finite time, which is typically constructed such that desired dynamics are obtained in this sliding mode.

The time it takes to reach the surface, the so-called reaching time, is an important closed-loop performance characteristic. Obtaining upper bounds for it is important, especially when disturbances are present, because these may otherwise delay convergence in a prohibitive way. Such bounds have extensively been studied in literature, see, e.g., [6], [7], [8], [9]. In [9], these approaches are compared. Each imposes certain restrictions on the perturbations, and the bounds differ in conservativeness. For small perturbations, the tightest bounds are obtained with the approach recently proposed in [9], which is based on a reaching time computation for the unperturbed case. Regarding perturbations, it is more restrictive than [7], however, which is based on a family of Lyapunov functions. The latter typically yields more conservative bounds, though, even when selecting the Lyapunov function in an optimal way.

The present contribution revisits this Lyapunov function family based estimate. As a motivation, a formal analysis of its conservativeness is first presented. The problem of finding the best reaching time bound obtainable using the Lyapunov function family is then considered. An optimization problem for this bound is derived, which may efficiently be solved numerically by means of semidefinite programming.

The paper is structured as follows. In Section II, some preliminaries including the considered Lyapunov function family and the reaching time estimate in [7] are briefly reviewed. Section III formally states the problem of finding the best reaching time bound obtainable using the considered Lyapunov function family. This problem's solution—the paper's main result—is given in Section IV. Its derivation is discussed in the following two sections: Section V derives a semidefinite program for the best reaching time bound that may be obtained with a given Lyapunov function, and Section VI shows how to minimize this bound with respect to all considered Lyapunov functions. Section VII, finally, compares results obtained using the proposed technique with existing results from literature and simulations. Section VIII concludes the paper.

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II. Preliminaries

This section discusses some notational conventions and then introduces the super-twisting algorithm and its reaching time function. It furthermore gives an overview of the considered Lyapunov function family and the existing reaching time estimate that is based on it.

A. Notational Conventions

The abbreviation $\lfloor y \rceil^p := |y|^p \operatorname{sign}(y)$ is commonly used throughout the paper, and $\lfloor y \rceil^0$ is written for the sign of y. The trace of a matrix $\mathbf{M}$ is denoted by $\operatorname{tr} \mathbf{M}$. For a symmetric matrix $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$, $\lambda_{\min}(\mathbf{M})$ and $\lambda_{\max}(\mathbf{M})$ denote its smallest and largest eigenvalue, and $\mathbf{M} \succeq 0$ or $\mathbf{M} \succ 0$ denote positive semidefiniteness or positive definiteness of $\mathbf{M}$, respectively.

B. Super-Twisting Algorithm

Consider a sliding variable σ governed by $\dot{\sigma} = u + w_1 + w_2$ with a control input u and perturbations w_1 and w_2, which satisfy $|w_1| \le K |\sigma|^{\frac{1}{2}}$ and $|\dot{w}_2| \le L$, respectively. The super-twisting control law

$$u(t) = -k_1 \lfloor \sigma(t) \rceil^{\frac{1}{2}} - k_2 \int_0^t \lfloor \sigma(\tau) \rceil^0 \, \mathrm{d}\tau. \tag{1}$$

is used to steer σ to zero in finite time. With the state variables $x_1 := \sigma$, $x_2 := w_2 - k_2 \int_0^t \lfloor \sigma(\tau) \rceil^0 \, \mathrm{d}\tau$ and perturbations $\delta_1 := w_1 |\sigma|^{-\frac{1}{2}}$, $\delta_2 := \dot{w}_2$, the closed loop is then given by the system

$$\dot{x}_1 = -k_1 \lfloor x_1 \rceil^{\frac{1}{2}} + x_2 + \delta_1 |x_1|^{\frac{1}{2}}, \tag{2a}$$

$$\dot{x}_2 = -k_2 \lfloor x_1 \rceil^0 + \delta_2, \tag{2b}$$

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with δ_1 and δ_2 bounded by

$$|\delta_1| \leq K, \qquad |\delta_2| \leq L. \tag{2c}$$

Solutions of this perturbed nonlinear system are understood in the sense of Filippov, see [10], and its state is aggregated in the vector $\mathbf{x} := [x_1 \quad x_2]^{\mathrm{T}}$.

The reaching time function of system (2) with perturbation bounds K, L is denoted by $T_{K,L}$ and defined as

$$T_{K,L}(\mathbf{x}_0) := \max_{\substack{|\delta_1| \leq K \\ |\delta_2| \leq L}} \min\{\tau : \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(t) = \mathbf{0} \ \forall t \geq \tau\}, \tag{3}$$

i.e., it yields the maximum time it takes for trajectories with initial state $\mathbf{x}_0$ to reach the origin for any perturbation.

C. Family of Quadratic Lyapunov Functions

In [7], [11], a family of quadratic Lyapunov functions

$$V = \mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}, \quad \text{with } \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \succ 0 \tag{4}$$

is proposed. Therein, the matrix $\mathbf{P}$ is positive definite and $\mathbf{z} = [z_1 \quad z_2]^{\mathrm{T}} = \mathbf{g}(\mathbf{x})$, with the function $\mathbf{g}$ given by

$$\mathbf{g}(\mathbf{x}) := \begin{bmatrix} \lfloor x_1 \rceil^{\frac{1}{2}} & x_2 \end{bmatrix}^{\mathrm{T}}. \tag{5}$$

The time derivative of V along the trajectories of system (2) satisfies the inequality

$$\dot{V} \leq -\frac{1}{|z_1|}\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z}, \quad \text{with } \mathbf{Q} \succ 0, \tag{6}$$

provided that the positive definite matrix $\mathbf{Q}$ and some positive constants Θ_1, Θ_2 satisfy the Riccati inequality

$$\begin{aligned} 0 \succeq \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} + \mathbf{P}\left(\frac{K}{4\Theta_1}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}} + \frac{L}{\Theta_2}\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}\right)\mathbf{P} \\ + (\Theta_1 K + \Theta_2 L)\,\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}, \end{aligned} \tag{7}$$

cf. [11], with the abbreviations

$$\mathbf{A} := \begin{bmatrix} -\frac{k_1}{2} & \frac{1}{2} \\ -k_2 & 0 \end{bmatrix}, \qquad \mathbf{e}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{8}$$

As discussed, e.g., in [7], [11], this inequality can be rewritten by means of the Schur complement as the equivalent linear matrix inequality $\mathbf{M} \succeq 0$ with

$$\mathbf{R} := -\mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\mathbf{A} - \mathbf{Q}, \tag{9a}$$

$$\mathbf{M} := \begin{bmatrix} \mathbf{R} - (\Theta_1 K + \Theta_2 L)\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}} & K\mathbf{P}\mathbf{e}_1 & L\mathbf{P}\mathbf{e}_2 \\ K\mathbf{e}_1^{\mathrm{T}}\mathbf{P} & 4\Theta_1 K & 0 \\ L\mathbf{e}_2^{\mathrm{T}}\mathbf{P} & 0 & \Theta_2 L \end{bmatrix}. \tag{9b}$$

D. Reaching Time Estimation

The Lyapunov function V defined in (4) is known to satisfy a differential inequality of the form

$$\dot{V} \leq -\frac{2}{\alpha}\sqrt{V} \tag{10}$$

with $\alpha > 0$. In [7], this fact is used to compute an upper bound for the reaching time of system (2). Rewriting (10) yields

$$\frac{\mathrm{d}\sqrt{V}}{\mathrm{d}t} \leq -\frac{1}{\alpha}. \tag{11}$$

This shows that the time for V to converge to zero, and thus the reaching time with initial state $\mathbf{x}_0$, is bounded by

$$T_{K,L}(\mathbf{x}_0) \leq \alpha\sqrt{V(\mathbf{x}_0)} = \alpha\sqrt{\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0)}. \tag{12}$$

In [7], a bound of the form (10) based on eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$ is given, which yields the reaching time estimate

$$T_{K,L}(\mathbf{x}_0) \leq \tilde{T}_{K,L}(\mathbf{x}_0) := \tilde{\alpha}(\mathbf{P},\mathbf{Q})\sqrt{\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0)} \tag{13}$$

with

$$\tilde{\alpha}(\mathbf{P},\mathbf{Q}) := \frac{2\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{Q})\sqrt{\lambda_{\min}(\mathbf{P})}}. \tag{14}$$

III. Problem Statement

The considered problem is motivated by the following result, which shows that the eigenvalue-based reaching time bound (13) can be quite conservative for large values of k_1.

Proposition 1: Consider system (2) with positive parameters k_1, k_2 and non-negative perturbation bounds K, L. If

$$k_1 \geq K + \sqrt{8(k_2 + L)}, \tag{15}$$

then the ratio of $\tilde{T}_{K,L}$ given in (13) and the reaching time function $T_{K,L}$ for initial states of the form $\mathbf{x}_0 = c\mathbf{e}_2$ with $c \neq 0$ and $\mathbf{e}_2$ as in (8) is bounded from below by

$$\frac{\tilde{T}_{K,L}(c\mathbf{e}_2)}{T_{K,L}(c\mathbf{e}_2)} \geq k_1 + \frac{1}{k_1} \tag{16}$$

for any positive definite matrices $\mathbf{P}$, $\mathbf{Q}$ that satisfy the Riccati inequality (7) with any positive scalars Θ_1, Θ_2.

Proof: Given in the appendix. ■

To improve the reaching time estimation, the task of finding the best bound that may be obtained using the quadratic Lyapunov function family is considered. This involves first finding a value of α such that the differential inequality (10) for a given Lyapunov function is as tight as possible. Then, the resulting reaching time bound is minimized with respect to all Lyapunov functions from the family.

To formally specify these problems, the function

$$\beta(\mathbf{z},\mathbf{P},\mathbf{Q}) := \frac{2\,|z_1|\,\sqrt{\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}}}{\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z}} \tag{17}$$

is introduced and the time derivative bound (6) is written as

$$\dot{V} \leq -\frac{\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z}}{|z_1|} = -\frac{2}{\beta(\mathbf{z},\mathbf{P},\mathbf{Q})}\sqrt{V}. \tag{18}$$

The differential inequality (10) holds for any α that satisfies $\alpha \geq \beta(\mathbf{z},\mathbf{P},\mathbf{Q})$ for all $\mathbf{z} \neq \mathbf{0}$. To find the smallest reaching time bound for given matrices $\mathbf{P}$, $\mathbf{Q}$ one seeks the smallest possible value of α that fulfills this relation, i.e.,

$$\overline{\alpha}(\mathbf{P},\mathbf{Q}) = \sup_{\mathbf{z}} \beta(\mathbf{z},\mathbf{P},\mathbf{Q}). \tag{19}$$

The best obtainable reaching time bound $\overline{T}_{K,L}$ is then given by the minimum of (12) for this value of α with respect to all positive definite matrices $\mathbf{P}$, $\mathbf{Q}$ and positive scalars Θ_1, Θ_2 that satisfy the Riccati inequality (7):

$$\overline{T}_{K,L}(\mathbf{x}_0) := \inf_{\substack{\mathbf{P},\mathbf{Q} \\ \text{s.t. (7)}}} \overline{\alpha}(\mathbf{P},\mathbf{Q})\sqrt{\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0)}. \tag{20}$$

This paper considers the task of computing $\overline{\alpha}$ and $\overline{T}_{K,L}$.

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IV. Main Result

Finding $\overline{T}_{K,L}$ may be achieved by solving the following *scalar* nonlinear optimization problem, whose objective function may be computed by means of a semidefinite program. Its derivation is discussed in Sections V and VI.

Theorem 1: Consider system (2) with given positive parameters k_1, k_2 and non-negative perturbation bounds K, L. Using the abbreviations $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$ in (8), define the matrix $\mathbf{M}(\mathbf{P}, \mathbf{Q}, \Theta_1, \Theta_2)$ as in (9) and the matrix

$$\mathbf{N}(\mathbf{P}, \mathbf{Q}, \gamma, \kappa) := \begin{bmatrix} \gamma\mathbf{Q} - \mathbf{P} & \gamma\mathbf{e}_1 \\ \gamma\mathbf{e}_1^{\mathrm{T}} & \kappa \end{bmatrix} \tag{21}$$

with $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{2\times 2}$ and $\Theta_1, \Theta_2, \gamma, \kappa \in \mathbb{R}$. Furthermore, let the function $\mathbf{g}(\mathbf{x})$ be defined as in (5), and, for a given initial state $\mathbf{x}_0 \in \mathbb{R}^2$, let $f(\gamma, \mathbf{x}_0)$ be the solution of the semidefinite program

$$f(\gamma, \mathbf{x}_0) := \min_{\substack{\mathbf{P},\mathbf{Q},\\ \Theta_1,\Theta_2,\kappa}} \kappa \tag{22a}$$

subject to

$$\mathbf{M}(\mathbf{P}, \mathbf{Q}, \Theta_1, \Theta_2) \succeq 0, \qquad \mathbf{P}, \mathbf{Q} \succeq 0, \tag{22b}$$

$$\mathbf{N}(\mathbf{P}, \mathbf{Q}, \gamma, \kappa) \succeq 0, \qquad \mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0) = 1. \tag{22c}$$

If positive definite matrices $\mathbf{P}$, $\mathbf{Q}$ satisfying the Riccati inequality (7) exist, then this optimization problem is feasible for sufficiently large values of $\gamma > 0$, and the best obtainable reaching time bound $\overline{T}_{K,L}$ as defined in (20) is given by the solution of the scalar nonlinear optimization problem

$$\overline{T}_{K,L}(\mathbf{x}_0) = \min_{\gamma \geq 0} \sqrt{f(\gamma, \mathbf{x}_0)}. \tag{23}$$

Furthermore, this value bounds the reaching time function of system (2) from above, i.e., $T_{K,L}(\mathbf{x}_0) \leq \overline{T}_{K,L}(\mathbf{x}_0)$.

Proof: Given in Sections V and VI-A. ■

Remark 1: Note that unlike (22), the overall optimization problem for finding $\overline{T}_{K,L}$ is not a semidefinite program. This is due to the bilinear term $\gamma\mathbf{Q}$ occuring in the matrix $\mathbf{N}$ in (21). Despite this fact, the solution may efficiently be found numerically by performing a line search with respect to the scalar decision variable γ. A useful proposition for this purpose, which permits to constrain the search to values of γ in a finite interval, is stated in Section VI-B.

V. Optimal Differential Inequality Bound

This section considers the task of obtaining the optimal differential inequality bound $\overline{\alpha}$ by maximizing $\beta(\mathbf{z}, \mathbf{P}, \mathbf{Q})$ defined in (17) with respect to $\mathbf{z}$, i.e., the optimization problem (19). This problem is given by

$$\overline{\alpha}(\mathbf{P}, \mathbf{Q}) = \sup_{\mathbf{z}} \beta(\mathbf{z}, \mathbf{P}, \mathbf{Q}) = \sup_{\mathbf{z}} \frac{2\,|z_1|\,\sqrt{\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}}}{\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z}}. \tag{24}$$

It yields the best reaching time bound of the form (12) for a given Lyapunov function from the considered family.

Since the objective function is invariant with respect to any nonzero scaling of $\mathbf{z}$, i.e., $\beta(\lambda\mathbf{z}, \mathbf{P}, \mathbf{Q}) = \beta(\mathbf{z}, \mathbf{P}, \mathbf{Q})$ for any $\lambda \neq 0$, it is sufficient to restrict considerations to nonzero values of $\mathbf{z}$ that satisfy

$$\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z} = 2z_1. \tag{25}$$

Squaring the objective function then yields the expression

$$\beta(\mathbf{z}, \mathbf{P}, \mathbf{Q})^2\big|_{\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z}=2z_1} = \mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}, \tag{26}$$

which is to be maximized subject to the constraint (25).

Upper bounds for this non-convex maximization problem may be found by considering its Lagrangian dual, see, e.g., [12]. Introducing the Lagrange function $\mathcal{L}(\mathbf{z}, \gamma)$ as

$$\mathcal{L}(\mathbf{z}, \gamma) := \mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z} - \gamma(\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z} - 2z_1) \tag{27}$$

with the Lagrange multiplier γ, one has

$$\overline{\alpha}(\mathbf{P}, \mathbf{Q})^2 = \sup_{\mathbf{z}} \inf_{\gamma} \mathcal{L}(\mathbf{z}, \gamma) \leq \inf_{\gamma} \sup_{\mathbf{z}} \mathcal{L}(\mathbf{z}, \gamma). \tag{28}$$

For a given value of γ, the supremum of $\mathcal{L}(\mathbf{z}, \gamma)$ with respect to $\mathbf{z}$ may be found analytically. To that end, first order necessary and second order sufficient optimality conditions for a maximum are obtained as

$$\frac{1}{2}\frac{\partial \mathcal{L}(\mathbf{z}, \gamma)}{\partial \mathbf{z}} = -\mathbf{z}^{\mathrm{T}}(\gamma\mathbf{Q} - \mathbf{P}) + \gamma\mathbf{e}_1^{\mathrm{T}} = 0, \tag{29a}$$

$$\frac{1}{2}\frac{\partial^2 \mathcal{L}(\mathbf{z}, \gamma)}{\partial \mathbf{z}^2} = -(\gamma\mathbf{Q} - \mathbf{P}) \prec 0. \tag{29b}$$

Solving (29a) for $\mathbf{z}$ and substituting into (27) shows that

$$\sup_{\mathbf{z}} \mathcal{L}(\mathbf{z}, \gamma) = \gamma^2\mathbf{e}_1^{\mathrm{T}}(\gamma\mathbf{Q} - \mathbf{P})^{-1}\mathbf{e}_1, \tag{30}$$

if the matrix $\gamma\mathbf{Q} - \mathbf{P}$ is positive definite, whereas the problem is unbounded if this matrix is negative semidefinite or indefinite. Recall now that the matrix $\mathbf{N}$ defined in (21) is positive definite if and only if $\gamma\mathbf{Q} - \mathbf{P}$ is positive definite and its Schur complement is positive, i.e., if $\gamma\mathbf{Q} - \mathbf{P} \succ 0$ and

$$\kappa - (\gamma\mathbf{e}_1^{\mathrm{T}})(\gamma\mathbf{Q} - \mathbf{P})^{-1}(\gamma\mathbf{e}_1) > 0. \tag{31}$$

Using (28) and (30), one thus obtains an upper bound for the optimal differential inequality bound $\overline{\alpha}$ by solving the semidefinite program

$$\overline{\alpha}^2 \leq \inf_{\substack{\gamma\\ \gamma\mathbf{Q}-\mathbf{P}\succ 0}} \gamma^2\mathbf{e}_1^{\mathrm{T}}(\gamma\mathbf{Q} - \mathbf{P})^{-1}\mathbf{e}_1 = \inf_{\substack{\gamma,\kappa\\ \mathbf{N}\succ 0}} \kappa = \min_{\substack{\gamma,\kappa\\ \mathbf{N}\succeq 0}} \kappa. \tag{32}$$

The following proposition shows that this optimization problem yields not only an upper bound for $\overline{\alpha}$, but in fact $\overline{\alpha}$ itself, i.e., that equality holds in (28).

Proposition 2: Let positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ satisfying the Riccati inequality (7) be given, and let κ^* be the solution of the semidefinite program

$$\kappa^* = \min_{\gamma,\kappa} \kappa \tag{33a}$$

subject to

$$\mathbf{N}(\mathbf{P}, \mathbf{Q}, \gamma, \kappa) = \begin{bmatrix} \gamma\mathbf{Q} - \mathbf{P} & \gamma\mathbf{e}_1 \\ \gamma\mathbf{e}_1^{\mathrm{T}} & \kappa \end{bmatrix} \succeq 0. \tag{33b}$$

Then, the optimal differential inequality bound $\overline{\alpha}$ defined in (19) is given by $\overline{\alpha}(\mathbf{P}, \mathbf{Q}) = \sqrt{\kappa^*}$. Furthermore, the reaching time function $T_{K,L}$ of system (2) is bounded by

$$T_{K,L}(\mathbf{x}_0) \leq \sqrt{\kappa^*\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0)}, \tag{34}$$

with the function $\mathbf{g}(\mathbf{x})$ defined in (5).

Proof: Given in the appendix. ■

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VI. Optimal Lyapunov Function Selection

Proposition 2 yields the best obtainable reaching time bound for a *given* Lyapunov function from the considered family. This section considers the optimal selection of the Lyapunov function such as to obtain the optimal bound $\overline{T}_{K,L}$ defined in (20), i.e., the optimization problem

$$\overline{T}_{K,L}(\mathbf{x}_0) = \inf_{\substack{\mathbf{P},\mathbf{Q} \\ \text{s.t. } (7)}} \overline{\alpha}(\mathbf{P},\mathbf{Q})\sqrt{\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0)}. \tag{35}$$

An equivalent optimization problem with a bilinear matrix inequality and otherwise linear objective function and constraints is derived and its numerical solution is discussed.

A. Optimization Problem

Squaring the objective function and using Proposition 2 to obtain $\overline{\alpha}^2$, one can write the optimization problem (35) as

$$\overline{T}_{K,L}(\mathbf{x}_0)^2 = \min_{\substack{\mathbf{P},\mathbf{Q}\succ 0 \\ \Theta_1,\Theta_2>0}} \min_{\kappa,\gamma} \kappa \mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0) \tag{36a}$$

subject to

$$0 \preceq \begin{bmatrix} \gamma\mathbf{Q}-\mathbf{P} & \gamma\mathbf{e}_1 \\ \gamma\mathbf{e}_1^{\mathrm{T}} & \kappa \end{bmatrix} = \mathbf{N}(\mathbf{P},\mathbf{Q},\gamma,\kappa), \tag{36b}$$

$$\begin{aligned} 0 \succeq\ & \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} + \mathbf{P}\left(\frac{K}{4\Theta_1}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}} + \frac{L}{\Theta_2}\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}\right)\mathbf{P} \\ & + (\Theta_1 K + \Theta_2 L)\,\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}. \end{aligned} \tag{36c}$$

This problem contains the bilinear terms $\gamma\mathbf{Q}$ and $\kappa\mathbf{P}$, and is constrained by the nonlinear matrix Riccati inequality (36c). As discussed in Section II-C, the Riccati inequality can be replaced by the equivalent linear matrix inequality

$$\mathbf{M}(\mathbf{P},\mathbf{Q},\Theta_1,\Theta_2) \succeq 0, \tag{37}$$

with the matrix $\mathbf{M}$ defined in (9), by applying the Schur complement. To furthermore eliminate the bilinearity $\kappa\mathbf{P}$, note that for any positive scalar λ one has

$$\mathbf{N}(\lambda\mathbf{P},\lambda\mathbf{Q},\gamma,\lambda^{-1}\kappa) = \mathbf{D}_\lambda^{\mathrm{T}}\mathbf{N}(\mathbf{P},\mathbf{Q},\gamma,\kappa)\mathbf{D}_\lambda, \tag{38a}$$
$$\mathbf{M}(\lambda\mathbf{P},\lambda\mathbf{Q},\lambda\Theta_1,\lambda\Theta_2) = \lambda\mathbf{M}(\mathbf{P},\mathbf{Q},\Theta_1,\Theta_2), \tag{38b}$$
$$(\lambda^{-1}\kappa)\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}(\lambda\mathbf{P})\mathbf{g}(\mathbf{x}_0) = \kappa\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0), \tag{38c}$$

with the diagonal matrix $\mathbf{D}_\lambda = \operatorname{diag}(\sqrt{\lambda},\sqrt{\lambda},\sqrt{\lambda}^{-1})$. Decision variables scaled in this way thus satisfy all constraints without changing the value of the objective function. Therefore, as discussed for example in [13], one can restrict considerations to matrices $\mathbf{P}$ that satisfy

$$\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0) = 1. \tag{39}$$

One thus obtains the following optimization problem with the linear objective function κ, the linear equality constraint (39), the linear matrix inequality (37), and the matrix inequality (36b), which is bilinear in γ and $\mathbf{Q}$:

$$\overline{T}_{K,L}(\mathbf{x}_0)^2 = \min_{\substack{\mathbf{P},\mathbf{Q}\succeq 0 \\ \gamma,\kappa,\Theta_1,\Theta_2\geq 0}} \kappa \tag{40a}$$

subject to

$$\mathbf{M}(\mathbf{P},\mathbf{Q},\Theta_1,\Theta_2) \succeq 0, \qquad \mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0) = 1, \tag{40b}$$
$$\mathbf{N}(\mathbf{P},\mathbf{Q},\gamma,\kappa) \succeq 0. \tag{40c}$$

Theorem 1 may now be proven. Due to (38b), positive definite matrices $\mathbf{P}$, $\mathbf{Q}$ satisfying the Riccati inequality (7), or equivalently (37), also satisfy (40b) after appropriate scaling. Thus, the optimization problem (40) is feasible for sufficiently large values of γ; it is furthermore equivalent to (22), (23), which completes the proof. ∎

B. Numerical Solution

Due to the bilinear term $\gamma\mathbf{Q}$ in the matrix $\mathbf{N}$ given in (36b), the optimization problem (40) is not a convex problem. One way to solve the problem numerically is to perform a line search with respect to $\gamma \geq 0$, as suggested in Theorem 1. In the course of this procedure, it is useful to bound the location of the minimum from above. In the following, a way for finding such a bound by solving another semidefinite program is discussed; this permits to limit the search for the optimal value of γ to a finite interval of the form $[0,\overline{\gamma}]$.

Proposition 3: Let the function $f(\gamma,\mathbf{x}_0)$ be defined as in Theorem 1, and let an initial state $\mathbf{x}_0 \in \mathbb{R}^2$ and a positive constant γ_0 be given. Suppose that the semidefinite program (22) is feasible for $\gamma = \gamma_0$. Then, the solution $\overline{\gamma}$ of

$$\overline{\gamma} := \max_{\mathbf{P},\mathbf{Q},\Theta_1,\Theta_2} \operatorname{tr}(\mathbf{Q}) f(\gamma_0,\mathbf{x}_0) \tag{41a}$$

subject to

$$\mathbf{M}(\mathbf{P},\mathbf{Q},\Theta_1,\Theta_2) \succeq 0, \qquad \mathbf{P} \succeq 0, \tag{41b}$$
$$\mathbf{g}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{P}\mathbf{g}(\mathbf{x}_0) = 1, \qquad \mathbf{Q} \succeq 0 \tag{41c}$$

bounds the location of the minimum of f from above, i.e.,

$$\overline{\gamma} \geq \arg\min_{\gamma\geq 0} \sqrt{f(\gamma,\mathbf{x}_0)}. \tag{42}$$

Proof: Consider any solution $\mathbf{P},\mathbf{Q},\kappa$ of the optimization problem (22) with $\gamma > \overline{\gamma}$. Then, one has due to (31)

$$\begin{aligned} f(\gamma,\mathbf{x}_0) = \kappa &\geq \gamma^2\mathbf{e}_1^{\mathrm{T}}(\gamma\mathbf{Q}-\mathbf{P})^{-1}\mathbf{e}_1 \geq \frac{\gamma^2}{\lambda_{\max}(\gamma\mathbf{Q}-\mathbf{P})} \\ &\geq \frac{\gamma^2}{\operatorname{tr}(\gamma\mathbf{Q}-\mathbf{P})} \geq \frac{\gamma^2}{\operatorname{tr}(\gamma\mathbf{Q})} = \frac{\gamma}{\operatorname{tr}\mathbf{Q}} \\ &\geq \gamma\overline{\gamma}^{-1} f(\gamma_0,\mathbf{x}_0) > f(\gamma_0,\mathbf{x}_0). \end{aligned} \tag{43}$$

Thus, γ cannot be the minimum of the function f. ∎

VII. Numerical Comparisons

This section compares the reaching time bound proposed in Theorem 1 to the eigenvalue based bound (13) and to the reaching time function based bound proposed in [9]. The semidefinite programs were solved[1] using YALMIP [14] and SEDUMI [15], and for the line search the MATLAB function `fminsearch` was used. For the eigenvalue based approach, the matrices $\mathbf{P}$, $\mathbf{Q}$ are selected as proposed in [7], i.e., by minimizing[2] $\tilde{\alpha}$ given in (14) with respect to all matrices that satisfy $\lambda_{\min}(\mathbf{Q}) = 1$ and the Riccati inequality (7).

[1]All computations were done using MATLAB version R2013b, YALMIP version R20150918, and SEDUMI version 1.3 in standard configuration.

[2]This minimization may also be performed using semidefinite programming by minimizing $2v$ subject to (37), $\mathbf{Q} \succeq \mathbf{I}$, $\lambda_1\mathbf{I} \succeq \mathbf{P} \succeq \lambda_2\mathbf{I}$, and

$$\begin{bmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & v \end{bmatrix} \succeq 0. \tag{44}$$

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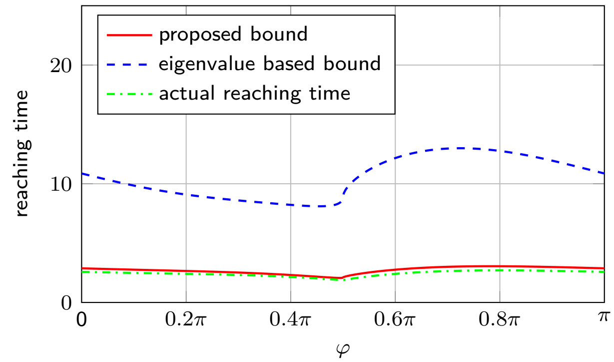

Fig. 1. Comparison of actual reaching time and reaching time bounds for the unperturbed system with parameters $k_1 = k_2 = 1$ and initial state $\mathbf{x}_0 = \begin{bmatrix}\cos\varphi & \sin\varphi\end{bmatrix}^{\mathrm{T}}$ as a function of φ.

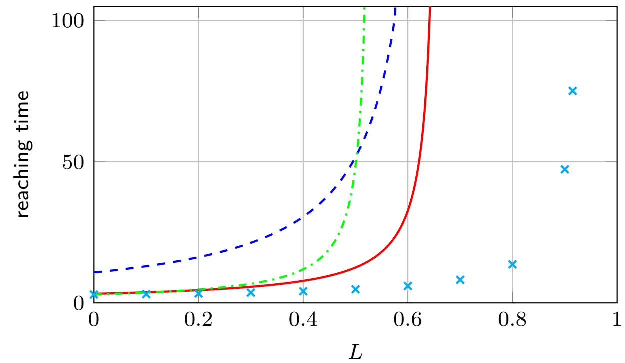

Fig. 3. Comparison of reaching time bounds with $k_1 = k_2 = 1$ and initial state $\mathbf{x}_0 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$, and simulation results with $\delta_2 = L\lfloor \dot{x}_1 \rceil^0$ as a function of the perturbation bound L for $K = 0$. For legend see Fig. 4.

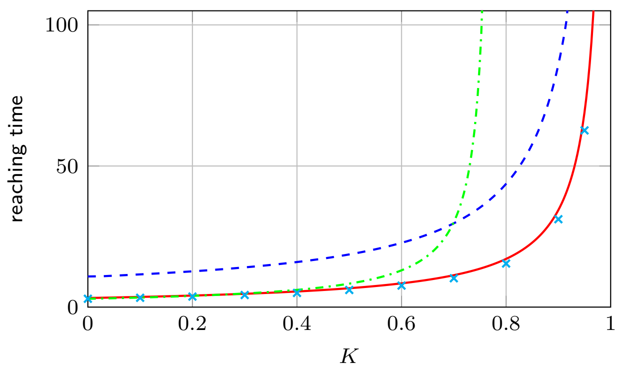

Fig. 2. Comparison of reaching time bounds with $k_1 = k_2 = 1$ and initial state $\mathbf{x}_0 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$, and simulation results with $\delta_1 = K\lfloor x_1 \rceil^0$ as a function of the perturbation bound K for $L = 0$. For legend see Fig 4.

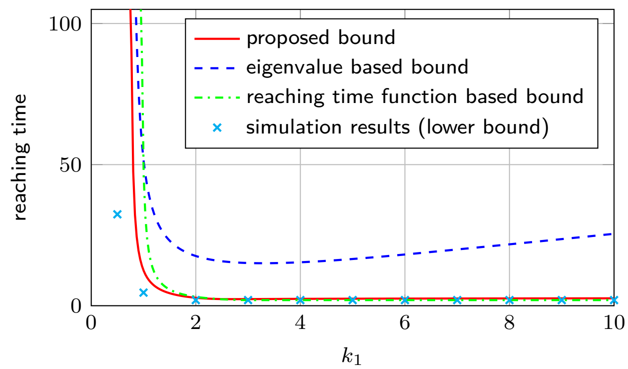

Fig. 4. Comparison of reaching time bounds with $k_2 = 1$ and initial state $\mathbf{x}_0 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$, and simulation results with $\delta_2 = L\lfloor x_1 \rceil^0$ as a function of the parameter k_1 for $K = 0$, $L = 0.5$.

Fig. 1 shows bounds obtained in the unperturbed case, i.e., for $K = L = 0$, and compares them to the actual reaching time for initial states on a half circle in the x_1-x_2-plane. One can see that the proposed approach significantly improves on the eigenvalue based reaching time bound and yields results that very close to the actual reaching time.

In Figs. 2 and 3, upper reaching time bounds for a particular initial state are shown as a function of the perturbation bounds K and L. They are compared to simulation results obtained for particular choices of perturbations, which constitute lower reaching time bounds. Compared to the other approaches, the proposed technique improves the obtained estimates and extends the range of permitted perturbations.

Fig. 4, finally, shows reaching time estimates and simulation results as a function of the parameter k_1. As suggested by Proposition 1, the eigenvalue based approach diverges with increasing k_1, while the proposed approach yields values close to actual reaching times for large values of k_1.

VIII. Conclusion and Outlook

This contribution considered the problem of finding the best reaching time bound for the super-twisting algorithm that may be obtained using a family of Lyapunov functions. Optimization problems were proposed for finding this bound for a given Lyapunov function and for finding the best bound obtainable with any Lyapunov function in the family. These may efficiently be solved numerically by means of semidefinite programming. Compared to established results from literature, the proposed approach increases the range of perturbations and yields significantly tighter bounds.

It should be highlighted that with this result, a matrix inequality for imposing a non-conservative upper bound on the super-twisting algorithm's reaching time is available. In future works, this fact enables tackling the problem of systematic parameter tuning subject to reaching time constraints without unnecessarily increasing the controller gains. Extending the results to more general polynomial rather than quadratic Lyapunov functions may also be studied in the future.

Appendix

Proof of Proposition 1: Since $T_{K,L}$ and $\tilde{T}_{K,L}$ are homogeneous functions of $\mathbf{x}_0$, and thus $T_{K,L}(c\mathbf{e}_2) = cT_{K,L}(\mathbf{e}_2)$, $\tilde{T}_{K,L}(c\mathbf{e}_2) = c\tilde{T}_{K,L}(\mathbf{e}_2)$, it suffices to consider $c = 1$, i.e., $\mathbf{x}_0 = \mathbf{e}_2$. It is first shown that trajectories with this initial state stay in the first quadrant $x_1, x_2 \geq 0$ for all time. To see this, introduce the state functions $v_1 = x_1$ and $v_2 = x_2 - \eta\lfloor x_1 \rceil^{\frac{1}{2}}$

with $\eta > 0$ and consider a subset of that quadrant characterized by $v_1, v_2 \geq 0$. In this subset, one has $x_1 \geq 0$ and, choosing $2\eta = k_1 - K$, one obtains for the time derivatives of v_1 and v_2 on the set's borders $v_1 = 0$ and $v_2 = 0$, respectively,

$$\dot{v}_1|_{v_1=0} = x_2 \geq \eta x_1^{\frac{1}{2}} \geq 0 \tag{45}$$

$$\begin{aligned}\dot{v}_2|_{v_2=0} &= -k_2 + \delta_2 - \frac{\eta(-k_1 x_1^{\frac{1}{2}} + \eta x_1^{\frac{1}{2}} + \delta_1 x_1^{\frac{1}{2}})}{2x_1^{\frac{1}{2}}} \\ &\geq -(k_2 + L) + \frac{(k_1 - K)^2}{8} \geq 0,\end{aligned} \tag{46}$$

provided that (15) is fulfilled. Thus, v_1 and v_2 never change sign, and trajectories stay in the considered subset of the first quadrant. In that quadrant, the time derivative of x_2 is bounded by $\dot{x}_2 \leq -k_2 + L$ according to (2b), and equality holds with the particular perturbation $\delta_2 = L$. Since x_2 converges to zero without changing sign, the reaching time for $\mathbf{x}_0 = \mathbf{e}_2 = [0 \quad 1]^{\mathrm{T}}$ is thus given by

$$T_{K,L}(\mathbf{e}_2) = \frac{1}{k_2 - L}. \tag{47}$$

A lower bound of the eigenvalue-based reaching time estimate $\tilde{T}_{K,L}$ given in (13) for the initial state $\mathbf{x}_0 = \mathbf{e}_2$ will now be established. To that end, the abbreviation $\mathbf{v} := \begin{bmatrix}1 & k_1\end{bmatrix}^{\mathrm{T}}$ is introduced. Using the Riccati inequality (7) and the components of $\mathbf{P}$ as defined in (4) one then obtains the inequalities

$$0 \geq \mathbf{e}_2^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{e}_2 = p_{12}, \tag{48}$$

$$\begin{aligned}0 &\geq \mathbf{v}^{\mathrm{T}}(\Theta_2 L \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}} + \frac{L}{\Theta_2}\mathbf{P}\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}\mathbf{P} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q})\mathbf{v} \\ &= \Theta_2 L + \frac{L}{\Theta_2}(\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{e}_2)^2 - 2k_2(p_{12} + k_1 p_{22}) + \mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} \\ &\geq 2L\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{e}_2 - 2k_2(p_{12} + k_1 p_{22}) + \mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} \\ &= -2(k_2 - L)(p_{12} + k_1 p_{22}) + \mathbf{v}^{\mathrm{T}}\mathbf{Q}\mathbf{v} \\ &\geq -2(k_2 - L)k_1 p_{22} + (1 + k_1^2)\lambda_{\min}(\mathbf{Q}).\end{aligned} \tag{49}$$

Since $\mathbf{g}(\mathbf{e}_2) = \mathbf{e}_2$, the estimate (13) is bounded by

$$\begin{aligned}\tilde{T}_{K,L}(\mathbf{e}_2) &= \tilde{\alpha}(\mathbf{P}, \mathbf{Q})\sqrt{\mathbf{e}_2^{\mathrm{T}}\mathbf{P}\mathbf{e}_2} = \frac{2\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{Q})\sqrt{\lambda_{\min}(\mathbf{P})}}\sqrt{p_{22}} \\ &\geq \frac{2\sqrt{\lambda_{\max}(\mathbf{P})}}{\lambda_{\min}(\mathbf{Q})}\sqrt{p_{22}} \geq \frac{2p_{22}}{\lambda_{\min}(\mathbf{Q})} \\ &\geq \frac{1 + k_1^2}{(k_2 - L)k_1} = \frac{k_1 + \frac{1}{k_1}}{k_2 - L}.\end{aligned} \tag{50}$$

Combining (47) and (50) yields the proposition's claim. ∎

Proof of Proposition 2: Since $\overline{\alpha} \leq \sqrt{\kappa^*}$ has already been established in (32), $\overline{\alpha} \geq \sqrt{\kappa^*}$ remains to be shown. To that end, consider the dual of the semidefinite program (33)

$$\kappa^* = \max_{\mathbf{X} \succeq 0} \operatorname{tr} \mathbf{X} \begin{bmatrix}\mathbf{P} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0\end{bmatrix} \tag{51a}$$

subject to

$$\operatorname{tr} \mathbf{X} \begin{bmatrix}\mathbf{Q} & \mathbf{e}_1 \\ \mathbf{e}_1^{\mathrm{T}} & 0\end{bmatrix} = 0, \qquad \operatorname{tr} \mathbf{X} \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1\end{bmatrix} = 1 \tag{51b}$$

with decision variable $\mathbf{X} \in \mathbb{R}^{3\times 3}$. It yields the same optimal value as (33), because strictly feasible points, i.e., positive scalars γ and κ such that $\mathbf{N}$ is positive definite, exist, cf. [12]. According to [16, Theorem 2.1], the semidefinite program (51) has an optimal solution matrix $\mathbf{X}^*$, whose rank satisfies

$$(\operatorname{rank} \mathbf{X}^* + 1) \operatorname{rank} \mathbf{X}^* \leq 2m, \tag{52}$$

where $m = 2$ is the number of equality constraints. Thus, $\operatorname{rank} \mathbf{X}^* = 1$, and $\mathbf{X}^*$ may be written as $\mathbf{X}^* = \mathbf{q}\mathbf{q}^{\mathrm{T}}$, with $\mathbf{q} := \begin{bmatrix}\mathbf{z}^{\mathrm{T}} & v\end{bmatrix}^{\mathrm{T}}$, a vector $\mathbf{z} \in \mathbb{R}^2$, and a scalar v. Substituting this optimal solution $\mathbf{X}^*$ into (51) shows that

$$\kappa^* = \mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}, \qquad \mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z} + 2vz_1 = 0, \qquad v^2 = 1, \tag{53}$$

and thus $\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z} = |2vz_1| = 2|z_1|$ holds. With this $\mathbf{z}$ one has

$$\begin{aligned}\overline{\alpha}(\mathbf{P}, \mathbf{Q}) \geq \beta(\mathbf{z}, \mathbf{P}, \mathbf{Q}) &= \frac{2|z_1|\sqrt{\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}}}{\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z}} \\ &= \sqrt{\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{z}} = \sqrt{\kappa^*},\end{aligned} \tag{54}$$

which proves the proposition. ∎

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