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Accepted Manuscript

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Journal: IEEE Transactions on Automatic Control
DOI: 10.1109/TAC.2018.2794411

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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2794411, IEEE Transactions on Automatic Control

A Lyapunov Function for an Extended Super-Twisting Algorithm

Richard Seeber, Markus Reichhartinger and Martin Horn

***Abstract*—Recently, an extension of the super-twisting algorithm for relative degrees $m \geq 1$ has been proposed. However, as of yet no Lyapunov functions for this algorithm exist. This paper discusses the construction of Lyapunov functions by means of the sum-of-squares technique for $m = 1$. Sign definiteness of both Lyapunov function and its time derivative is shown in spite of numerically obtained—and hence possibly inexact—sum-of-squares decompositions. By choosing the Lyapunov function to be positive semidefinite, the finite time attractivity of the system's multiple equilibria is shown. A simple modification of this semidefinite function yields a positive definite Lyapunov function as well. Based on this approach, a set of the algorithm's tuning parameters ensuring finite-time convergence and stability in the presence of bounded uncertainties is proposed. Finally, a generalization of the approach for $m > 1$ is outlined.**

***Index Terms*—sliding-mode control, positive semidefinite Lyapunov function, multiple equilibria, polynomial methods, convex programming.**

I. INTRODUCTION

Sliding-mode control is a well-known control technique enabling the robust handling of large classes of unknown disturbances acting on a plant. A common task in sliding mode control is to steer a function of the system state $\mathbf{x}$ – the (scalar) sliding variable $\sigma(\mathbf{x})$ – to zero in finite time and keep it there. Writing u for the control input, $\sigma^{(l)}$ for the l-th time derivative of σ and m for the relative degree of σ with respect to u, the problem is that of finding a control law of the form $u = u(\sigma, \ldots, \sigma^{(m-1)})$ to steer the trajectories of the system governed by the differential equation $\sigma^{(m)} = u$ to zero.

Several solutions to this problem have been proposed: among them are, most notably, the twisting and the sub-optimal algorithm, which handle the case $m = 2$ with a discontinuous control signal [13], [5], [1], and the super-twisting algorithm, which gives a continuous control signal for $m = 1$ [13]. More recently, algorithms for higher relative degrees have been studied [7], [14], [15], with current research focusing on control laws yielding a continuous control signal [8], [10], [19], [26], [12]. In [2] one such continuous sliding mode algorithm is proposed in the form of an extension of the super-twisting algorithm for $m \geq 1$. Like the super-twisting algorithm (and variants thereof), it is designed to result in a homogeneous closed-loop system with negative homogeneity degree, yielding finite time convergence.

Unlike other algorithms, this system has *multiple equilibria*. This fact makes the stability proof more involved, because finite-time attractivity and stability of the equilibria has to be shown rather than finite-time stability of the origin. Usually, stability of an equilibrium is proven using a positive definite, radially unbounded Lyapunov function whose time-derivative along the solutions of the system is negative definite. Here, however, the time derivative of any possible Lyapunov function is zero on the set of equilibria. It can thus at best be negative semidefinite, which usually requires applying some kind of invariance principle, see e.g. [21].

In the present paper this problem is solved by constructing a *positive semidefinite* rather than a positive definite Lyapunov function. Unlike similar approaches in [9], the presented approach does not require any prior knowledge or assumptions regarding the stability of the equilibria.

Regarding the construction of Lyapunov functions in general, several techniques have been proposed in literature for the super-twisting algorithm or variants thereof [20], [18], as well as for other classes of homogeneous sliding mode algorithms [24]. For polynomial systems, the sum-of-squares technique offers a systematic way to construct polynomial Lyapunov functions [23]. In [22] a way is shown to apply this technique to non-polynomial systems that can be transformed to polynomial systems (possibly by introducing polynomial state constraints in the process). Here, this sum-of-squares approach is used to construct the Lyapunov function.

The extended super-twisting algorithm, when applied to a first order sliding variable σ whose dynamics are given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = u + \Delta, \qquad |\dot{\Delta}| \leq L, \tag{1}$$

with a Lipschitz continuous, time-varying perturbation $\Delta(t)$, consists of applying the control law

$$u = -\lambda_1 \left\lfloor \int_0^t \sigma(\tau)\,\mathrm{d}\tau \right\rceil^{\frac{1}{3}} - \lambda_2 \left\lfloor \sigma \right\rceil^{\frac{1}{2}} - \alpha \int_0^t \left\lfloor \sigma(\tau) \right\rceil^0 \mathrm{d}\tau. \tag{2}$$

Therein λ_1, λ_2 and α are constant controller parameters, and $\lfloor y \rceil^q$ is an abbreviation for

$$\lfloor y \rceil^q := |y|^q \operatorname{sign}(y). \tag{3}$$

Introducing state variables $x_1 := \int_0^t \sigma(\tau)\,\mathrm{d}\tau$, $x_2 := \sigma(t)$, and $x_3 := -\alpha \int_0^t \lfloor \sigma(\tau) \rceil^0 \mathrm{d}\tau + \Delta$ the closed-loop system

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The financial support by the Austrian Federal Ministry of Science, Research and Economy and the National Foundation for Research, Technology and Development is gratefully acknowledged.

consisting of (1) and (2) can be written as the homogeneous differential inclusion

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \tag{4a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\lambda_1 \lfloor x_1 \rceil^{\frac{1}{3}} - \lambda_2 \lfloor x_2 \rceil^{\frac{1}{2}} + x_3, \tag{4b}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} \in -\alpha \lfloor x_2 \rceil^0 + [-L, L] \tag{4c}$$

with

$$\lfloor x_2 \rceil^0 = \begin{cases} \{1\} & \text{for } x_2 > 0, \\ [-1, 1] & \text{for } x_2 = 0, \\ \{-1\} & \text{for } x_2 < 0. \end{cases} \tag{5}$$

Solutions of this differential inclusion (4) are understood as locally absolutely continuous functions of time t satisfying the inclusion for almost all t. The right-hand side of (4) is non-empty, convex, compact and upper semi-continuous. Therefore, solutions exist and are extendable in time [6].

An equilibrium is a point in state space at which a constant solution exists.[1] One can see that the system has multiple equilibria, in any of which $\sigma = x_2 = 0$ holds; they are given by all $\mathbf{x} := \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$ in the set

$$\mathcal{S} := \left\{ \mathbf{x} \;\middle|\; x_2 = x_3 - \lambda_1 \lfloor x_1 \rceil^{\frac{1}{3}} = 0 \right\}. \tag{6}$$

As of yet, no Lyapunov functions for this system (nor for applications of the algorithm for $m > 1$) are known to the authors.[2] In this paper a positive semidefinite Lyapunov function constructed by means of the sum-of-squares technique is used to prove for a given parameter setting that

1) all trajectories of (4) are bounded and converge in finite time to the set of equilibria $\mathcal{S}$,
2) all trajectories of (4) starting in $\mathcal{S}$ are constant, and
3) every $\mathbf{x} \in \mathcal{S}$ is a Lyapunov stable equilibrium of (4).

Furthermore, the constructed Lyapunov function will allow to estimate the finite convergence time.

The present paper is organized as follows: Section II discusses the proof technique and the choice of the Lyapunov function candidate. In Section III the unperturbed system is considered. Computational issues arising in the process of numerically solving the sum-of-squares problem are first discussed; then a formal proof of the claimed properties is given in Subsection III-B. Section IV extends the results to the perturbed case. In Section V numerical results valid for a set of parameters of the perturbed system are given. In Section VI one possible way to extend this approach to the case $m > 1$ is presented in the form of an outlook. Section VII summarizes and concludes the paper.

II. Lyapunov Candidate and Proof Technique

Due to the presence of multiple equilibria, a positive definite Lyapunov function V could at best have a negative semidefinite time derivative $\dot{V}$. Using such a function, a statement on the attractivity of $\mathcal{S}$, the type of convergence and the convergence time could thus not easily be made. Therefore, a *positive semidefinite* Lyapunov function V with negative semidefinite $\dot{V}$ is constructed, both being zero if and only if $\mathbf{x} \in \mathcal{S}$. This function permits to prove finite-time attractivity of $\mathcal{S}$ and estimate the convergence time. To show boundedness of trajectories, the semidefinite Lyapunov function is then modified slightly to yield a positive definite, radially unbounded Lyapunov function; and to show stability of equilibria other than the origin, the semidefinite Lyapunov function along with a convergence time estimate turns out to be useful.

A. Lyapunov Candidate

Consider a candidate Lyapunov function $V(\mathbf{z}(\mathbf{x}))$ that is a multivariate polynomial with respect to the variables

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} := \begin{bmatrix} \lfloor x_1 \rceil^{\frac{1}{3}} \\ \lfloor x_2 \rceil^{\frac{1}{2}} \\ x_3 - \lambda_1 \lfloor x_1 \rceil^{\frac{1}{3}} \end{bmatrix}. \tag{7}$$

With this choice of variables the set of equilibria $\mathcal{S}$ is described by $z_2 = z_3 = 0$. To ensure that V is zero in this set, it is thus sufficient to use only monomials in its construction that are divisible by either z_2 or z_3.

In order to apply the sum-of-squares technique also the derivative

$$\dot{V} = \frac{\partial V}{\partial \mathbf{z}} \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \tag{8}$$

needs to be expressed in terms of a polynomial in $\mathbf{z}$. To see that this is possible, apply the chain-rule to obtain that at each time instant with $x_2 \neq 0$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \begin{bmatrix} \frac{1}{3} |z_1|^{-2} \lfloor z_2 \rceil^2 \\ \frac{1}{2} |z_2|^{-1} (-\lambda_1 z_1 - \lambda_2 z_2 + z_3) \\ -\alpha \lfloor z_2 \rceil^0 - \frac{1}{3} \lambda_1 |z_1|^{-2} \lfloor z_2 \rceil^2 + \delta \end{bmatrix} \tag{9}$$

holds with some[3] $\delta \in [-L, L]$. For any $y \in \mathbb{R}$ and integer k

$$y^k = \begin{cases} |y|^k & \text{for } k \text{ even} \\ \lfloor y \rceil^k & \text{for } k \text{ odd} \end{cases} \tag{10}$$

holds. Hence, with the abbreviation $\varphi := \delta \lfloor x_2 \rceil^0$, (9) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} &= |z_2|^{-1} z_1^{-2} \begin{bmatrix} \frac{1}{3} z_2^3 \\ -\frac{1}{2} \lambda_2 z_1^2 z_2 + \frac{1}{2} z_1^2 z_3 \\ -\alpha z_1^2 z_2 - \frac{1}{3} \lambda_1 z_2^3 + \varphi z_1^2 z_2 \end{bmatrix} \\ &=: |z_2|^{-1} z_1^{-2} \mathbf{f}_\varphi(\mathbf{z}), \end{aligned} \tag{11}$$

which is polynomial in $\mathbf{z}$ up to the positive scaling factor $|z_2|^{-1} z_1^{-2}$, with $\varphi \in [-L, L]$ occuring as a parameter[4].

Note that the functions $\mathbf{z}$ and hence in general V are non-differentiable with respect to $\mathbf{x}$ on the manifold $\mathcal{M} := \{\mathbf{x} \mid x_1 x_2 = 0\}$. This is problematic, because $\dot{V} \leq 0$ in this case does not imply that V is non-increasing. The following lemmas, which form the basis for the stability proof in Section III-B, solve this problem.

[1] Note that by this definition at least one, but not necessarily all trajectories starting in an equilibrium are constant.

[2] Although a stability proof is outlined in [2], no Lyapunov function can be constructed using it.

[3] Note that δ may have a different value for each time instant; in the context of the original closed loop (1), (2) one can see that $\delta = \dot{\Delta}$.

[4] In the following discussions it will only be necessary to consider $\mathbf{f}_\varphi$ for a few specific constant values of the parameter φ.

B. Proof Technique

Lemma 1: Consider system (4) and let $\mathcal{S}$ be the set of its equilibria as given in (6). Let I be a compact interval and let $\mathbf{x}(t)$ be a trajectory of the system with $\mathbf{x}(t) \notin \mathcal{S}$ for all $t \in I$. Then the function $\mathbf{z}(\mathbf{x}(t))$ given in (7) is absolutely continuous on I.

Proof: given in the appendix. ■

As a consequence, the Lyapunov function V, being a polynomial in $\mathbf{z}$, is guaranteed to be absolutely continuous on intervals satisfying the lemma's condition as well. Hence it is—loosely speaking—sufficient to consider $\dot{V}$ only where it is defined. The following lemma and its corollary formalize this. Note that in contrast to similar results given in [3], the lemma does not require V to be positive definite.

Lemma 2: Consider system (4) and let $\mathcal{S}$ be the set of its equilibria as given in (6). Suppose there exists a function $V(\mathbf{x})$, polynomial in $\mathbf{z}(\mathbf{x})$, with $V(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} \in \mathcal{S}$, and its time-derivative $\dot{V}$ along the trajectories of system (4) satisfying, where $\dot{V}$ is defined,

$$\dot{V} \leq -\gamma V^{\rho} \tag{12}$$

with $\gamma > 0$ and $\rho \in [0, 1)$. Then V is non-increasing with respect to time, and for every trajectory $\mathbf{x}(t)$ there exists a unique time instant T bounded by

$$T \leq \frac{V^{1-\rho}(\mathbf{x}(0))}{\gamma(1-\rho)} =: \overline{T} \tag{13}$$

such that $\mathbf{x}(t) \notin \mathcal{S}$ for $t < T$ and $\mathbf{x}(t) = \mathbf{x}(T) \in \mathcal{S}$ for $t \geq T$.

Proof: given in the appendix. ■

Corollary 1: Suppose that the conditions of Lemma 2 are satisfied. Then *any* function $U(\mathbf{x})$, polynomial in $\mathbf{z}(\mathbf{x})$, whose time-derivative $\dot{U}$ along the trajectories of system (4) satisfies $\dot{U} \leq 0$ where $\dot{U}$ is defined, is non-increasing with respect to time.

Proof: given in the appendix. ■

III. Unperturbed Case

As a first step, the unperturbed system, i.e. (4) for $L = 0$, is considered. Due to the aforementioned reasoning it is—loosely speaking—sufficient to search for Lyapunov functions for the vector field $\mathbf{f}_0$, i.e. $\mathbf{f}_\varphi$ from (11) with the parameter $\varphi = 0$. To this end, consider as a Lyapunov candidate a polynomial $V(\mathbf{z})$ of even degree $l = 2r$; due to the homogeneity of $\mathbf{f}_0$, it suffices to consider a homogeneous polynomial.

To ensure the required positive semidefiniteness of V, the sum-of-squares technique is applied. This is done by ensuring that V satisfies the equality constraint

$$V(\mathbf{z}) = \mathbf{m}_1(\mathbf{z})^{\mathrm{T}} \mathbf{P} \mathbf{m}_1(\mathbf{z}) \tag{14}$$

with a suitable vector of homogeneous degree r monomials $\mathbf{m}_1$, and a matrix $\mathbf{P}$ constrained to the cone of positive semidefinite matrices, i.e.,

$$\mathbf{P} \succeq 0. \tag{15}$$

Writing $\dot{V}$ for the time derivative of V along the trajectories of system (4), one can formally introduce the polynomial

$$W(\mathbf{z}) := \frac{\partial V}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z}) = |z_2| \, z_1^2 \dot{V}(\mathbf{z}), \tag{16}$$

which is nothing else but the Lie derivative of V with respect to the vector field $\mathbf{f}_0$. If this polynomial fulfills

$$W(\mathbf{z}) = -\mathbf{m}_2(\mathbf{z})^{\mathrm{T}} \mathbf{Q} \mathbf{m}_2(\mathbf{z}) \tag{17}$$

with another suitable monomial vector $\mathbf{m}_2$ and

$$\mathbf{Q} \succeq 0, \tag{18}$$

then $\dot{V}$ is negative semidefinite in the sense that $\dot{V} \leq 0$ holds wherever it is defined.

Semidefiniteness constraints (15) and (18) along with equality constraints (14) and (17) represent a semidefinite program, whose solution can, in theory, be obtained using state-of-the-art software packages.

A. Computational Issues

In practice, the solution of the semidefinite program has to be computed numerically. Due to finite precision arithmetics, solutions obtained this way in general do not satisfy the equality constraints exactly, but only up to nonzero residuals. In the problem at hand, instead of (14) and (17)

$$V(\mathbf{z}) = \mathbf{m}_1(\mathbf{z})^{\mathrm{T}} \mathbf{P} \mathbf{m}_1(\mathbf{z}) + \mu(\mathbf{z}), \tag{19a}$$

$$W(\mathbf{z}) = -\mathbf{m}_2(\mathbf{z})^{\mathrm{T}} \mathbf{Q} \mathbf{m}_2(\mathbf{z}) - \nu(\mathbf{z}). \tag{19b}$$

will hold with residual polynomials μ and ν. Consequently, the non-negativity of V and the non-positivity of its time derivative cannot be ascertained, making the approach in this form unsuitable for a rigorous proof.

This is an issue of the sum-of-squares technique in general. It is also investigated in [17], where the following is proven.

Theorem 1 ([17]): A polynomial g with inexact sum-of-squares decomposition

$$g(\mathbf{z}) = \mathbf{m}(\mathbf{z})^{\mathrm{T}} \mathbf{R} \mathbf{m}(\mathbf{z}) + \kappa(\mathbf{z}), \tag{20}$$

with monomial vector $\mathbf{m}$ and $\mathbf{R} \in \mathbb{R}^{q \times q}$ also admits an exact decomposition $g(\mathbf{z}) = \mathbf{m}(\mathbf{z})^{\mathrm{T}} \tilde{\mathbf{R}} \mathbf{m}(\mathbf{z})$ with $\tilde{\mathbf{R}} \succ 0$ and is hence non-negative, if the error polynomial κ can be written as a quadratic form

$$\kappa(\mathbf{z}) = \mathbf{m}(\mathbf{z})^{\mathrm{T}} \mathbf{K} \mathbf{m}(\mathbf{z}) \tag{21}$$

for some matrix $\mathbf{K}$ and the smallest eigenvalue of $\mathbf{R}$ is greater than $q \cdot \|\kappa\|_\infty$, i.e.,

$$\mathbf{R} \succeq \mathbf{I} \cdot q \|\kappa\|_\infty . \tag{22}$$

Therein $\|\kappa\|_\infty$ denotes the maximum absolute coefficient of the polynomial κ (i.e., the infinity norm of a vector containing its coefficients), and $\mathbf{I}$ is the identity matrix.

This means that the sum-of-squares technique can be used to prove sign definiteness of V and W, if the smallest eigenvalues of the matrices $\mathbf{P}$ and $\mathbf{Q}$ in (19) are sufficiently large compared to the coefficients of residual polynomials μ and ν. To apply the theorem, two preconditions need to be fulfilled:

First, the monomial vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ as well as the monomials used in V need to be chosen diligently, to permit the matrices $\mathbf{P}$ and $\mathbf{Q}$ to take positive definite values. As stated previously, all monomials in V are required to be divisible by either z_2 or z_3. Therefore, $V(\mathbf{z}) = W(\mathbf{z}) = 0$ for $\mathbf{z} \in \mathcal{S}$

and the matrices $\mathbf{P}$ and $\mathbf{Q}$ have to be singular, unless the monomials in the vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ also have this property[5].

And second, the residual polynomial ν must be representable as a quadratic form of the monomial vector $\mathbf{m}_2$. Writing this out and taking into account the divisibility property of the monomials in $\mathbf{m}_2$, one obtains

$$\begin{aligned}\nu(\mathbf{z}) = \mathbf{m}_2(\mathbf{z})^{\mathrm{T}}\mathbf{K}\mathbf{m}_2(\mathbf{z}) &= \sum_{j=1}^{q}\Big[z_2\gamma_j(\mathbf{z}) + z_3\eta_j(\mathbf{z})\Big]^2 \\ &= \sum_{j=1}^{q}\Big[z_2^2\gamma_j^2(\mathbf{z}) + z_2z_3\gamma_j(\mathbf{z})\eta_j(\mathbf{z}) + z_3^2\eta_j^2(\mathbf{z})\Big] \qquad (23)\end{aligned}$$

with polynomials γ_j and η_j. One can see that this requires all monomials in ν to be divisible by either z_2^2, z_3^2 or z_2z_3. As W is given by (19b), this means that also W must only contain such monomials.

These conditions yield equality constraints on coefficients of V that, despite the application of Theorem 1, still need to be satisfied *exactly*. Compared to the original equality constraints this can be done rather easily, however. Writing W as

$$W(\mathbf{z}) = \frac{\partial V}{\partial z_1}f_{0,1}(\mathbf{z}) + \frac{\partial V}{\partial z_2}f_{0,2}(\mathbf{z}) + \frac{\partial V}{\partial z_3}f_{0,3}(\mathbf{z}) \qquad (24)$$

with $f_{0,i}$ denoting the components of $\mathbf{f}_0$ one can see by looking at (11) that this is the case if $\frac{\partial V}{\partial z_2}$ and $\frac{\partial V}{\partial z_3}$ each are divisible by either z_2 or z_3. For the homogeneous polynomial V of degree $2r$ this is fulfilled if it does not contain the monomials $z_1^{2r-1}z_2$ and $z_1^{2r-1}z_3$.

A specific polynomial V with degree $2r = 6$ is now chosen

$$\begin{aligned}V = {} & c_1z_1z_2z_3^4 + c_2z_1z_2^2z_3^3 + c_3z_1z_2^5 + c_4z_1z_3^5 + c_5z_1^2z_2z_3^3 \\ & + c_6z_1^2z_2^2z_3^2 + c_7z_1^2z_2^4 + c_8z_1^2z_3^4 + c_9z_1^3z_2z_3^2 + c_{10}z_1^3z_3^3 \\ & + c_{11}z_1^4z_2z_3 + c_{12}z_1^4z_2^2 + c_{13}z_1^4z_3^2 + c_{14}z_2z_3^5 + c_{15}z_2^2z_3^4 \\ & + c_{16}z_2^6 + c_{17}z_3^6, \qquad (25)\end{aligned}$$

with c_i being free coefficients. In accordance with the previous considerations, all monomials are divisible by z_2 or z_3 and the monomials $z_1^5z_2$ and $z_1^5z_3$ are not present in V (along with some other monomials that proved unnecessary for solving the problem). The monomial vectors are chosen as

$$\mathbf{m}_1 = \begin{bmatrix}z_1^2z_2 & z_1z_2^2 & z_2^3 & z_1^2z_3 & z_1z_2z_3 & z_1z_3^2 & z_2z_3^2 & z_3^3\end{bmatrix}^{\mathrm{T}} \qquad (26a)$$

$$\mathbf{m}_2 = \begin{bmatrix}z_1^3z_2 & z_1^2z_2^2 & z_1z_2^3 & z_1^3z_3 & z_1^2z_2z_3 & z_1^2z_3^2 \\ & & z_1z_2z_3^2 & z_1z_3^3 & z_2^2z_3^2 & z_2^3z_3 & z_2^4\end{bmatrix}^{\mathrm{T}}. \qquad (26b)$$

One may check that each monomial in V is representable as a product of two monomials from $\mathbf{m}_1$. For W, however, this is not yet the case: one finds that despite the above efforts a single monomial, $z_2^3z_3^5$, is not a product of any two monomials in $\mathbf{m}_2$. By means of a symbolic computation the coefficient of this monomial in W is found to be $\frac{1}{3}c_4 - 2\lambda_1c_{17}$. Hence, by making the substitution

$$c_4 = 6\lambda_1c_{17} \qquad (27)$$

all numerical issues can be dealt with by applying Theorem 1.

B. Formal Proof

The system properties claimed in Section I are now formally proven:

Theorem 2: Consider system (4) with $L = 0$ and given system parameters λ_1, λ_2 and α. Suppose that there exist coefficients c_i of the polynomial (25) such that the (inexact) sum-of-squares decomposition (19), with the polynomial W defined in (16) and monomial vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ being given by (26), fulfills

$$\mathbf{P} \succeq 8\left\|\mu\right\|_\infty \cdot \mathbf{I}, \qquad \mathbf{Q} \succeq 11\left\|\nu\right\|_\infty \cdot \mathbf{I}. \qquad (28)$$

Then all trajectories of the system are bounded and reach the set of equilibria $\mathcal{S}$ given by (6) in finite time. Furthermore, each equilibrium is Lyapunov stable.

Remark 1: The conditions of this theorem constitute a semidefinite program; for given system parameters, a computer program such as SeDuMi may thus be used to check if they are satisfied.

Proof: By assumption and by the choice of monomials, the conditions of Theorem 1 are fulfilled for both V and W. Hence, positive definite matrices $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ exist such that the exact sum-of-squares decompositions

$$V(\mathbf{z}) = \mathbf{m}_1(\mathbf{z})^{\mathrm{T}}\tilde{\mathbf{P}}\mathbf{m}_1(\mathbf{z}), \qquad (29a)$$

$$W(\mathbf{z}) = -\mathbf{m}_2(\mathbf{z})^{\mathrm{T}}\tilde{\mathbf{Q}}\mathbf{m}_2(\mathbf{z}) \qquad (29b)$$

hold. Consequently, $V \geq 0$ and, where it is defined, $|z_2|^{-1}z_1{}^{-2}W = \dot{V} \leq 0$.

To prove finite time attractivity of $\mathcal{S}$ it will be shown that an inequality of the form (12) holds with $\gamma > 0$ and $q = \frac{5}{6}$. To that end, consider the expressions $W(\mathbf{z}) + \eta_1z_1^2V(\mathbf{z})$ and $V(\mathbf{z}) - \eta_2^6z_2^6$ with constant parameters $\eta_1, \eta_2 > 0$. By observing that the vector $z_1\mathbf{m}_1$ is given by the first eight monomials in $\mathbf{m}_2$, i.e.

$$z_1\mathbf{m}_1(\mathbf{z}) = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\mathbf{m}_2(\mathbf{z}), \qquad (30)$$

and by writing $\mathbf{e}_i$ for the i-th standard basis vector of suitable dimensions one obtains

$$W + \eta_1z_1^2V = -\mathbf{m}_2^{\mathrm{T}}\underbrace{\left(\tilde{\mathbf{Q}} - \eta_1\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\tilde{\mathbf{P}}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\right)}_{=:\mathbf{R}_1}\mathbf{m}_2, \qquad (31a)$$

$$V - \eta_2^6z_2^6 = \mathbf{m}_1^{\mathrm{T}}\underbrace{\left(\tilde{\mathbf{P}} - \eta_2^6\mathbf{e}_3\mathbf{e}_3^{\mathrm{T}}\right)}_{=:\mathbf{R}_2}\mathbf{m}_1. \qquad (31b)$$

By choosing η_1 and η_2 small enough, both matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ may be assured to be positive definite, yielding with (16) the inequalities

$$|z_2|\,z_1^2\dot{V} + \eta_1z_1^2V \leq 0, \qquad (32a)$$

$$V - \eta_2^6z_2^6 \geq 0. \qquad (32b)$$

The latter yields $\eta_2|z_2| \leq V^{\frac{1}{6}}$, and by substituting into the former one obtains

$$\dot{V} \leq -\frac{\eta_1}{|z_2|}V \leq -\eta_1\eta_2V^{\frac{5}{6}}. \qquad (33)$$

[5]For $\mathbf{m}_2$ this would need to be the case even if V were chosen to be positive definite. This is due to $\mathbf{f}_\varphi(\mathbf{z})$ and hence $W(\mathbf{z})$ always being zero for $\mathbf{z} \in \mathcal{S}$.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2794411, IEEE Transactions on Automatic Control

From Lemma 2 one thus obtains that for any trajectory there exists a time T bounded by

$$T \leq \frac{6}{\eta_1 \eta_2} V^{\frac{1}{6}}(\mathbf{x}(0)) \tag{34}$$

such that for $t \geq T$ the relation $\mathbf{x}(t) \in \mathcal{S}$ holds.

To show boundedness of the trajectories, the positive semidefinite Lyapunov function V is modified slightly to obtain the positive definite and radially unbounded candidate Lyapunov function $U = V + \varepsilon z_1^6$ with a constant, positive parameter ε. Its time-derivative, where it is defined, is given by

$$\begin{aligned}\dot{U} &= |z_2|^{-1} z_1^{-2} \left(W + 2\varepsilon z_1^5 z_2^3\right) \\ &= -|z_2|^{-1} z_1^{-2} \mathbf{m}_2^{\mathrm{T}} \underbrace{\left[\tilde{\mathbf{Q}} - \varepsilon(\mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} + \mathbf{e}_2\mathbf{e}_1^{\mathrm{T}})\right]}_{=:\mathbf{R}} \mathbf{m}_2. \end{aligned} \tag{35}$$

For sufficiently small ε the matrix $\mathbf{R}$ is positive definite and $\dot{U} \leq 0$. Corollary 1 then guarantees that U is non-increasing, which proves that the radially unbounded function U and hence also all trajectories are bounded with respect to time.

Finally, fix any equilibrium $\mathbf{v} \in \mathcal{S}$; Lyapunov stability of $\mathbf{v}$ is now shown directly. Let any neighbourhood $\mathcal{N}$ of $\mathbf{v}$ be given; it will be shown that another neighbourhood $\mathcal{G} \subseteq \mathcal{N}$ exists such that all trajectories starting in $\mathcal{G}$ stay in $\mathcal{N}$. To construct $\mathcal{G}$, parametrize a family of neighborhoods of $\mathbf{v}$ by

$$\mathcal{G}(\varepsilon) = \left\{\mathbf{x} \mid V(\mathbf{x}) < \varepsilon^6, |x_3 - v_3| < \varepsilon\right\} \tag{36}$$

with $\varepsilon > 0$. Due to the obvious bound $|\dot{x}_3| \leq \alpha$ for the time-derivative of x_3 and the finite reaching time T from (34), any trajectory $\mathbf{x}(t)$ starting in this set satisfies

$$|x_3(t) - v_3| < \varepsilon + \alpha T \leq \varepsilon + \frac{6\alpha\varepsilon}{\eta_1\eta_2} \tag{37}$$

for all t. Since V is non-increasing, trajectories starting in $\mathcal{G}(\varepsilon)$ are for all t contained in the set $\mathcal{G}([1 + 6\alpha\eta_1^{-1}\eta_2^{-1}]\varepsilon)$. By choosing ε small enough, this set is contained in the given neighborhood $\mathcal{N}$, concluding the proof. ∎

IV. Perturbed Case

Now consider the perturbed case, i.e. system (4) with a perturbation bound $L > 0$. The sum-of-squares approach allows to ensure that finite-time attractivity of $\mathcal{S}$ and Lyapunov stability of each equilibrium also hold in this case. To this end, the constraint (17) of the semidefinite program is replaced by

$$\frac{\partial V}{\partial \mathbf{z}} \mathbf{f}_{-M}(\mathbf{z}) =: W_1(\mathbf{z}) = -\mathbf{m}_2(\mathbf{z})^{\mathrm{T}} \mathbf{Q}_1 \mathbf{m}_2(\mathbf{z}) \tag{38a}$$

$$\frac{\partial V}{\partial \mathbf{z}} \mathbf{f}_{M}(\mathbf{z}) =: W_2(\mathbf{z}) = -\mathbf{m}_2(\mathbf{z})^{\mathrm{T}} \mathbf{Q}_2 \mathbf{m}_2(\mathbf{z}) \tag{38b}$$

with $M \geq L$, which ensures the non-positivity of $\dot{V}$ for both $\varphi = -M$ and $\varphi = M$. From a standard convexity argument it follows that $\dot{V}$ is non-positive also for all other permitted values of φ:

$$\frac{\partial V}{\partial \mathbf{z}} \mathbf{f}_{\varphi}(\mathbf{z}) = (1-\psi) W_1(\mathbf{z}) + \psi W_2(\mathbf{z}) \leq 0 \tag{39}$$

with $\psi = \frac{M+\varphi}{2M}$ satisfying $0 \leq \psi \leq 1$ for $|\varphi| \leq L \leq M$. One thus obtains

Theorem 3: Consider system (4) with given parameters λ_1, λ_2, α and perturbation bound $L \geq 0$. Suppose that for some $M \geq L$ there exist coefficients c_i of the polynomial (25) such that the inexact decomposition (19a) and (possibly inexact versions of) decompositions (38) hold, each satisfying the conditions of Theorem 1. Then all trajectories of the system are bounded and reach the set of equilibria $\mathcal{S}$ in finite time. Furthermore, each equilibrium is Lyapunov stable.

Remark 2: Checking this theorem's conditions amounts to the solution of a semidefinite program, as does finding a supremum for M such that the conditions are satsified. In Section V-B this is demonstrated in the course of an example.

Proof: The proof is the same as that of Theorem 2, with small changes: Relation (29b) is replaced by (38); by using the previously mentionend convexity argument one thus shows that the positive semidefinite Lyapunov function V and the positive definite Lyapunov function U satsify the differential inequalities $\dot{V} \leq -\gamma V^{\frac{5}{6}}$ with $\gamma > 0$ and $\dot{U} \leq 0$, respectively. Finite-time convergence, with a convergence time estimate of the same structure as (34), and boundedness of the trajectories readily follow. To show Lyapunov stability, α is replaced by $\alpha + L$ in the last part of the proof of Theorem 2. ∎

V. Numerical Results

This section shows some numerical results that are obtained with the presented technique.

A. Unperturbed Case

The sum-of-squares problem for the unperturbed case is given by constraints (14), (15), (17) and (18), with V given in (25). Using the MATLAB toolboxes YALMIP [16], which also was helpful in choosing the monomial vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, and SEDUMI [25] this problem is solved numerically[6]. To ensure positivity of V for $\mathbf{z} \notin \mathcal{S}$, which is necessary for the positive definiteness of $\mathbf{P}$ and $\mathbf{Q}$, the constraints $c_{16} \geq 1$ and $c_{17} \geq 1$ are added.

As an example, the parameter values

$$\lambda_1 = 2 \qquad \lambda_2 = 2 \qquad \alpha = 1 \tag{40}$$

are considered, for which one solution of the sum-of-squares problem is obtained as follows: the coefficients of V are

$$\begin{aligned} &c_1 = -81 && c_2 = 261 && c_3 = -545 && c_4 = 24 \\ &c_5 = -400 && c_6 = 786 && c_7 = 1271 && c_8 = 120 \\ &c_9 = -607 && c_{10} = 268 && c_{11} = -1064 && c_{12} = 1025 \\ &c_{13} = 914 && c_{14} = -6 && c_{15} = 29 && c_{16} = 810 \\ &c_{17} = 2; \end{aligned} \tag{41}$$

the minimum eigenvalues of matrices $\mathbf{P}$ and $\mathbf{Q}$ exceed 0.4 and 0.09, respectively; and the coefficient bounds of the corresponding residual polynomials are

$$\|\mu\|_\infty \leq 4.4 \times 10^{-10}, \qquad \|\nu\|_\infty \leq 1.5 \times 10^{-9}. \tag{42}$$

The conditions of Theorem 2 can easily be checked to be satisfied, which certifies that the unperturbed system with

[6] The following results are obtained using MATLAB version R2013b, YALMIP version 20150919, and SEDUMI version 1.3.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2794411, IEEE Transactions on Automatic Control

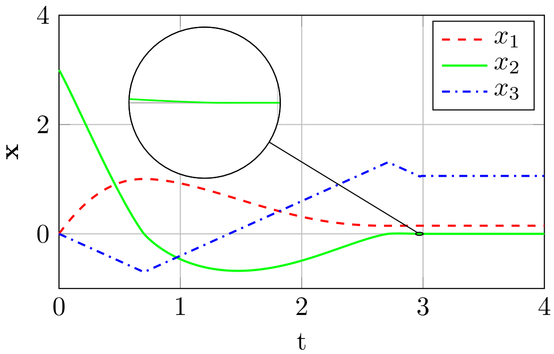

Fig. 1. Time evolution of state variables $\mathbf{x}$ from simulation of system (4) without perturbation ($L = 0$).

the considered parameters has the properties claimed in that theorem.

For demonstration purposes, results from a numerical simulation of system (4) are shown. Fig. 1 depicts the time evolution of state variables with initial condition $\mathbf{x}(0) = \begin{bmatrix}0 & 3 & 0\end{bmatrix}^{\mathrm{T}}$ and without perturbation, i.e. $L = 0$. In Fig. 2 the corresponding value of the semidefinite Lyapunov function V from (25) with coefficients (41) is shown. One can see that it is non-negative and strictly decreases to zero; also, it can be seen to be non-differentiable when either x_1 or x_2 are zero.

To obtain the corresponding convergence time estimate, one may numerically verify that $\mathbf{R}_1, \mathbf{R}_2 \succ 0$ holds in the proof of Theorem 2 for $\eta_1 = 0.2$ and $\eta_2 = 3$. Relation (34) thus yields $T \leq 52.9$.

B. Perturbation Bound

The maximum perturbation bound, i.e. the robustness of the system guaranteed by the obtained Lyapunov function may be computed by solving the semidefinite program

$$M^* = \max M \quad \text{subject to (38)}. \tag{43}$$

A numerical computation with the previously considered coefficients (40) and (41) yields $L^* \approx 0.2699$, and one may verify that for, e.g.

$$L \leq M = 0.26, \tag{44}$$

the sum-of-squares decompositions fulfill the conditions of Theorem 1 and thus the stability and attractivity guarantees of Theorem 3 hold.

C. Scaling of Gains

As is common practice with similar sliding-mode algorithms [14], it is possible to scale the controller parameters if the disturbance bound is larger than M obtained previously in (44), i.e. if $L > M$. By means of the state transform $\mathbf{v} = LM^{-1}\mathbf{x}$ it can be seen, that all statements stay valid for an arbitrary positive disturbance bound L, provided that the controller parameters are modified according to

$$\tilde{\lambda}_1 = \lambda_1 \left(\frac{L}{M}\right)^{\frac{2}{3}}, \quad \tilde{\lambda}_2 = \lambda_2 \left(\frac{L}{M}\right)^{\frac{1}{2}}, \quad \tilde{\alpha} = \alpha \left(\frac{L}{M}\right). \tag{45}$$

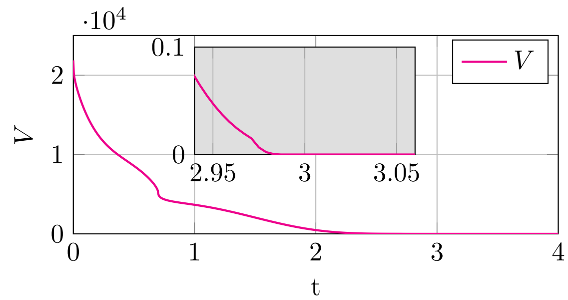

Fig. 2. Time evolution of positive semidefinite Lyapunov function V given in (25), (41) for the simulation of system (4) shown in Fig. 1.

VI. Generalizations to Higher Relative Degrees

As stated initially, the extended super-twisting algorithm in [2] is presented for arbitrary relative degree m. The approach presented in the paper may be generalized to higher orders of the algorithm, albeit with some modifications. The present section outlines this generalization for the unperturbed case.

The control law applied to an unperturbed chain of integrators with relative degree m, i.e. $\frac{\mathrm{d}^m \sigma}{\mathrm{d}t^m} = u$, is given by

$$\begin{aligned} u = &-\lambda_1 \left\lfloor \int_0^t \sigma(\tau)\,\mathrm{d}\tau \right\rceil^{\frac{1}{m+2}} - \sum_{i=0}^{m-1} \lambda_{i+2} \left\lfloor \frac{\mathrm{d}^i \sigma}{\mathrm{d}t^i} \right\rceil^{\frac{1}{m+1-i}} \\ &- \alpha \int_0^t \left\lfloor \frac{\mathrm{d}^{m-1}\sigma}{\mathrm{d}t^{m-1}} \right\rceil^0 \mathrm{d}\tau. \end{aligned} \tag{46}$$

Choosing the state variables $x_1 := \int_0^t \left\lfloor \frac{\mathrm{d}^{m-1}\sigma}{\mathrm{d}t^{m-1}} \right\rceil^0 \mathrm{d}\tau$, $x_i := \frac{\mathrm{d}^{m+1-i}\sigma}{\mathrm{d}t^{m+1-i}}$ for $i = 2, \dots, m+1$, and $x_{m+2} := \int_0^t \sigma(\tau)\,\mathrm{d}\tau$ (note that the order of state variables is reversed compared to the previous sections[7]), the closed-loop system can be written as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \lfloor x_2 \rceil^0, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -\sum_{i=1}^{m+2} \beta_i \lfloor x_i \rceil^{\frac{1}{i}}, \tag{47a}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_2, \qquad \dots \qquad \frac{\mathrm{d}x_{m+2}}{\mathrm{d}t} = x_{m+1}, \tag{47b}$$

where

$$\beta_i := \begin{cases} \alpha & i = 1 \\ \lambda_{m+3-i} & i \geq 2. \end{cases} \tag{48}$$

In the following, $m > 1$ and $\alpha > 0$ is assumed. To apply the sum-of-squares technique, a function V that is a polynomial in[8]

$$\mathbf{v} := \begin{bmatrix} z_1 & \dots & z_{m+2} & w_1 & \dots & w_{\lfloor \frac{m}{2} \rfloor} \end{bmatrix}^{\mathrm{T}} \tag{49}$$

is considered, where

$$z_i = \begin{cases} x_1 + \frac{\beta_{m+2}}{\beta_1} \lfloor x_{m+2} \rceil^{\frac{1}{m+2}} & i = 1 \\ \lfloor x_i \rceil^{\frac{1}{i}} & i \geq 2, \end{cases} \tag{50a}$$

$$w_i = |x_2|^{\frac{1}{4}} \, |x_{2i+2}|^{\frac{1}{4i+4}}. \tag{50b}$$

[7]This allows to write the following results in a more compact form.

[8]Note that $\lfloor y \rfloor$ is written for the floor function, i.e., the largest integer $k \leq y$.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2794411, IEEE Transactions on Automatic Control

m	n	p
1	3	3
2	5	11
3	6	15
4	8	23
5	9	29

TABLE I
ORDER n AND POLYNOMIAL DEGREE p OF $\mathbf{f}$ IN (52) OR (11) FOR SOME VALUES OF RELATIVE DEGREE m.

Application of the chain-rule yields

$$|z_2| \frac{\partial z_i}{\partial \mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{cases} z_2 + \frac{\beta_{m+2}}{\beta_1} |z_2| \frac{\mathrm{d}z_{m+2}}{\mathrm{d}t} & i = 1 \\ -\frac{1}{2} \sum_{i=1}^{m+1} \beta_i z_i & i = 2 \\ \frac{1}{i} w_{\frac{i-3}{2}}{}^2 z_i^{-i+1} z_{i-1}^{i-2} & i \geq 3, i \text{ odd} \\ \frac{1}{i} w_{\frac{i-2}{2}}{}^2 z_i^{-i} z_{i-1}^{i-1} & i \geq 4, i \text{ even}, \end{cases} \tag{51a}$$

$$|z_2| \frac{\partial w_i}{\partial \mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\frac{1}{4} w_i z_2^{-1} \sum_{j=1}^{m+1} \beta_j z_j + \frac{1}{4} \frac{1}{i+1} w_i^3 z_{2i+2}^{-2i-3} z_{2i+1}^{2i+1}, \tag{51b}$$

taking into account that w_0—though it is not a state variable—satisfies $w_0^2 = z_2^2$. These equations can be written in the form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = |z_2|^{-1} \prod_{i=3}^{m+2} z_i^{-2\left\lfloor \frac{i}{2} \right\rfloor} \prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} z_{2j+2}^{-2} \cdot \mathbf{f}(\mathbf{v}) \tag{52}$$

with a polynomial vector field $\mathbf{f}$.

Note that (in contrast to the case $m = 1$) state variables w_i occur, which increase – compared to (47) – the order of $\mathbf{f}$ to

$$n = m + 2 + \left\lfloor \frac{m}{2} \right\rfloor. \tag{53}$$

The polynomial degree of $\mathbf{f}$ is given by

$$p := \deg \mathbf{f} = 3 + 2 \left(\left\lfloor \frac{m}{2} \right\rfloor + \sum_{i=3}^{m+2} \left\lfloor \frac{i}{2} \right\rfloor \right) = \begin{cases} m + 1 + \frac{(m+2)^2}{2} & m \geq 2, m \text{ even} \\ m + \frac{(m+1)(m+3)}{2} & m \geq 3, m \text{ odd} \end{cases} \tag{54}$$

Table I shows both polynomial degree p and system order n of the vector field $\mathbf{f}$ for some values of the relative degree m.

The sum-of-squares technique may be applied to search for Lyapunov functions for the polynomial vector field $\mathbf{f}$. In doing so, it can be taken into account that the definition of state variables w_i in (50b) is equivalent to the polynomial constraints $w_i \geq 0$, $w_i^4 = z_2^2 z_{2i+2}^2$ for $i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor$. This way, a Lyapunov function for the original system (47) may (conceivably) be obtained.

For $m = 2$, for example, the closed-loop system is

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \lfloor x_2 \rceil^0 \tag{55a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\beta_1 x_1 - \beta_2 \lfloor x_2 \rceil^{\frac{1}{2}} - \beta_3 \lfloor x_3 \rceil^{\frac{1}{3}} - \beta_4 \lfloor x_4 \rceil^{\frac{1}{4}} \tag{55b}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_2,$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t} = x_3, \tag{55c}$$

and the coordinates $\mathbf{v}$ are given by

$$\mathbf{v} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & w_1 \end{bmatrix} \tag{56}$$

$$= \begin{bmatrix} x_1 - \frac{\beta_4}{\beta_1} \lfloor x_4 \rceil^{\frac{1}{4}} & \lfloor x_2 \rceil^{\frac{1}{2}} & \lfloor x_3 \rceil^{\frac{1}{3}} & \lfloor x_4 \rceil^{\frac{1}{4}} & |x_2|^{\frac{1}{4}} |x_4|^{\frac{1}{8}} \end{bmatrix}. \tag{57}$$

One thus obtains the polynomial vector field

$$\mathbf{f}(\mathbf{v}) = \begin{bmatrix} z_2^3 z_3^2 z_4^6 - \frac{1}{4} \frac{\beta_4}{\beta_1} z_2^2 z_3^5 z_4^2 w_1^2 \\ -\frac{1}{2} z_2^2 z_3^2 z_4^6 (\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3) \\ \frac{1}{3} z_2^5 z_4^6 \\ \frac{1}{4} z_2^2 z_3^5 z_4^2 w_1^2 \\ \frac{1}{8} z_2^2 z_3^5 z_4 w_1^3 - \frac{1}{4} z_2 z_3^2 z_4^6 w_1 (\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3) \end{bmatrix}. \tag{58}$$

When applying the sum-of-squares technique, the polynomial constraints $w_1 \geq 0$, $w_1^4 = z_1^2 z_3^2$ may be taken into account.

VII. SUMMARY

This paper considered an extended super-twisting algorithm for sliding variables with relative degrees greater than one. The sum-of-squares technique was used to obtain for the first time a Lyapunov function for one instance of this algorithm. In doing so, the problem that a numerically obtained sum-of-squares decomposition of a polynomial does not necessarily prove its non-negativity had to be considered. By carefully choosing variables and monomials involved in the decomposition this problem was solved. A positive semidefinite function proving finite time attractivity of the set of equilibria as well as a positive definite Lyapunov function proving boundedness of the trajectories was thus obtained. The validity of of the approach subject to perturbations with sufficiently small upper bound was shown. A particular set of parameters was considered and the scaling of parameters to deal with larger perturbations was discussed. Finally, the basic steps in applying the technique to higher order systems were outlined.

APPENDIX

Proof of Lemma 1: The functions $x_1(t)$, $x_2(t)$ and $x_3(t)$ are absolutely continuous by virtue of being a solution of the differential inclusion; it is hence sufficient to show absolute continuity of $\lfloor x_1 \rceil^{\frac{1}{3}}$ and $\lfloor x_2 \rceil^{\frac{1}{2}}$. This can be done by reasoning similar to [20]: On closed time intervals where x_1 or x_2 are non-zero, they are differentiable and hence Lipschitz, which implies their absolute continuity. Thus only zero-crossings are considered, starting with x_2. If $x_2(T) = 0$ in some interval $[T - \varepsilon, T + \varepsilon]$ with $\varepsilon > 0$, then (4b) evaluated at $t = T$ reads

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\lambda_1 \lfloor x_1 \rceil^{\frac{1}{3}} + x_3. \tag{59}$$

The right-hand side of this equation is non-zero because $\mathbf{x}(T) \notin \mathcal{S}$ by assumption. By choosing ε small enough, $x_2(t)$ can be guaranteed to be monotonous on the considered interval. Absolute continuity then follows from the fact that the composition of the absolutely continuous function $\lfloor . \rceil^{\frac{1}{2}}$ with the absolutely continuous and monotonous function $x_2(t)$ is guaranteed to be absolutely continuous [20], [4].

For x_1 the proof is similar though slightly more involved. Again, consider a zero $x_1(T) = 0$ on some closed interval

$[T-\varepsilon, T+\varepsilon]$. If $x_2(T) \neq 0$, then x_1 is locally monotonous and the reasoning is as above. Otherwise one has, again at time instant T, $\frac{\mathrm{d}x_1}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = x_3$. As $x_3(T) = 0$ contradicts the assumption $\mathbf{x}(T) \notin \mathcal{S}$, the function $x_1(t)$ is either locally convex or concave and is clearly never monotonous on the entire interval. It can however be guaranteed to be monotonous on each of the closed intervals $[T-\varepsilon, T]$ and $[T, T+\varepsilon]$ by appropriate choice of ε, which implies absolute continuity on both intervals. The proof is concluded by noting that I can be written as a finite union of the considered types of closed intervals, as infinitely many zeros of x_1 or x_2 on the compact interval I would cluster at some time $\tau \in I$ with $\mathbf{x}(\tau) \in \mathcal{S}$.

To see that this is indeed the case, assume that zeros of x_2 cluster at τ but $\mathbf{x}(\tau) \notin \mathcal{S}$. (For zeros of x_1 the argument is similar.) Since $\mathbf{x}(t)$ is continuous, it follows that $x_2(\tau) = 0$, and since $\mathbf{x}(\tau) \notin \mathcal{S}$ the right-hand side of (4b) is not equal to zero; without loss of generality it is assumed to be negative. Continuity of $\mathbf{x}(t)$ then implies existence of $R, \varepsilon > 0$ such that $\frac{\mathrm{d}x_2}{\mathrm{d}t} \leq -R$ holds for almost all $t \in (\tau - \varepsilon, \tau) =: J$. As $\mathbf{x}(t)$ is also absolutely continuous, one may integrate this differential inequality to obtain $x_2(t) \geq (\tau - t)R > 0$ for $t \in J$, contradicting the assumption that τ is a cluster of zeros of x_2. ■

Proof of Lemma 2: First prove by contradiction that V is non-increasing with respect to time: Assume that a trajectory exists such that $V(t_3) > V(t_1) \geq 0$ for some $t_3 > t_1$. Continuity of V then guarantees the existence of $t_2 \in [t_1, t_3]$ such that $V(t_3) > V(t_2) > 0$ and $V(t) \neq 0$ for $t \in [t_2, t_3] =: I$ hold. One thus has $\mathbf{x}(t) \notin \mathcal{S}$ on I, and Lemma 1 together with the structure of V guarantees absolute continuity of V on I. One may thus integrate $\dot{V} \leq 0$ to obtain the contradiction

$$V(t_3) = V(t_2) + \int_{t_2}^{t_3} \dot{V} \,\mathrm{d}\tau \leq V(t_2) \tag{60}$$

proving that V is non-increasing. An immediate consequence is that the existence of T such that $\mathbf{x}(T) \in \mathcal{S}$, i.e. $V(T) = 0$, implies $\mathbf{x}(t) \in \mathcal{S}$, i.e. $V(t) = 0$, for all $t > T$. In that case one may write $\mathbf{x}(t) = \begin{bmatrix} \theta^3(t) & 0 & \lambda_1 \theta(t) \end{bmatrix}$ for $t \geq T$ with an absolutely continuous $\theta(t)$; by substitution into (4a) one finds that $\theta(t)$ and hence $\mathbf{x}(t)$ is constant.

The bound on T is now also proven by contradiction. Assume that for some trajectory $\mathbf{x}(t) \notin \mathcal{S}$ holds for all $t \in [0, t_0] =: J$ with $t_0 \geq \overline{T}$ from (13). Then V is absolutely continuous on J by virtue of Lemma 1. The comparison lemma, see e.g. [11], may thus be applied to the differential inequality (12) to obtain $V(T) = 0 < V(t_0)$ with $T \leq \overline{T}$; as $\overline{T} \leq t_0$, this contradicts the fact that V is non-increasing. ■

Proof of Corollary 1: Consider any trajectory $\mathbf{x}(t)$ and let T be given as in Lemma 2. Absolute continuity of $U(t)$ is guaranteed by Lemma 1 on any compact subinterval of $[0, T)$. That U is non-increasing is thus obtained from integrating its time-derivative for $t < T$ and from the fact that $\mathbf{x}(t)$ and hence $U(t)$ are constant for $t \geq T$. ■

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their critical comments, which have greatly improved the manuscript.

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