

**A Novel Method to Estimate the Reaching Time of
the Super-Twisting Algorithm**
Accepted Manuscript

Authors: Richard Seeber, Martin Horn, Leonid Fridman

Journal: IEEE Transactions on Automatic Control

DOI: 10.1109/TAC.2018.2812789

Copyright Notice

© 2018 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.

A Novel Method to Estimate the Reaching Time of the Super-Twisting Algorithm

Richard Seeber, Martin Horn and Leonid Fridman

Abstract—The super-twisting algorithm is a well known technique in the field of sliding mode control or observation. In this contribution, an exact analytic expression for this algorithm’s finite reaching time in the unperturbed case is derived. Based on this derivation, a novel estimation for the upper bound of the algorithm’s reaching time the presence of perturbations is presented. The considered perturbations may be composed of additive components that are either Lipschitz continuous in time or Hölder continuous in the sliding variable. Both analytically and in the course of numerical examples the new strategy is shown to yield significant improvements compared to existing reaching time estimates.

Index Terms—sliding mode control; reaching time; Lyapunov function

I. INTRODUCTION

When dealing with real-world control problems, the influence of unknown external disturbances is omnipresent. During control design, their influence on the closed-loop behavior is usually to be minimized. Sliding mode control offers one way to achieve this goal: a surface in state-space that corresponds to the desired system behavior is chosen, and the system is forced to stay on this sliding surface even in the presence of a certain class of disturbances. When initialized outside of this sliding surface, a certain finite time is required to reach it. An upper bound of this so-called reaching time is thus an important characteristic variable of a sliding mode control algorithm.

The subject of this contribution is the well-known super-twisting algorithm [1], [2], which was designed to replace the discontinuous control signal of first order sliding mode controllers by a continuous one. In theory, it allows to completely reject Lipschitz perturbations, achieving a second order sliding mode in finite time, i.e. the output and its derivative are robustly driven to zero in finite time. It is often used to alleviate the chattering effect that is incurred by the use of sliding mode control in the presence of unmodeled actuator dynamics [3], [4]. In a discrete-time setting it ensures a quadratic precision of the output with respect to the sampling step due to its homogeneity properties. It has also widely been

Richard Seeber and Martin Horn are with the Christian Doppler Laboratory for Model Based Control of Complex Test Bed Systems, Institute of Automation and Control, Graz University of Technology, Graz, Austria; email addresses: richard.seeber@tugraz.at, martin.horn@tugraz.at

Leonid Fridman is with the Engineering Faculty, National Autonomous University of Mexico (UNAM), Ciudad Universitaria, Mexico; email address: lfridman@unam.mx

The financial support by the Austrian Federal Ministry of Science, Research and Economy and the National Foundation for Research, Technology and Development is gratefully acknowledged.

used for robust control, observation and exact differentiation [5].

The estimation (i.e. upper-bounding) of reaching times has repeatedly been studied in sliding mode control, see e.g. [6]–[9]. For the twisting-algorithm [1], for example, the exact reaching time in the unperturbed case has been obtained in [10] in the form of a Lyapunov function. For the super-twisting algorithm, reaching time estimations based on various Lyapunov functions are proposed in [6], [7]. The latter of these has recently been used in [11] for parameter design given a prescribed reaching time bound. They have also proven useful in the context of applying the super-twisting algorithm when the control signal is limited [12]. The results often are conservative, however, and to the best knowledge of the authors no techniques exist that yield exact reaching times.

In this contribution, after the discussion of some preliminaries in Section II, an analytic expression for the reaching time of the super-twisting algorithm without perturbation is derived in Section III. This result is then used in Section IV to obtain a novel and structurally surprisingly simple reaching time estimate for the perturbed case. The performance of this estimate compared to existing estimation strategies is discussed in Section V in the course of several numerical examples. Section VI concludes the paper. Proofs for the presented theorems are given in an appendix.

II. PRELIMINARIES

Consider a sliding surface characterized by $\sigma = 0$ satisfying the differential equation $\dot{\sigma} = u + \Delta$ with some control input u and a perturbation Δ . The perturbation shall be composed of two components $\Delta = \Delta_1 + \Delta_2$ with the first one bounded by $|\Delta_1| \leq K\sqrt{|\sigma|}$, and the second one being Lipschitz continuous with time-derivative bounded by $|\dot{\Delta}_2| \leq L$. To steer σ to zero in spite of the perturbation, the super-twisting control law

$$u(t) = -k_1\sqrt{|\sigma(t)|}\text{sign}(\sigma(t)) - k_2 \int_0^t \text{sign}(\sigma(\tau)) d\tau \quad (1)$$

with positive parameters k_1, k_2 is used. Introducing state-variables $x_1 := \sigma, x_2 := \Delta_2 - k_2 \int_0^t \text{sign}(\sigma(\tau)) d\tau$ and using the abbreviation $[x]^p := |x|^p \text{sign}(x)$ the closed loop system may be written as

$$\dot{x}_1 = -k_1 [x_1]^{\frac{1}{2}} + x_2 + \delta_1 [x_1]^{\frac{1}{2}}, \quad (2a)$$

$$\dot{x}_2 = -k_2 [x_1]^0 + \delta_2 \quad (2b)$$

with new perturbations $\delta_1 := |x_1|^{-\frac{1}{2}} \Delta_1, \delta_2 := \dot{\Delta}_2$ bounded by

$$|\delta_1| \leq K, \quad |\delta_2| \leq L. \quad (2c)$$

Solutions of system (2) are understood in the sense of Filippov [13], i.e. as absolutely continuous solutions of a differential inclusion obtained by replacing δ_1, δ_2 by sets and $[x_1]^0$ by a set-valued function. It has been shown in [14] that system (2) without perturbation, i.e. the system with $K = L = 0$, is globally finite time stable iff $k_1 > 0$ and $k_2 > 0$. For the perturbed case, several sufficient conditions for global finite time stability are given in literature, see e.g. [2], [15].

Consider, for given bounds K, L , the reaching time of system (2) as a function of the initial state \mathbf{x}_0 . This quantity is denoted by $T_{K,L}$ and is formally defined as

$$T_{K,L}(\mathbf{x}_0) := \max\{\tau_{\mathbf{x}} \mid \mathbf{x}(t) \text{ satisfies (2), } \mathbf{x}(0) = \mathbf{x}_0\}. \quad (3a)$$

Therein $\tau_{\mathbf{x}}$, given by

$$\tau_{\mathbf{x}} := \min(\{\tau \in \mathbb{R} \mid \mathbf{x}(t) = \mathbf{0} \forall t \geq \tau\} \cup \{\infty\}), \quad (3b)$$

denotes the reaching time of a given trajectory $\mathbf{x}(t)$, i.e. the first time instant after which $\mathbf{x}(t)$ is identically zero, or infinity if no such time exists. The reaching time $T_{K,L}$ of system (2) is given by the maximum of all its solutions' reaching times or, in other words, by the maximum time it takes for $\mathbf{x}(t)$ to reach the origin for *any* perturbation satisfying (2c).

In the course of the following derivations the abbreviation

$$\mathbf{z}(\mathbf{x}) := \begin{bmatrix} [x_1]^{\frac{1}{2}} & x_2 \end{bmatrix}^T \quad (4)$$

and its inverse

$$\mathbf{z}^{-1}(z_1, z_2) = \begin{bmatrix} [z_1]^2 & z_2 \end{bmatrix}^T \quad (5)$$

are commonly used; the two components of the vector $\mathbf{z}(\mathbf{x})$ are sometimes also abbreviated by z_1 and z_2 . The derivations are based on the fact, remarked in [15], that the system dynamics (2) can formally be written with respect to \mathbf{z} in the form

$$\dot{\mathbf{z}} = \frac{1}{|z_1|} \mathbf{A} \mathbf{z} + \frac{1}{2} \mathbf{e}_1 \delta_1 + \mathbf{e}_2 \delta_2, \quad (6)$$

with $\mathbf{e}_1 = [1 \ 0]^T$, $\mathbf{e}_2 = [0 \ 1]^T$ and matrix \mathbf{A} given by

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2}k_1 & \frac{1}{2} \\ -k_2 & 0 \end{bmatrix}. \quad (7)$$

Note that \mathbf{A} is Hurwitz iff k_1 and k_2 are positive.

III. REACHING TIME FUNCTION FOR UNPERTURBED CASE

Consider system (2) in the unperturbed case, i.e. with $K = L = 0$. In the following the reaching time function $T_{0,0}$ of this system is computed. The derivation is based on the transformed system (6); the results are stated in the form of a theorem, for which a more rigorous proof of the obtained result is given in the appendix. After the theorem, a closed-form expression is derived for $T_{0,0}$.

A. Derivation

We first try to find an absolutely continuous function $\mathbf{z}(t)$ that satisfies (6) for almost all t . To this end, consider the following ansatz for a solution of (6)

$$\mathbf{z}(t) = e^{\mathbf{A}\alpha(t)} \mathbf{z}_0 \quad (8)$$

with $\mathbf{z}_0 := \mathbf{z}(\mathbf{x}_0)$ and a to-be-determined function α . This function should satisfy $\alpha(0) = 0$ in order to have \mathbf{z}_0 as the initial value of $\mathbf{z}(t)$, i.e. $\mathbf{z}(0) = \mathbf{z}_0$. In addition to this condition, the following differential equation for α is obtained by substituting (8) into (6) with $\delta_1 = \delta_2 = 0$:

$$\dot{\alpha} = \frac{1}{|z_1|} = \frac{1}{|\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}_0|}. \quad (9)$$

Therein $\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}_0$ is the first component of the vector $e^{\mathbf{A}\alpha} \mathbf{z}_0$. Solving (9) by separation of variables and integration yields the relation

$$\int_0^{\alpha(t)} |\mathbf{e}_1^T e^{\mathbf{A}\kappa} \mathbf{z}_0| d\kappa = t. \quad (10)$$

Now consider the reaching time $T_{0,0}(\mathbf{x}_0)$; by definition $\mathbf{x}(t)$ and hence $\mathbf{z}(t)$ vanishes as t approaches this value. It is clear from (8) that this can only happen (for non-zero initial conditions) as $\alpha(t)$ tends to infinity. This suggests that

$$\lim_{t \rightarrow T_{0,0}(\mathbf{x}_0)} \alpha(t) = \infty \quad (11)$$

has to hold. By applying this limit to both sides of (10), one obtains an expression for the reaching time and, as a byproduct of this derivation, a semi-explicit solution for the trajectories of the unperturbed super-twisting algorithm. Both are given in the following

Theorem 1 (Reaching time and semi-explicit solution of the unperturbed super-twisting algorithm). *Consider system (2) with $k_1 > 0$, $k_2 > 0$ and $K = L = 0$. Given an initial state \mathbf{x}_0 , the finite reaching time is given by*

$$T_{0,0}(\mathbf{x}_0) := \int_0^\infty |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x}_0)| d\alpha \quad (12)$$

and the unique solution is

$$\mathbf{x}(t) = \begin{cases} \mathbf{z}^{-1}(e^{\mathbf{A}\alpha(t)} \mathbf{z}(\mathbf{x}_0)) & t < T_{0,0}(\mathbf{x}_0) \\ \mathbf{0} & t \geq T_{0,0}(\mathbf{x}_0) \end{cases} \quad (13)$$

where $\alpha(t)$ is the strictly increasing, absolutely continuous function that is implicitly defined by (10), and $\mathbf{z}(\cdot)$, $\mathbf{z}^{-1}(\cdot)$ and \mathbf{A} are given by (4), (5) and (7), respectively.

Formal proofs for this theorem as well as for all further theorems are given in the appendix.

B. Closed Form Expression

In the following, a closed-form expression for the reaching time function $T_{0,0}(\mathbf{x})$ is obtained by solving the integral in (12). Note that the expression

$$f(\alpha) := \mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x}), \quad (14)$$

whose absolute value appears in the integral, changes sign whenever it is zero. The domain of integration is thus divided into segments where the sign of $f(\alpha)$ is constant; in each segment the integration may then be done analytically. The segments are separated by the zeros of $f(\alpha)$. To compute

those, one needs to distinguish whether the eigenvalues s_1, s_2 of the matrix \mathbf{A}

$$s_1 = -\frac{k_1}{4} + \frac{\sqrt{k_1^2 - 8k_2}}{4}, \quad s_2 = -\frac{k_1}{4} - \frac{\sqrt{k_1^2 - 8k_2}}{4} \quad (15)$$

are real or complex, because zeros consequently occur either at most once or periodically, respectively.

With imaginary and real part of the eigenvalues ordered according to $\text{Im}\{s_1\} \geq \text{Im}\{s_2\}$ and $0 > \text{Re}\{s_1\} \geq \text{Re}\{s_2\}$ the first (i.e. smallest) non-negative zero α_1 of $f(\alpha)$ is, as a function of \mathbf{x} , given by

$$\alpha_1 = \begin{cases} 0 & x_1 = 0 \\ -\frac{2}{c+2s_1} & c < -2 \text{Re}\{s_1\}, s_1 = s_2 \\ \frac{1}{s_1-s_2} \log \frac{c+2s_2}{c+2s_1} & c < -2 \text{Re}\{s_1\}, s_1 \neq s_2 \\ \frac{1}{s_1-s_2} [2\pi j + \log \frac{c+2s_2}{c+2s_1}] & c \geq -2 \text{Re}\{s_1\}, s_1 \notin \mathbb{R} \\ \infty & c \geq -2 \text{Re}\{s_1\}, s_1 \in \mathbb{R} \end{cases}$$

with the abbreviation $c = [x_1]^{-\frac{1}{2}} x_2$. (16)

Therein the complex logarithm

$$\log y := \ln |y| + j \arg y \quad \text{with } -\pi \leq \arg y < \pi \quad (17)$$

is used. The remaining zeros are given by

$$\alpha_i = \alpha_1 + (i-1) \frac{\pi}{\text{Im}\{s_1\}} \quad (18)$$

for $i = 2, \dots, \infty$. Note that this is correct even for real eigenvalues; in that case $\alpha_i = \infty$ for $i \geq 2$, meaning that no further (finite) zeros exist.

With the additional abbreviation $\alpha_0 := 0$ the non-negative reals are partitioned into intervals (α_i, α_{i+1}) , $i = 0, \dots, \infty$ (some of which may be empty); on each of these intervals the integrand in (12) is non-zero. The integration in (12) hence yields

$$\begin{aligned} T_{0,0}(\mathbf{x}) &= \sum_{i=0}^{\infty} \int_{\alpha_i}^{\alpha_{i+1}} |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x})| d\alpha \\ &= \left| \sum_{i=0}^{\infty} (-1)^{i+1} \int_{\alpha_i}^{\alpha_{i+1}} \mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x}) d\alpha \right| \\ &= \frac{1}{k_2} \left| \mathbf{e}_2^T \mathbf{z}(\mathbf{x}) + 2 \sum_{i=1}^{\infty} (-1)^i \mathbf{e}_2^T e^{\mathbf{A}\alpha_i} \mathbf{z}(\mathbf{x}) \right|. \end{aligned} \quad (19)$$

In the case $k_1^2 \geq 8k_2$, the eigenvalues of \mathbf{A} are real and all but the first term of the sum are zero. Otherwise, for complex eigenvalues $s_{1,2}$, a geometric series is obtained by substituting (18) into (19) and noting that

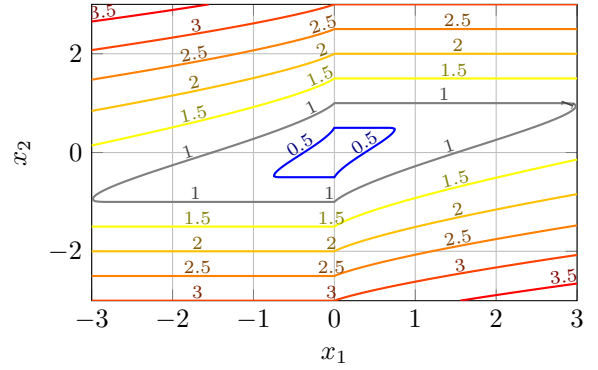
$$e^{\mathbf{A} \frac{\pi}{\text{Im}\{s_1\}}} = -e^{\frac{\text{Re}\{s_1\}}{\text{Im}\{s_1\}} \pi} \mathbf{I} = -e^{-\frac{k_1 \pi}{\sqrt{8k_2 - k_1^2}}} \mathbf{I}. \quad (20)$$

Computing the sum in both cases yields the reaching time function

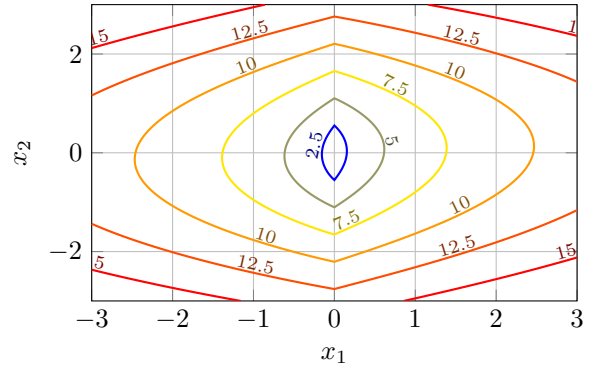
$$T_{0,0}(\mathbf{x}) = \frac{1}{k_2} \left| x_2 - \frac{2}{1-\lambda} \mathbf{e}_2^T e^{\mathbf{A}\alpha_1(\mathbf{x})} \mathbf{z}(\mathbf{x}) \right| \quad (21a)$$

with $\alpha_1(\mathbf{x})$ given in (16) and with the abbreviation

$$\lambda := \begin{cases} 0 & k_1^2 \geq 8k_2 \\ e^{-\frac{k_1 \pi}{\sqrt{8k_2 - k_1^2}}} & k_1^2 < 8k_2. \end{cases} \quad (21b)$$



(a) $k_1 = 4$, \mathbf{A} has real eigenvalues



(b) $k_1 = 0.4$, \mathbf{A} has complex eigenvalues

Fig. 1. Level lines of the unperturbed super-twisting algorithm's reaching time function $T_{0,0}(\mathbf{x})$ with $k_2 = 1$ for two values of k_1

C. Numerical Example

Consider the two parameter sets $k_1 = 4, k_2 = 1$ and $k_1 = 0.4, k_2 = 1$. In the former case the eigenvalues of the matrix \mathbf{A} are real, while in the latter case they are complex. Figure 1 shows the level lines of the reaching time function $T_{0,0}(\mathbf{x})$ obtained in these two cases.

IV. REACHING TIME ESTIMATE FOR PERTURBED CASE

Consider now system (2) with perturbation bounds $K, L \geq 0$. Using the previously derived reaching-time function $T_{0,0}(\mathbf{x})$ as a Lyapunov function, one obtains the following reaching time estimate for the perturbed system:

Theorem 2. Consider system (2) with $k_1 > 0, k_2 > 0$ and $K, L \geq 0$. Let $T_{0,0}(\mathbf{x})$ denote its unperturbed reaching time function given in (12) and define positive constants \bar{K}, \bar{L} by

$$\bar{K} := \frac{2}{T_{0,0}(\mathbf{e}_1)}, \quad \bar{L} := \frac{1}{T_{0,0}(\mathbf{e}_2)}. \quad (22)$$

If the relation

$$\bar{K}^{-1} K + \bar{L}^{-1} L < 1 \quad (23)$$

holds, then the perturbed system is globally finite time stable and an upper bound for its reaching time function $T_{K,L}(\mathbf{x})$ is given by

$$\bar{T}_{K,L}(\mathbf{x}_0) := \frac{T_{0,0}(\mathbf{x}_0)}{1 - K\bar{K}^{-1} - L\bar{L}^{-1}} \geq T_{K,L}(\mathbf{x}_0). \quad (24)$$

Remark 3. One can see that $K < \bar{K}$ and $L < \bar{L}$ are necessary conditions for this theorem to be applicable. The quantities \bar{K} and \bar{L} may hence be interpreted as maximum perturbation bounds. It is interesting to note that according to (22) and (12) they are given by

$$\bar{K} = \frac{2}{\int_0^\infty |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{e}_1| d\alpha}, \quad \bar{L} = \frac{1}{\int_0^\infty |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{e}_2| d\alpha} \quad (25)$$

which can be seen to be the reciprocal of the \mathcal{L}_∞ -gain $\int_0^\infty |g_i(\tau)| d\tau$, $i = 1, 2$, of linear time-invariant systems with impulse responses

$$g_1(t) = \frac{1}{2} \mathbf{e}_1^T e^{\mathbf{A}t} \mathbf{e}_1, \quad g_2(t) = \mathbf{e}_1^T e^{\mathbf{A}t} \mathbf{e}_2 \quad (26)$$

and corresponding transfer functions

$$G_1(s) = \frac{1}{2} \mathbf{e}_1^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{e}_1 = \frac{s}{2s^2 + k_1s + k_2}, \quad (27a)$$

$$G_2(s) = \mathbf{e}_1^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{e}_2 = \frac{1}{2s^2 + k_1s + k_2}. \quad (27b)$$

Both maximum perturbation bounds \bar{K} and \bar{L} may be given in closed form by using the expressions (16) and (21) previously derived. This is straightforward for \bar{L} , as one then has $\alpha_1(\mathbf{e}_2) = 0$. Substitution into (21) and (22) yields

$$\bar{L} = \begin{cases} k_2 & k_1^2 \geq 8k_2 \\ k_2 \tanh \frac{\pi}{2} \frac{k_1}{\sqrt{8k_2 - k_1^2}} & k_1^2 < 8k_2. \end{cases} \quad (28a)$$

For \bar{K} the computation is much more lengthy and thus not given here in detail; one eventually obtains

$$\bar{K} = \begin{cases} \sqrt{\frac{k_2}{2}} \cdot \left(\frac{\frac{k_1}{\sqrt{k_1^2 - 8k_2}} + 1}{\frac{k_1}{\sqrt{k_1^2 - 8k_2}} - 1} \right)^{\frac{k_1}{2\sqrt{k_1^2 - 8k_2}}} & k_1^2 > 8k_2 \\ \sqrt{\frac{k_2}{2}} \cdot e & k_1^2 = 8k_2 \\ \sqrt{\frac{k_2}{2}} \cdot \frac{2 \sinh \frac{\pi}{2} \frac{k_1}{\sqrt{8k_2 - k_1^2}}}{e^{\frac{k_1}{\sqrt{8k_2 - k_1^2}} \arctan \frac{k_1}{\sqrt{8k_2 - k_1^2}}}} & k_1^2 < 8k_2. \end{cases} \quad (28b)$$

Remark 4. An important special case is $K = 0$, i.e. the case of a Lipschitz continuous perturbation Δ with Lipschitz constant L . In this case condition (23) of Theorem 2 reduces to $L < \bar{L}$ and the upper reaching time bound in (24) is

$$\bar{T}_{0,L}(\mathbf{x}_0) = \frac{\bar{L}}{\bar{L} - L} T_{0,0}(\mathbf{x}_0). \quad (29)$$

As L is typically given, one is often interested in conditions on the parameters k_1, k_2 such that $L < \bar{L}$ holds. One can check that inequality (28a) is equivalent to

$$k_1 > \frac{\sqrt{32k_2} \operatorname{arctanh} \frac{L}{k_2}}{\sqrt{\pi^2 + 4 \left(\operatorname{arctanh} \frac{L}{k_2} \right)^2}}, \quad k_2 > L. \quad (30)$$

Up to now an *upper* bound $\bar{T}_{K,L}(\mathbf{x})$ for the reaching time function $T_{K,L}(\mathbf{x})$ has been considered. A *lower* bound for this function is given by the time it takes for x_2 to reach zero for the particular perturbations $\delta_1 = 0$, $\delta_2 = L[x_1]^0$: from (2b) one obtains this lower bound as

$$\underline{T}_{K,L}(\mathbf{x}) := \frac{|x_2|}{k_2 - L} \leq T_{K,L}(\mathbf{x}). \quad (31)$$

The following theorem shows that for large enough values of k_1 the upper bound $\bar{T}_{K,L}$ is either equal to this lower bound $\underline{T}_{K,L}$ or tends to it asymptotically:

Theorem 5. For any $K \geq 0$ and any $k_2 > L \geq 0$, the upper bound $\bar{T}_{K,L}(\mathbf{x})$ given in (24) and the lower bound $\underline{T}_{K,L}(\mathbf{x})$ given in (31) of the reaching time function $T_{K,L}(\mathbf{x})$ of system (2) satisfy

$$\lim_{k_1 \rightarrow \infty} \bar{T}_{K,L}(\mathbf{x}) = \underline{T}_{K,L}(\mathbf{x}). \quad (32)$$

Additionally, if $K = 0$ and, given \mathbf{x} , the parameter k_1 satisfies

$$k_1 \geq \begin{cases} \sqrt{8k_2} & x_1 = 0 \\ \sqrt{8k_2} & [x_1]^{-\frac{1}{2}} x_2 \geq \sqrt{2k_2} \\ \frac{2k_2}{[x_1]^{-\frac{1}{2}} x_2} + [x_1]^{-\frac{1}{2}} x_2 & 0 < [x_1]^{-\frac{1}{2}} x_2 < \sqrt{2k_2} \\ \infty & [x_1]^{-\frac{1}{2}} x_2 \leq 0, \end{cases} \quad (33)$$

equality holds instead of the limit, i.e.

$$\bar{T}_{0,L}(\mathbf{x}) = \underline{T}_{0,L}(\mathbf{x}) = T_{0,L}(\mathbf{x}). \quad (34)$$

Remark 6. Note that the first part of this theorem, i.e. relation (32), is independent of condition (33), which is not satisfied for any value of k_1 if $[x_1]^{-\frac{1}{2}} x_2 \leq 0$.

Remark 7. For the unperturbed case, the second part of this theorem, i.e. equality of $T_{0,0}(\mathbf{x})$ to a value independent of x_1 , can also be seen in Figure 1a in the form of straight level line segments. This is not the case for Figure 1b, as $k_1 \geq \sqrt{8k_2}$ is a necessary condition for (33) to hold.

V. COMPARISON TO EXISTING RESULTS

Several publications deal with the estimation of reaching times [6]–[8]; their results are compared to the presented approach in this section. As all of them consider only the case of Lipschitz perturbations, i.e. $K = 0$, the comparisons are restricted to this case as well.

In [6] Polyakov et al. propose a family of strict Lyapunov functions defined for $x_1 x_2 \neq 0$ by

$$V(\mathbf{x}) = \frac{k^2([x_1 x_2]^0)}{4k_1^4} \left(\frac{8x_2 [x_1]^0}{g([x_1 x_2]^0)} + \frac{k_1^2 k_0 ([x_1 x_2]^0)}{e^{-m(\mathbf{x})} s(\mathbf{x})^{-\frac{1}{2}}} \right)^2. \quad (35)$$

Therein

$$g(\mu) = \frac{8(k_2 - L\mu)}{k_1^2}, \quad (36)$$

and $m(\mathbf{x})$, $s(\mathbf{x})$, $k_0(\mu)$ and $k(\mu) > 0$ are functions defined in [6]; the last two of these depend on a parameter \bar{k} . They are designed such that V is continuously extendable to the set $x_1 x_2 = 0$. It is proven that

$$\dot{V} \leq -\eta \sqrt{V} \quad (37)$$

holds with

$$\eta = \min_{\mu \in \{-1, 1\}} k(\mu) \quad (38)$$

under the two conditions that $g(1) > 1$, i.e.

$$L < k_2 - \frac{k_1^2}{8}, \quad (39a)$$

and that the parameter \bar{k} satisfies the condition

$$\bar{k} \in \left(\frac{2}{g(1)} + \frac{1}{\sqrt{g(1)}} \exp \frac{-\frac{\pi}{2} - \arctan \frac{1}{\sqrt{g(1)-1}}}{\sqrt{g(1)-1}}, \frac{1}{\sqrt{g(-1)}} \exp \frac{\frac{\pi}{2} - \arctan \frac{1}{\sqrt{g(-1)-1}}}{\sqrt{g(-1)-1}} \right). \quad (39b)$$

Differential inequality (37) yields the reaching time estimate

$$T_{0,L}(\mathbf{x}) \leq \frac{2}{\eta} \sqrt{V(\mathbf{x})}. \quad (40)$$

In [7] Dávila et al. discuss an estimation by means of selecting from a family of strict Lyapunov functions an optimal one. This Lyapunov function family is proposed in [14] and is given by the quadratic form

$$V(\mathbf{x}) = \mathbf{z}(\mathbf{x})^T \mathbf{P} \mathbf{z}(\mathbf{x}). \quad (41)$$

If the positive definite matrices $\mathbf{P} \succ 0$ and $\mathbf{Q} \succ 0$ satisfy the Riccati equation

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} + L \mathbf{e}_1 \mathbf{e}_1^T + L \mathbf{P} \mathbf{e}_2 \mathbf{e}_2^T \mathbf{P} + \mathbf{Q} = \mathbf{0}, \quad (42)$$

then the time derivative \dot{V} of V along the trajectories of the perturbed system (2) is bounded by

$$\dot{V}(\mathbf{x}) \leq - \frac{\mathbf{z}(\mathbf{x})^T \mathbf{Q} \mathbf{z}(\mathbf{x})}{|z_1|}. \quad (43)$$

It is furthermore shown that (37) holds with

$$\eta = \frac{\lambda_{\min}(\mathbf{Q}) \sqrt{\lambda_{\min}(\mathbf{P})}}{\lambda_{\max}(\mathbf{P})}, \quad (44)$$

where λ_{\min} and λ_{\max} denote the matrices' smallest and largest eigenvalue, respectively. This yields a reaching time estimate in the same form as (40). It is well known and also remarked in [15] that such matrices \mathbf{P} , \mathbf{Q} satisfying the Riccati equation (42) exist iff

$$L < \bar{L}^* = \frac{1}{\sup_{\omega} |G_2(j\omega)|} = \begin{cases} k_2 & k_1^2 \geq 4k_2 \\ \frac{k_1 \sqrt{8k_2 - k_1^2}}{4} & k_1^2 < 4k_2, \end{cases} \quad (45)$$

i.e. provided that L is less than an upper bound \bar{L}^* given by the reciprocal of the \mathcal{L}_2 -gain (i.e. the \mathcal{H}_∞ -norm) of the transfer function $G_2(s)$ defined in (27b). Considering that the perturbation bound \bar{L} from (28a) is—as mentioned in Remark 3—the reciprocal of the \mathcal{L}_∞ -gain of $G_2(s)$, the inequality $\bar{L} \leq \bar{L}^*$ holds. Thus, in general the bound \bar{L}^* obtained using the quadratic Lyapunov function is better (i.e. larger) than or at least as good as the bound \bar{L} derived using the reaching time function as a Lyapunov function.

In [8] Utkin computes an upper reaching time bound under the condition that

$$\gamma := \sqrt[3]{\frac{16e(k_2 + L)^3}{k_1^4(k_2 - L)}} < 1 \quad (46)$$

holds. It is obtained by means of a geometric series as

$$T_{0,L}(\mathbf{x}) \leq \frac{|x_2|}{k_2 - L} + 4 \frac{k_2 |x_1|^{\frac{1}{2}}}{k_1(k_2 - L)} + \frac{k_2 |x_1^*|^{\frac{1}{2}}}{k_1(k_2 - L)} \frac{1}{1 - \gamma} \left(2 + \frac{k_1^2 \gamma}{k_2 + L} \right). \quad (47a)$$

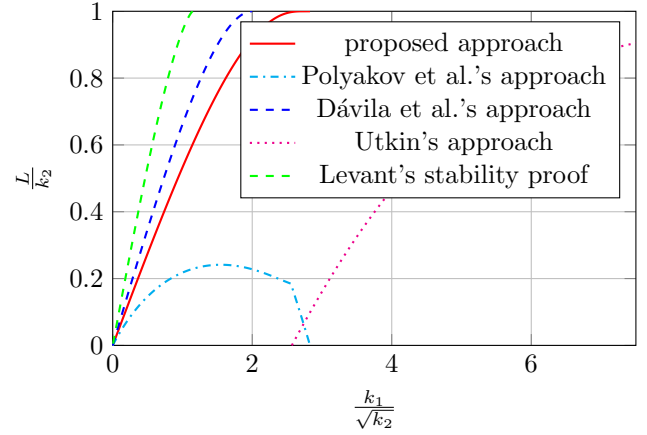


Fig. 2. Upper bound on L (normalized to $k_2 = 1$) imposed by the proposed approach with $K = 0$, by other reaching time estimation approaches, and by Levant's geometric stability proof

Therein the abbreviation

$$|x_1^*|^{\frac{1}{2}} := \max \left(|x_1|^{\frac{1}{2}}, \frac{2(k_2 + L) |x_1|^{\frac{1}{2}} + k_1 |x_2|}{k_1^2} \right) \quad (47b)$$

denotes the upper bound for $|x_1|^{\frac{1}{2}}$ at the first time instant where x_2 vanishes. It is furthermore proven that this estimate has the same asymptotic property as shown in the first part of Theorem 5, i.e. that it converges to the lower reaching time bound with increasing k_1 .

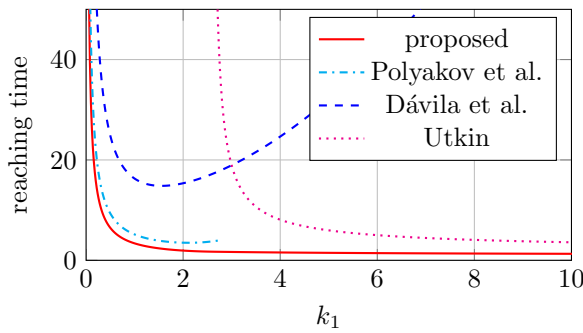
All discussed conditions amount to (implicit or explicit) upper bounds on L , which are compared in Figure 2. For completeness, the numerically obtained bound imposed by the conditions of Levant's geometric stability proof [2, Theorem 1] is also shown. One can see that this yields the best (i.e. largest) upper bound. Among the techniques for reaching time estimation, one can see that Dávila et al.'s approach yields the largest parameter region; the approach proposed in the present paper is slightly more conservative as discussed previously, and the other approaches are even more conservative.

The upper reaching time bounds obtained using the different approaches are now compared. This requires choosing parameters \bar{k} and \mathbf{Q} of Polyakov et al.'s and Dávila et al.'s Lyapunov function, respectively. For the former, numerical evidence suggests that the best (i.e. smallest) bounds are obtained when \bar{k} is very close to the infimum of the interval in (39b). For the latter it is argued in [7] that choosing \mathbf{Q} as a scalar multiple of the identity, i.e. $\mathbf{Q} = \beta \mathbf{I}$ with $\beta > 0$, leads to certain optimality properties regarding this estimate in the unperturbed case. For the perturbed case, choice of \mathbf{Q} by means of nonlinear numerical optimization strategies is suggested. Here, in order to enable a reproducible comparison, an intuitive heuristic (though suboptimal) way to select \mathbf{Q} in the perturbed case is derived and used.

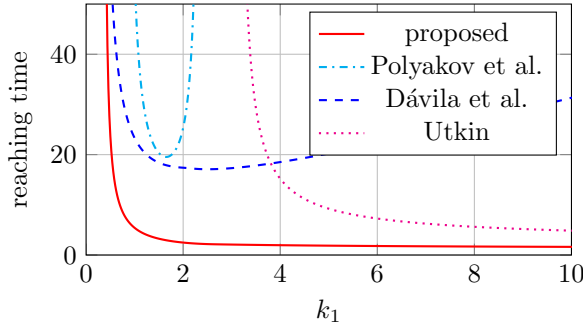
In particular, \mathbf{Q} is chosen as a multiple of the identity matrix

$$\mathbf{Q} = \beta(L) \mathbf{I} \quad (48)$$

with the positive-valued function β designed in such a way that the Riccati equation has a positive definite solution for



(a) unperturbed case $L = 0$



(b) perturbed case $L = 0.2$

Fig. 3. Upper reaching time bounds obtained using different approaches for $k_2 = 1$, $\mathbf{x}_0 = [1 \ -1]^T$ and $K = 0$ for two values of L

all $L < \bar{L}^*$ given in (45). Additionally, $\beta(0)$ is required to be finite, in order to obtain the “optimal” estimate for $L = 0$. As argued in Appendix B, an intuitive way to achieve this is to select

$$\beta(L) = \frac{1}{L^*} \max \left[\min \left(\frac{k_1^2 - 4k_2}{4}, \frac{k_2^2 - L^2}{k_1^2 + 1} \right), \chi_2 \right]. \quad (49)$$

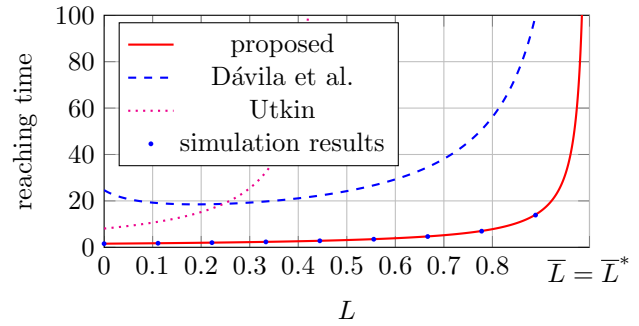
Therein χ_2 is the largest root of the polynomial

$$q(\chi) = \chi^2 + \left(\frac{k_1^2}{2} + 2k_2 + 1 \right) \chi + \frac{k_1^2}{16} (k_1^2 - 8k_2) + L^2 \quad (50)$$

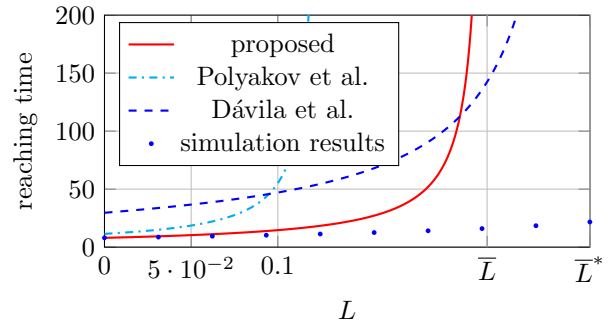
which is guaranteed to be real for $L < \bar{L}^*$.

In Figure 3 the approaches are compared for varying values of the parameter k_1 in the unperturbed and one perturbed case. In the unperturbed case one can see that Polyakov et al.’s estimate is very close to the exact reaching time obtained using the proposed approach; it is, however, in this case only applicable for a bounded subset of values for k_1 which satisfy (39a). In the perturbed case it is significantly more conservative, because the border of the admissible parameter region (see Figure 2) is approached. Dávila et al.’s approach yields satisfactory results for small values of k_1 in both the unperturbed and the perturbed case; as k_1 grows it diverges, however. In both cases the proposed estimate yields significantly better (i.e. lower) estimates than the other approaches. As described in Theorem 5, for large values of k_1 it converges to 1 and 1.25 in the two respective cases—a property shared by Utkin’s approach as mentioned before.

Figure 4a compares the approaches for varying values of L in a case where \mathbf{A} has real eigenvalues: for $k_1 = 4$, $k_2 = 1$.



(a) $k_1 = 4$, \mathbf{A} has real eigenvalues



(b) $k_1 = 0.4$, \mathbf{A} has complex eigenvalues

Fig. 4. Upper reaching time bounds for $k_2 = 1$, $\mathbf{x}_0 = [1 \ -1]^T$ and $K = 0$ obtained using the different approaches, and actual reaching times from a simulation with $\delta_1 = 0$, $\delta_2 = L [x_2]^0$ for two values of k_1

Additionally, simulation results obtained with the perturbations given by $\delta_1 = 0$, $\delta_2 = L [x_2]^0$ are shown. One can see that the upper bounds obtained with the proposed approach are very close to the simulation results, which constitute lower bounds. Polyakov et al.’s approach is not applicable in this case, and also the other approaches yield more conservative results.

Figure 4b shows the same comparison for $k_1 = 0.4$, a case where \mathbf{A} has complex eigenvalues. One can see that the estimate obtained with the proposed approach converges to the true reaching time for vanishing L . Additionally, for small values of L estimates much closer to the simulation results are obtained: Polyakov et al.’s estimate, though that also yields quite good results for very small values of L , is consistently larger than the proposed one, and Utkin’s approach is not applicable for this set of parameters. Only for large values of L Dávila et al.’s approach eventually takes the lead due to the larger set of admissible parameters shown in Figure 2. The estimate in that case is rather conservative, however; in the depicted case it is larger than sixfold the lower bound.

Figure 5 demonstrates the dependence of the reaching time estimate on k_2 in a perturbed case. Dávila et al.’s approach shows qualitatively the same behavior as in the case of increasing k_1 shown in Fig. 3b. The same appears to be true for the proposed estimate; it should be noted, however, that no conclusions regarding its asymptotic behavior can be drawn from Theorem 5 here. The behavior of Utkin’s and Polyakov et al.’s approaches is interchanged, in the sense that now the latter converges to a finite value as k_2 tends to infinity.

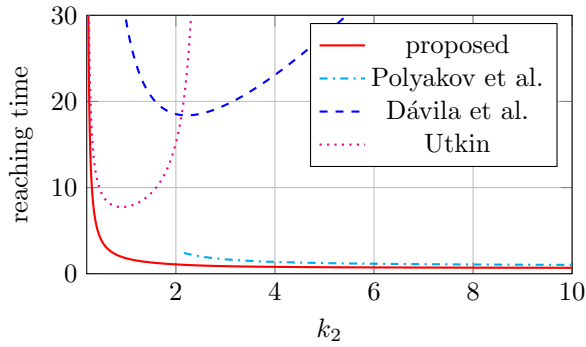


Fig. 5. Upper reaching time bounds obtained using the different approaches for $k_1 = 4$, $\mathbf{x}_0 = [1 \ -1]^T$ in the perturbed case with $K = 0$, $L = 0.2$

VI. CONCLUSION

In this paper, an analytic expression for the reaching time of the super-twisting algorithm without perturbations was derived. This result was subsequently used to provide a new upper reaching time bound for the perturbed algorithm. A comparison to three established estimates exhibited significant improvements: Especially for large values of the parameter k_1 and moderately small values of the perturbation's Lipschitz constant L much improved reaching time bounds that are also very close to the simulation results were obtained. Additionally, the estimate tends to the true worst-case reaching time of the algorithm if either k_1 tends to infinity or the perturbation bounds vanish.

APPENDIX A PROOFS

Proof of Theorem 1. Consider first relation (10), which gives in fact explicitly the inverse function of $\alpha(t)$, i.e. t as a function of α . This inverse is strictly increasing and bounded, with the last property being guaranteed by standard results from linear systems theory and the matrix \mathbf{A} being Hurwitz. It is furthermore absolutely continuous, and its derivative vanishes only at the countably many points that satisfy $\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}_0 = 0$. Hence, by a theorem due to Zareckii, see e.g. [16], the function $\alpha(t)$ is absolutely continuous as well.

Now consider the trajectories given in (13). The inclusion is trivially fulfilled for $t \geq T_{0,0}(\mathbf{x}_0)$. For $t < T_{0,0}(\mathbf{x}_0)$, one observes that $\mathbf{z}(\mathbf{x}(t))$ is the function given in (8) and thus by construction satisfies (6) whenever $x_1(t) \neq 0$, i.e. almost everywhere. Consequently, $\mathbf{x}(t)$ fulfills (2) for almost all t as well and hence is a solution of this differential inclusion.

Uniqueness of all segments of this solution with $x_1 > 0$ or $x_1 < 0$ follows from the fact that the right-hand side of (2) in these cases satisfies a one-sided Lipschitz condition [13]. For $x_1 = 0$, $x_2 \neq 0$ the inclusion reads

$$\dot{x}_1 = x_2 \neq 0, \quad \dot{x}_2 \in [-k_2, k_2], \quad (51)$$

implying that zero crossings of x_1 occur at isolated time instants. The case $x_1 = x_2 = 0$ finally is the equilibrium, which the system can not leave. The unique solution segments can hence be pieced together and the solution is unique globally (for $t \geq 0$). \square

Proof of Theorem 2. Consider the expression in (12) as a Lyapunov candidate

$$V(\mathbf{x}) = \int_0^\infty |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x})| d\alpha. \quad (52)$$

This function is not differentiable for $x_1 = 0$, as is also obvious in Figure 1. For $x_1 \neq 0$ one obtains the following inequality for its time derivative along the trajectories of the perturbed system (2):

$$\begin{aligned} \dot{V}(\mathbf{x}) &= |x_1|^{-\frac{1}{2}} \int_0^\infty [\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x})]^0 \mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{A} \mathbf{z}(\mathbf{x}) d\alpha \\ &\quad + \frac{\partial V}{\partial x_1} |x_1|^{\frac{1}{2}} \delta_1 + \frac{\partial V}{\partial x_2} \delta_2 \\ &= -|x_1|^{-\frac{1}{2}} |\mathbf{e}_1^T \mathbf{z}(\mathbf{x})| + \delta_1 |x_1|^{\frac{1}{2}} \frac{\partial V}{\partial x_1} + \delta_2 \frac{\partial V}{\partial x_2} \\ &= -1 + \delta_1 |x_1|^{\frac{1}{2}} \frac{\partial V}{\partial x_1} + \delta_2 \frac{\partial V}{\partial x_2}. \end{aligned} \quad (53)$$

Using (2c) and (12), one may furthermore verify that

$$\begin{aligned} \delta_1 |x_1|^{\frac{1}{2}} \frac{\partial V}{\partial x_1} &= \frac{1}{2} \delta_1 \int_0^\infty [\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x})]^0 \mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{e}_1 d\alpha \\ &\leq \frac{1}{2} K \int_0^\infty |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{e}_1| d\alpha = K \frac{T_{0,0}(\mathbf{e}_1)}{2}, \end{aligned} \quad (54a)$$

$$\begin{aligned} \delta_2 \frac{\partial V}{\partial x_2} &= \delta_2 \int_0^\infty [\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{z}(\mathbf{x})]^0 \mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{e}_2 d\alpha \\ &\leq L \int_0^\infty |\mathbf{e}_1^T e^{\mathbf{A}\alpha} \mathbf{e}_2| d\alpha = LT_{0,0}(\mathbf{e}_2). \end{aligned} \quad (54b)$$

hold. It is noteworthy that both of these inequalities are tight, i.e. equality holds for $\mathbf{x} = \mathbf{z}(\mathbf{e}_1) = \mathbf{e}_1$ or $\mathbf{x} = \mathbf{z}(\mathbf{e}_2) = \mathbf{e}_2$, respectively. By combining (53), (54) and (22) one obtains

$$\dot{V}(\mathbf{x}) \leq -1 + K\bar{K}^{-1} + L\bar{L}^{-1}. \quad (55)$$

Now consider one particular solution $\mathbf{x}(t)$ of the system with initial value \mathbf{x}_0 , i.e. let effectively the perturbation be fixed, and denote the reaching time of this solution by $\tau_{\mathbf{x}}$. Let T be a nonnegative time instant with $T < \tau_{\mathbf{x}}$. As $x_1 = 0$ can occur in the compact interval $[0, T]$ only at finitely many isolated time instants, integration immediately yields

$$\begin{aligned} V(\mathbf{x}(T)) &= V(\mathbf{x}(0)) + \int_0^T \dot{V}(\mathbf{x}(t)) dt \\ &\leq V(\mathbf{x}(0)) + (K\bar{K}^{-1} + L\bar{L}^{-1} - 1)T. \end{aligned} \quad (56)$$

Both sides of this inequality are continuous functions of T , so taking the limit $T \rightarrow \tau_{\mathbf{x}}$ and noting that $V(\mathbf{x}) = T_{0,0}(\mathbf{x})$ yields

$$\tau_{\mathbf{x}} \leq \frac{T_{0,0}(\mathbf{x}_0)}{1 - K\bar{K}^{-1} - L\bar{L}^{-1}}. \quad (57)$$

This inequality holds for any trajectory of the system and hence proves the estimate (29). \square

Proof of Theorem 5. It is first shown that the theorem's claims hold in the unperturbed case $K = L = 0$. In this case $\bar{T}_{K,L}(\mathbf{x}) = T_{0,0}(\mathbf{x})$; it will thus be shown that

$$\lim_{k_1 \rightarrow \infty} T_{0,0}(\mathbf{x}) = \underline{T}_{0,0}(\mathbf{x}) = \frac{|x_2|}{k_2} \quad (58)$$

and that equality rather than the limit, i.e. $k_2 T_{0,0}(\mathbf{x}) = |x_2|$ holds under the conditions given in the theorem.

If $x_1 = 0$ or $|x_1|^{-\frac{1}{2}} x_2 > 0$ the inequality (33) implies $k_1^2 \geq 8k_2$ (for case 3 this is verified by computing the minimum). Hence \mathbf{A} has real eigenvalues $s_{1,2}$ and one may check that (16) yields either $\alpha_1(\mathbf{x}) = \infty$ or $\alpha_1(\mathbf{x}) = 0$ when k_1 satisfies (33). One thus obtains $k_2 T_{0,0}(\mathbf{x}) = |x_2|$ from (21).

Let now $|x_1|^{-\frac{1}{2}} x_2 \leq 0$; in this case $\alpha_1(\mathbf{x})$ only takes finite values. As only the limit for $k_1 \rightarrow \infty$ is of interest, assume $k_1^2 \geq 8k_2$, such that the matrix \mathbf{A} has real eigenvalues $s_2 \leq s_1 = \frac{k_2}{2s_2} < 0$. Then the limit $k_1 \rightarrow \infty$ is equivalently obtained for $s_2 \rightarrow -\infty$. With

$$\mathbf{T} := \begin{bmatrix} 1 & \frac{s_1}{k_2} \\ -\frac{k_2}{s_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2s_2} \\ -\frac{k_2}{s_2} & 1 \end{bmatrix} \quad (59)$$

the matrix

$$\mathbf{A} := \mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \begin{bmatrix} s_2 & 0 \\ 0 & \frac{k_2}{2s_2} \end{bmatrix} \quad (60)$$

is diagonal and one furthermore has

$$\lim_{s_2 \rightarrow -\infty} \mathbf{T} = \mathbf{I}, \quad \lim_{s_2 \rightarrow -\infty} e^{\frac{k_2}{2s_2} \alpha_1(\mathbf{x})} = 1. \quad (61)$$

The latter relation is obtained by using L'Hôpital's rule and $\alpha_1(\mathbf{x})$ as well as c from case 3 of (16) to show that

$$\lim_{s_2 \rightarrow -\infty} \frac{k_2}{2s_2} \alpha_1(\mathbf{x}) = \lim_{s_2 \rightarrow -\infty} \frac{k_2}{2s_2} \frac{1}{\frac{k_2}{2s_2} - s_2} \log \frac{c + 2s_2}{c + \frac{k_2}{s_2}} = 0 \quad (62)$$

holds. Consequently one has

$$\begin{aligned} \lim_{s_2 \rightarrow -\infty} \mathbf{e}_2^T e^{\mathbf{A} \alpha_1(\mathbf{x})} \mathbf{z}(\mathbf{x}) &= \lim_{s_2 \rightarrow -\infty} \mathbf{e}_2^T \mathbf{T} e^{\mathbf{A} \alpha_1(\mathbf{x})} \mathbf{T}^{-1} \mathbf{z}(\mathbf{x}) \\ &= \lim_{s_2 \rightarrow -\infty} \mathbf{e}_2^T e^{\frac{k_2}{2s_2} \alpha_1(\mathbf{x})} \mathbf{z}(\mathbf{x}) = x_2 \end{aligned} \quad (63)$$

which after substitution into (21) completes the proof for the unperturbed case.

For the perturbed case $K, L \geq 0$ one obtains

$$\lim_{k_1 \rightarrow \infty} \overline{K}^{-1} = 0, \quad \lim_{k_1 \rightarrow \infty} \overline{L}^{-1} = k_2^{-1} \quad (64)$$

from (22) using relation (58). The proof is completed by substituting (58) and (64) into (24) to obtain (32) and noting that equalities hold instead of all limits under the conditions given in the theorem. \square

APPENDIX B CHOICE OF \mathbf{Q}

It is well known that the Riccati equation (42) has a positive definite solution iff the associated Hamiltonian matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & L \mathbf{e}_2 \mathbf{e}_2^T \\ -\mathbf{Q} - L \mathbf{e}_1 \mathbf{e}_1^T & -\mathbf{A}^T \end{bmatrix} \quad (65)$$

has no eigenvalues on the imaginary axis [17]. The characteristic polynomial of this matrix with $\mathbf{Q} = \beta \mathbf{I}$ is given by

$$\begin{aligned} \det(s\mathbf{I} - \mathbf{H}) &= s^4 + \left(\beta L - \frac{k_1^2 - 4k_2}{4} \right) s^2 \\ &\quad + \frac{k_2^2 - L^2 - \beta L(k_1^2 + 1)}{4}. \end{aligned} \quad (66)$$

One may check that the eigenvalue condition is equivalent to either the inequalities

$$L\beta < \frac{k_1^2 - 4k_2}{4} \quad L\beta < \frac{k_2^2 - L^2}{k_1^2 + 1} \quad (67)$$

being both satisfied or the polynomial $q(\chi)$ defined in (50) satisfying $q(L\beta) < 0$. For $L < \overline{L}^*$ this polynomial is guaranteed to have real roots $\chi_1 < \chi_2$ with $\chi_1 < 0$ and the latter condition is thus equivalent to $L\beta \in (0, \chi_2)$. To satisfy these conditions, β is (intuitively) chosen as in (49).

ACKNOWLEDGEMENT

The authors would like to thank Prof. Nicolaos Dourdoumas (Graz University of Technology) for several useful suggestions and discussions.

REFERENCES

- [1] Arie Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6):1247–1263, 1993.
- [2] Arie Levant. Robust exact differentiation via sliding mode technique. *Automatica*, 34(3):379 – 384, 1998.
- [3] Igor Boiko. *Discontinuous Systems*. Birkhäuser, Boston, 2008.
- [4] Ulises P. Ventura and Leonid Fridman. Is it reasonable to substitute discontinuous SMC by continuous HOSMC? 2017. arXiv:1705.09711.
- [5] Yuri B. Shtessel, Christopher Edwards, Leonid Fridman, and Arie Levant. *Sliding Mode Control and Observation*. Springer, New York, 2014.
- [6] Andrey Polyakov and Alex Poznyak. Reaching time estimation for “super-twisting” second order sliding mode controller via Lyapunov function designing. *IEEE Transactions on Automatic Control*, 54(8):1951–1955, 2009.
- [7] Alejandro Dávila, Jaime A. Moreno, and Leonid Fridman. Optimal Lyapunov function selection for reaching time estimation of super twisting algorithm. In *Proceedings of the 48th IEEE Conference on Decision and Control (CDC)*, pages 8405–8410, 2009.
- [8] Vadim Utkin. On convergence time and disturbance rejection of super-twisting control. *IEEE Transactions on Automatic Control*, 58(8):2013–2017, 2013.
- [9] Yury Orlov, Yannick Aoustin, and Christine Chevallereau. Finite time stabilization of a perturbed double integrator—part I: Continuous sliding mode-based output feedback synthesis. *IEEE Transactions on Automatic Control*, 56(3):614–618, 2011.
- [10] Tonámel Sánchez and Jaime A. Moreno. Construction of Lyapunov functions for a class of higher order sliding modes algorithms. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 6454–6459, 2012.
- [11] Rosalba Galván-Guerra, Leonid Fridman, Juan E. Velázquez-Velázquez, Shyam Kamal, and Bijnan Bandyopadhyay. Continuous output integral sliding mode control for switched linear systems. *Nonlinear Analysis: Hybrid Systems*, 22:284–305, 2016.
- [12] Ismael Castillo, Martin Steinberger, Leonid Fridman, Jaime A. Moreno, and Martin Horn. Saturated super-twisting algorithm based on perturbation estimator. In *Proceedings of the 55th IEEE Conference on Decision and Control (CDC)*, pages 7325–7328, 2016.
- [13] Aleksei F. Filippov. *Differential Equations with Discontinuous Right-Hand Side*. Kluwer Academic Publishing, Dordrecht, The Netherlands, 1988.
- [14] Jaime A. Moreno and Marisol Osorio. Strict Lyapunov functions for the super-twisting algorithm. *IEEE Transactions on Automatic Control*, 57(4):1035–1040, 2012.
- [15] Jaime A. Moreno. Lyapunov approach for analysis and design of second order sliding mode algorithms. In Leonid Fridman, Jaime A. Moreno, and Rafael Iriarte, editors, *Sliding Modes After the First Decade of the 21st Century*, pages 113–149. Springer, 2011.
- [16] Silvia Spătaru. An absolutely continuous function whose inverse function is not absolutely continuous. *Note di Matematica*, 23(1):47–49, 2004.
- [17] Stephen Boyd, Venkatarmanan Balakrishnan, and Pierre Kabamba. A bisection method for computing the H_∞ norm of a transfer matrix and related problems. *Mathematics of Control, Signals and Systems*, 2(3):207–219, 1989.