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Christian Kühn

Schrödinger operators and singular infinite rank perturbation

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1 INTRODUCTION

Schrödinger equations and the corresponding Schrödinger operators are essential objects in quantum mechanics and have consequently attracted an enormous mathematical interest. In the thesis at hand we are concerned with the special class of Schrödinger operators with so called δ-interactions. Such operators are used, for example, to model photonic crystals or systems with short range interactions. We will realize these operators as singular perturbations of the free Laplacian.

A Schrödinger operator with a δ-interaction of strength $\frac{1}{\alpha}$, $\alpha \in \mathbb{R} \setminus \{0\}$, supported on a discrete set or on a manifold $\Sigma \subseteq \mathbb{R}^d$ is an operator associated with the formal differential expression $-\Delta - \frac{1}{\alpha}\delta_\Sigma$, where δ_Σ denotes the δ-distribution on Σ. The formal action of this operator is given by

$$u \mapsto -\Delta u - \frac{1}{\alpha} u|_\Sigma \delta_\Sigma. \tag{1.1}$$

Such an operator can be used as an idealized model of a Schrödinger operator with a potential which has relatively high values or even a singularity on Σ and which vanishes away from Σ. For example, already in [46], Kronig and Penney considered periodic rectangular potentials which become in the limit a sequence of equally distributed δ-point interactions in $\mathbb{R}$. A systematical mathematical treatment of δ-interactions which yields a justification for the replacement of classical potentials by δ-interactions can be found for example in the monograph [5] and the papers [7, 31, 33, 53, 60].

Within the study of δ-interactions it turns out that the codimension of the interaction support Σ is more important than the dimension of the Euclidean space $\mathbb{R}^d$ in which Σ is embedded. For example the way how to define a δ-interaction on a curve in $\mathbb{R}^2$ is more similar to the way how to define a δ-interaction on a surface in $\mathbb{R}^3$ than on a curve in $\mathbb{R}^3$. In particular the task of giving a proper definition of a Schrödinger operator with a δ-interaction becomes progressively more difficult with increasing codimension of the interaction support.

We will tackle this problem from a more abstract point of view and consider first singular perturbations of a selfadjoint operator A in a Hilbert space $\mathcal{H}^0$ which can be formally written as

$$\mathcal{A}_\vartheta = \tilde{A} - G\vartheta^{-1}G^*. \tag{1.2}$$

Here G is a continuous injective operator from another Hilbert space $\mathcal{G}$ into $\mathcal{H}^{-k} \setminus \mathcal{H}^{-k+1} \cup \{0\}$, where $\mathcal{H}^{-k}$ is an element in the chain of rigged Hilbert spaces

$$\ldots \supseteq \mathcal{H}^{-k} \supseteq \ldots \supseteq \mathcal{H}^{-1} \supseteq \mathcal{H}^0 \supseteq \mathcal{H}^1 \supseteq \ldots \supseteq \mathcal{H}^k \supseteq \ldots \tag{1.3}$$

generated by A with $\mathcal{H}^2 := \mathrm{dom} A$. The operator $\tilde{A} : \mathcal{H}^0 \to \mathcal{H}^{-2}$ is an extension of A and the parameter ϑ is an invertible operator in $\mathcal{G}$. For technical reasons we will assume $A \geq 1$.

Such singular perturbations were considered for example in [47, 61] for the case that G is a rank one map and in [23] for the case that G is a finite rank operator. The approach used in this thesis is an extended version of the one in [23] and allows also maps G with infinite rank, which is necessary to apply it to δ-interactions supported on manifolds. If the map G has finite rank our approach reduces to the one in [23]. The same idea was also used in [62]. For another concept to handle infinite dimensional perturbations see for example [22].

It turns out that the index k has a major impact on the way how to interpret the formal expression $\mathcal{A}_\vartheta$ in (1.2). If $k = 1$ one can define in a very intuitive way selfadjoint operators associated with $\mathcal{A}_\vartheta$. If $k = 2$ such an approach will just lead to operators which are restrictions of A. Hence it is not possible to define selfadjoint perturbations of A in this way. This problem can be solved by slightly modifying the expression $\mathcal{A}_\vartheta$ in (1.2) to

$$\tilde{\mathcal{A}}_\vartheta = \tilde{A} - G\vartheta^{-1}G^*P, \tag{1.4}$$

where P is a suitable projection. If $k > 2$ also such a modification will not lead to selfadjoint operators in $\mathcal{H}^0$. Roughly speaking this is caused by the fact that the difference between a nontrivial element in the range of G and a nontrivial element in the range of $\tilde{A}$ never belongs to $\mathcal{H}^0$. In other words the perturbation is too singular. We will call this case the *supersingular case*. To handle this situation we have to extend the space $\mathcal{H}^0$ to a larger Krein space $\tilde{\mathcal{K}}$. In this space we are able to define selfadjoint operators (with respect to the inner product of $\tilde{\mathcal{K}}$) whose action can be seen as a shifted version of the one resulting from $\mathcal{A}_\vartheta$.

For any k our approach leads to a generalized boundary triple which enables us to parameterize the operators A_ϑ corresponding to the expression $\mathcal{A}_\vartheta$ (or $\tilde{\mathcal{A}}_\vartheta$). Boundary triples and their generalizations have turned out to be a helpful tool in extension theory of symmetric operators. In particular we get a Krein type resolvent formula

$$(A_\vartheta - \lambda)^{-1} - (A - \lambda)^{-1} = \gamma(\lambda)\big[\vartheta - M(\lambda)\big]^{-1}\gamma(\overline{\lambda})^*, \quad \lambda \in \rho(A_\vartheta) \cap \rho(A), \tag{1.5}$$

which establishes a connection between the operator A_ϑ and the parameter ϑ via a holomorphic function M. This function M, the so called Weyl function, is the analog of the classical Titchmarsh-Weyl m-function from Sturm-Liouville theory. Together with Krein's resolvent formula the Weyl function allows in many cases a detailed analysis of the operator A_ϑ and its spectrum.

We will use the same strategy for Schrödinger operators with δ-interactions on a manifold Σ in $\mathbb{R}^d$. Therefore we have to identify the objects from the abstract approach described above in our situation. The operator A is given by $-\Delta_{\text{free}} + 1$, where $-\Delta_{\text{free}}$ is the free Laplacian in $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. The rigged Hilbert spaces in (1.3) generated by A become the Sobolev spaces $H^s(\mathbb{R}^d)$, $s \in \mathbb{Z}$, and the Hilbert space $\mathcal{G}$ is $L^2(\Sigma)$. The δ-distribution on Σ with weight function $h \in L^2(\Sigma)$ is defined by

$$\big(h\delta_\Sigma\big)\varphi := \int_\Sigma h \cdot \varphi|_\Sigma \, d\sigma, \qquad \varphi \in H^k(\mathbb{R}^d),$$

and belongs to a Sobolev space $H^{-k}(\mathbb{R}^d)$ of a certain negative order $-k$, depending on the codimension of Σ. Hence the operator

$$G : L^2(\Sigma) \to H^{-k}(\mathbb{R}^d), \quad h \mapsto h\delta_\Sigma,$$

fits into our scheme. Note that $G^* : H^k(\mathbb{R}^d) \to L^2(\Sigma)$ is given by $G^*u = u|_\Sigma$. On a purely formal level we have now for $\vartheta = \alpha \in \mathbb{R} \setminus \{0\}$

$$\mathcal{A}_\vartheta u = \big(\tilde{A} - G\vartheta^{-1}G^*\big)u = (-\Delta + 1)u - \alpha^{-1}u|_\Sigma \delta_\Sigma,$$

which coincides (up to the constant +1) with the mapping given in (1.1). The rigorous definition of the corresponding operator A_ϑ is done with the help of the generalized boundary triple resulting from the abstract approach. If the codimension of Σ is 1 this generalized boundary triple coincides with the one which was used in [12] to define Schrödinger operators with δ-interactions on boundaries of bounded C^∞-domains in $\mathbb{R}^d$. Hence these Schrödinger operators coincide with the operators A_ϑ (up to the constant +1). It was shown in [12] (see also Remark 4.1 in [18]) that their definition of a Schrödinger operator with δ-interaction coincides with the usual definition as the representing operator of the semi-bounded sesquilinear form

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)} - \langle \vartheta^{-1}u|_\Sigma, v|_\Sigma\rangle_{L^2(\Sigma)}, \qquad \operatorname{dom}\mathfrak{t} = H^1(\mathbb{R}^d).$$

This definition is used for example in [31, 33, 36, 44, 45, 65] as well as in the recent publication [26], see also the more general approach via Radon measures in [18], which contains the situation above as a special case. If the codimension is 4 or larger we are in the supersingular case $k > 2$ and the whole situation becomes more complicated because we have to extend the space $L^2(\mathbb{R}^d)$ to a larger Krein space. We leave it for future works to check how the operators obtained in this way are connected with operators introduced by other authors to handle such problems, e.g. in [21].

The main focus of our application is on the situation that Σ is a manifold of codimension 2, which corresponds (as well as the situation of codimension 3) to the case $k = 2$. The abstract approach yields a generalized boundary triple which enables us to parametrize operators A_ϑ corresponding to the expression $\tilde{\mathcal{A}}_\vartheta$ in (1.4). The challenging question which appears now is how we have to chose the parameters ϑ such that the resulting operators coincide with those operators, which are known in the literature as Schrödinger operators with δ-interactions.

Schrödinger operators with δ-interactions on curves in $\mathbb{R}^3$ were already considered in [17] for the special case of a straight line and in [48, 49] for smooth infinite or closed curves. Other works which deal with such operators are for example [15, 29, 32, 35, 43]. The definition of these operators is inspired by the case of a δ-point-interaction in $\mathbb{R}^2$ and uses a “boundary” condition at the curve. An alternative way to define these operators was given in [65] via a quadratic form in $L^2(\Sigma)$.

Our approach is a special case of [54] and strongly inspired by the one used in [63] to define δ-interactions on curves in $\mathbb{R}^3$. We will generalize it (after a small modification such that it fits better into our theoretical scheme) to δ-interactions on manifolds of codimension 2 in $\mathbb{R}^d$ for arbitrary d. The essential part of this approach is an operator which we will call "the generalized trace operator". With this generalized trace we are able to construct operators ϑ in $L^2(\Sigma)$ which parametrize Schrödinger operators with δ-interactions of an arbitrary given strength on Σ, cf. Definition 4.17. Furthermore, the generalized boundary triple which is used for this parametrization provides a Krein type resolvent formula as in (1.5). For an optimal utilization of this formula a deep understanding of the generalized trace is needed. As the properties of this operator depend on the space dimension and on the geometry of Σ we will concentrate for the spectral analysis again on the case of a closed curve in $\mathbb{R}^3$. We will show in Theorem 4.25 that the singular values of the resolvent difference

$$(-\Delta_{\Sigma,\alpha} - \lambda)^{-1} - (-\Delta_{\text{free}} - \lambda)^{-1}, \quad \lambda \in \rho(-\Delta_{\Sigma,\alpha}) \cap \rho(-\Delta_{\text{free}}),$$

counted with multiplicities satisfy

$$s_j(\lambda) = O\Big(\frac{1}{j^2 \ln j}\Big) \quad \text{as} \quad j \to \infty.$$

In particular, this implies that the resolvent difference belongs to the trace class, which was already shown in [19] (see also Remark 4.1 in [29] for a similar result in the case of a δ-interaction of periodic strength on a straight line in $\mathbb{R}^3$). Moreover, by using a Birman-Schwinger principle, we obtain in Theorem 4.26 estimates for the number of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$ similar to those in [43] (see also [19]). A more explicit estimate is given in Corollary 4.27 that leads to an asymptotic estimate similar to the one in Theorem 3.3 in [34]. In Theorem 4.28 we show that the lower bound of $-\Delta_{\Sigma,\alpha}$ is maximized if the curve Σ is a circle (by fixed length and strength). The proof is analog to the proof of the two-dimensional equivalent in [28, 30].

At the end of this introduction we will give a brief overview on the structure of this thesis. In Chapter 2 we provide some definitions and basic properties of boundary triples, Friedrichs extensions, Sobolev spaces and other concepts, which will be used in this work. Chapter 3 is devoted to the abstract approach. Starting with a selfadjoint operator $A \geq 1$ in a Hilbert space $\mathcal{H}^0$ we will construct in the first section of Chapter 3 the chain of Hilbert spaces from (1.3) and extend the operator A onto spaces $\mathcal{H}^s$ with negative index. Furthermore we construct a generalized boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ which depends on the index k. In Section 3.2 we discuss how we can parametrize the operators corresponding to the formal expression $\mathcal{A}_\vartheta$ in (1.2) with this triple if $k = 1$ and apply it to Schrödinger operators with δ-interactions supported on boundaries of C^∞-domains. In Section 3.3 we give a brief discussion of the case $k = 2$, but without applications. This will be done in the following chapter. In Section 3.4 we analyze the supersingular case $k > 2$. For this we extend the Hilbert space $\mathcal{H}^0$ to a larger Krein space and construct an ordinary boundary triple $(\mathcal{G}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$. Also in this case an application to δ-interactions is given.

Chapter 4 is devoted to Schrödinger operators with δ-interactions on manifolds of codimension 2 and uses the abstract approach from Chapter 3, in particular from Section 3.3. In Section 4.1 we investigate the generalized boundary triple in this case and the corresponding γ-field and Weyl function. In Section 4.2 we present some first spectral results for the operators A_ϑ corresponding to the formal expression $\tilde{A}_\vartheta$ in (1.4). The generalized trace is constructed in Section 4.3 and is used afterwards to identify the correct parameter ϑ such that the operator A_ϑ coincides (up to a constant) with the Schrödinger operators with δ-interactions on the manifold. In Section 4.4 we consider the special case that the manifold is a closed curve in $\mathbb{R}^3$ and provide a detailed spectral analysis.

Note that large parts of Chapter 4 and in particular of Section 4.4 where already published by the author in [8].

2 PRELIMINARIES

This chapter contains definitions and basic properties of boundary triples, Friedrichs extensions, Sobolev spaces and other concepts, which we will need in this thesis.

2.1 Notation and basic properties

By $\mathbb{R}$ and $\mathbb{C}$ we will denote the real and complex numbers, respectively. The natural numbers are denoted by $\mathbb{N}$, whereas $\mathbb{N}_0$ denotes the set of nonnegative integers. The set of integers is denoted by $\mathbb{Z}$.

All Hilbert and Krein spaces in this thesis are supposed to be separable.

All sesquilinear forms like scalar products or Krein products are linear in the first entry and antilinear in the second one.

A *linear relation* in a Hilbert or Krein space $\mathcal{H}$ is a linear subspace of $\mathcal{H} \times \mathcal{H}$.

We write elements in $\mathcal{H} \times \mathcal{H}$ as $\{u, u'\}$ or $\begin{bmatrix} u \\ u' \end{bmatrix}$ with $u, u' \in \mathcal{H}$.

If A is a linear relation in $\mathcal{H}$ then we denote by

(i) $\operatorname{dom} A := \{u \in \mathcal{H} : \exists u' \in \mathcal{H} \text{ with } \{u, u'\} \in A\}$ the *domian* of A,

(ii) $\operatorname{ran} A := \{u' \in \mathcal{H} : \exists u \in \mathcal{H} \text{ with } \{u, u'\} \in A\}$ the *range* of A,

(iii) $\ker A := \{u \in \mathcal{H} : \{u, 0\} \in A\}$ the *kernel* of A and by

(iv) $\operatorname{mul} A := \{u' \in \mathcal{H} : \{0, u'\} \in A\}$ the *multivalued part* of A.

All operators in this thesis are linear operators. If A is a linear operator in $\mathcal{H}$ then the graph of A is a linear relation in $\mathcal{H}$. As usual we will not distinguish between an operator and its graph.

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert or Krein spaces we denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the set of all bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$ whose domain is the whole space $\mathcal{H}$. Note that all these operators are closed. As usual we define $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$.

We define the *resolvent set* $\rho(A)$ and the *spectrum* $\sigma(A)$ of a linear relation A by

$$\rho(A) := \{\lambda \in \mathbb{C} : (A - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})\} \qquad \text{and} \qquad \sigma(A) := \mathbb{C} \setminus \rho(A).$$

Special subsets of $\sigma(A)$ are the *point spectrum* $\sigma_p(A)$, the *continuous spectrum* $\sigma_c(A)$, the *discrete spectrum* $\sigma_d(A)$ and the *essential spectrum* $\sigma_{\mathrm{ess}}(A)$, which are defined by

$$\begin{aligned}
\sigma_p(A) &:= \{\lambda \in \mathbb{C} : \ker(A-\lambda) \neq \{0\}\},\\
\sigma_c(A) &:= \{\lambda \in \sigma(A) : \ker(A-\lambda) = \{0\}, \overline{\operatorname{ran}(A-\lambda)} = \mathcal{H}\},\\
\sigma_d(A) &:= \{\lambda \in \sigma_p(A) : \dim\ker(A-\lambda) < \infty \text{ and } \exists\, \varepsilon > 0 \text{ with } B_\varepsilon(\lambda) \cap \sigma(A) = \{\lambda\}\},\\
\sigma_{\mathrm{ess}}(A) &:= \sigma(A) \setminus \sigma_d(A),
\end{aligned}$$

respectively. Note that $\rho(A) = \emptyset$ if A is not closed.

If A is a linear relation in the Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ then we define its *adjoint* by

$$A^* := \big\{\{v,v'\} \in \mathcal{H}\times\mathcal{H} : \langle v,u'\rangle_{\mathcal{H}} = \langle v',u\rangle_{\mathcal{H}} \text{ for all } \{u,u'\} \in A\big\}.$$

A is called *symmetric* if $A \subseteq A^*$ and A is called *selfadjoint* if $A = A^*$. If A is a densely defined operator these definitions coincide with the usual definitions of the adjoint operators.

Analogously, the *Krein space adjoint* of a linear relation A in the Hilbert space $\mathcal{K}$ with inner product $[\![\cdot,\cdot]\!]_{\mathcal{K}}$ is defined by

$$A^+ := \big\{\{v,v'\} \in \mathcal{K}\times\mathcal{K} : [\![v,u']\!]_{\mathcal{K}} = [\![v',u]\!]_{\mathcal{K}} \text{ for all } \{u,u'\} \in A\big\}.$$

A is called *symmetric* (*selfadjoint*) with respect to $[\![\cdot,\cdot]\!]_{\mathcal{K}}$, if $A \subseteq A^+$ ($A = A^+$).

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H}^1 \subseteq \mathcal{H}$ a subspace which is a Hilbert space by itself and denote by $\mathcal{H}^*$ and $(\mathcal{H}^1)^*$ the corresponding dual spaces. Then the inclusion $(\mathcal{H}^1)^* \supseteq \mathcal{H}^*$ holds. According to the Riesz representation theorem we can identify $\mathcal{H}^*$ with $\mathcal{H}$ and get the inclusion

$$\mathcal{H}^1 \subseteq \mathcal{H} \subseteq (\mathcal{H}^1)^*.$$

In this case the (sesquilinear) dual pairing $\langle\varphi,u\rangle_{\mathcal{H}^1,(\mathcal{H}^1)^*}$ coincides with the scalar product $\langle\varphi,u\rangle_{\mathcal{H}}$ for all $\varphi \in \mathcal{H}^1$ and $u \in \mathcal{H}$. If $\mathcal{G}$ is another Hilbert space, $\mathcal{G}^1 \subseteq \mathcal{G}$ a subspace which is a Hilbert space by itself and $G : \mathcal{H}^1 \to \mathcal{G}^1$ an operator, then the adjoint operator $G^* : (\mathcal{G}^1)^* \to (\mathcal{H}^1)^*$ is defined by $\langle G\varphi,u\rangle_{\mathcal{G}^1,(\mathcal{G}^1)^*} = \langle\varphi,G^*u\rangle_{\mathcal{H}^1,(\mathcal{H}^1)^*}$ for $\varphi \in \mathcal{G}^1$ and $u \in (\mathcal{G}^1)^*$. Analogously if $G : \mathcal{H} \to \mathcal{G}^1$, $G : (\mathcal{H}^1)^* \to \mathcal{G}^1$, etc. It will be clear from the context which Hilbert spaces will be identified with their dual spaces. In particular if $G : \mathcal{H} \to \mathcal{G}$ and both spaces are identified with their dual spaces this definition of the adjoint operator coincides with the one given above.

The following lemma provides a helpful decomposition of domains of linear operators.

Lemma 2.1. *Let A and T be operators in the Hilbert space $\mathcal{H}$ such that $A = A^* \subseteq T$ holds. Then the decomposition*

$$\operatorname{dom} T = \operatorname{dom} A \dotplus \ker(T-\lambda)$$

holds for all $\lambda \in \rho(A)$, where $\dotplus$ is the direct sum in the Hilbert space $\mathcal{H}$.

Proof. Let $u \in \operatorname{dom} T$ be arbitrary. As $\lambda \in \rho(A)$ we know $(A-\lambda)^{-1} \in \mathcal{L}(\mathcal{H})$. Hence $v := (A-\lambda)^{-1}(T-\lambda)u \in \operatorname{dom} A$ is well defined and satisfies $(A-\lambda)v = (T-\lambda)u$. As T is an extension of A we get also $(T-\lambda)v = (T-\lambda)u$, i.e. $w := u - v \in \ker(T-\lambda)$. Hence

$$u = v + w \in \operatorname{dom} A + \ker(T-\lambda)$$

and therefore $\operatorname{dom} T \subseteq \operatorname{dom} A + \ker(T-\lambda)$. The other inclusion is trivial as T is an extension of A. It remains to show, that the sum is direct. For this let $u \in \operatorname{dom} A \cap \ker(T-\lambda)$. As T is an extension of A we have $Tu = Au$. Hence

$$(A-\lambda)u = (T-\lambda)u = 0.$$

Due to $\lambda \in \rho(A)$ it follows $u = 0$. Hence $\operatorname{dom} A \cap \ker(T-\lambda) = \{0\}$. □

Furthermore we will need the following special case of the well-known min-max-principle. For the sake of completeness we will give a proof although similar proofs can be found in the literature, see for example the proof of Theorem 12.1 in [50].

Lemma 2.2. *Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$ which is bounded from above and which has no essential spectrum, i.e. $\sigma(A)$ just consists of isolated eigenvalues with finite multiplicities. Denote these eigenvalues in nonincreasing order and counted with multiplicity by ν_k, $k \in \mathbb{N}$. Then*

$$\nu_k = \max_{\substack{U \subseteq \operatorname{dom} A \\ \dim U = k}} \min_{u \in U \setminus \{0\}} \frac{\langle Au, u\rangle_{\mathcal{H}}}{\|u\|_{\mathcal{H}}^2}.$$

Of course it is assumed above that U in the maximum is a linear subspace of $\mathcal{H}$.

Proof. As A is selfadjoint with $\sigma(A) = \sigma_p(A)$ there exists an orthonormal basis $(u_n)_{n\in\mathbb{N}}$ of eigenvectors, i.e.

$$Au_n = \nu_n u_n, \qquad \langle u_n, u_m\rangle_{\mathcal{H}} = \delta_{n,m} \qquad \text{and} \qquad \overline{\operatorname{span}\{u_n : n \in \mathbb{N}\}} = \mathcal{H}.$$

For $k \in \mathbb{N}$ define $U_k := \operatorname{span}\{u_1, \dots, u_k\}$. Let $u = \sum_{j=1}^k \alpha_j u_j \in U_k$. Then

$$\begin{aligned}\langle Au, u\rangle_{\mathcal{H}} &= \sum_{j=1}^k \sum_{l=1}^k \alpha_j \overline{\alpha_l} \langle Au_j, u_l\rangle_{\mathcal{H}} = \sum_{j=1}^k \sum_{l=1}^k \alpha_j \overline{\alpha_l} \nu_j \underbrace{\langle u_j, u_l\rangle_{\mathcal{H}}}_{\delta_{j,l}} \\ &= \sum_{j=1}^k |\alpha_j|^2 \nu_j \langle u_j, u_j\rangle_{\mathcal{H}} \geq \nu_k \sum_{j=1}^k |\alpha_j|^2 \langle u_j, u_j\rangle_{\mathcal{H}} = \nu_k \|u\|_{\mathcal{H}}^2\end{aligned}$$

and hence $\min_{u \in U_k \setminus \{0\}} \frac{\langle Au, u\rangle_{\mathcal{H}}}{\|u\|_{\mathcal{H}}^2} \geq \nu_k$. On the other hand we have

$$\frac{\langle Au_k, u_k\rangle_{\mathcal{H}}}{\|u_k\|_{\mathcal{H}}^2} = \frac{\langle \nu_k u_k, u_k\rangle_{\mathcal{H}}}{\|u_k\|_{\mathcal{H}}^2} = \nu_k$$

and hence $\min_{u\in U_k\setminus\{0\}} \frac{\langle Au,u\rangle_{\mathcal{H}}}{\|u\|^2_{\mathcal{H}}} = \nu_k$. Let $U \neq U_k$ be an arbitrary subspace of $\mathcal{H}$ with dimension k. Hence $U \cap \overline{\operatorname{span}\{u_n : n \geq k\}} \neq \{0\}$. Let

$$u = \sum_{j=k}^{\infty} \alpha_j u_j \in \left(U \cap \overline{\operatorname{span}\{u_n : n \geq k\}}\right) \setminus \{0\}.$$

Hence

$$\langle Au,u\rangle_{\mathcal{H}} = \sum_{j=k}^{\infty} |\alpha_j|^2 \nu_j \langle u_j,u_j\rangle_{\mathcal{H}} \leq \nu_k \sum_{j=k}^{\infty} |\alpha_j|^2 \langle u_j,u_j\rangle_{\mathcal{H}} = \nu_k \|u\|^2_{\mathcal{H}}.$$

Hence $\min_{u\in U\setminus\{0\}} \frac{\langle Au,u\rangle_{\mathcal{H}}}{\|u\|^2_{\mathcal{H}}} \leq \nu_k$. As this is true for every subspace $U \neq U_k$ with $\dim U = k$ and $\min_{u\in U_k\setminus\{0\}} \frac{\langle Au,u\rangle_{\mathcal{H}}}{\|u\|^2_{\mathcal{H}}} = \nu_k$ we get

$$\max_{\substack{U\subseteq \operatorname{dom} A \\ \dim U = k}} \min_{u\in U\setminus\{0\}} \frac{\langle Au,u\rangle_{\mathcal{H}}}{\|u\|^2_{\mathcal{H}}} = \nu_k.$$

Note that all minima and maxima are attained. □

2.2 Ordinary and generalized boundary triples

In this section we will introduce the abstract concept of ordinary boundary triples. This concept goes back to [42] and [20] (see also [67] for a special cases of an ordinary boundary triples) and is used to describe extensions of a given symmetric operator. We will also define so called generalized boundary triples, cf. [25]. Another generalization of ordinary boundary triples (which contain generalized boundary triples) are quasi boundary triples, cf. [9].

We start with the definitions of ordinary and generalized boundary triples.

Definition 2.3. Let S be a closed symmetric linear relation in the Hilbert space $\mathcal{H}$. Let $\mathcal{G}$ be another Hilbert space and let $\Gamma_0, \Gamma_1 : S^* \to \mathcal{G}$ be linear mappings. The triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is called an *ordinary boundary triple* for S^* if

(i) $\Gamma := \binom{\Gamma_0}{\Gamma_1} : S^* \to \mathcal{G} \times \mathcal{G}$ is surjective and

(ii) the abstract Green's identity

$$\langle u', v\rangle_{\mathcal{H}} - \langle u, v'\rangle_{\mathcal{H}} = \langle \Gamma_1 \hat{u}, \Gamma_0 \hat{v}\rangle_{\mathcal{G}} - \langle \Gamma_0 \hat{u}, \Gamma_1 \hat{v}\rangle_{\mathcal{G}}$$

holds for all $\hat{u} = \{u, u'\}$ and $\hat{v} = \{v, v'\} \in S^*$.

Analogously we define an ordinary boundary triple for the case that S is a symmetric linear relation in the Krein space $\mathcal{K}$ (with $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ replaced by $[\![\cdot,\cdot]\!]_{\mathcal{K}}$ and S^* replaced by S^+).

Also in the next definition the Hilbert space can be replaced by a Krein space (see for example Definition 2.1 in [6]), but in the following we will just need it for Hilbert spaces.

Definition 2.4. Let S be a closed symmetric linear relation in the Hilbert space $\mathcal{H}$ and T be a linear relation in $\mathcal{H}$ with $\overline{T} = S^*$. Let $\mathcal{G}$ be another Hilbert space and let $\Gamma_0, \Gamma_1 : T \to \mathcal{G}$ be linear mappings. The triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is called a *generalized boundary triple* for S^* if

(i) Γ_0 is surjective,

(ii) $A := \ker \Gamma_0$ is selfadjoint and

(iii) the abstract Green's identity

$$\langle u', v\rangle_{\mathcal{H}} - \langle u, v'\rangle_{\mathcal{H}} = \langle \Gamma_1 \hat{u}, \Gamma_0 \hat{v}\rangle_{\mathcal{G}} - \langle \Gamma_0 \hat{u}, \Gamma_1 \hat{v}\rangle_{\mathcal{G}}$$

holds for all $\hat{u} = \{u, u'\}$ and $\hat{v} = \{v, v'\} \in T$.

Remark 2.5. In the following we will call the maps Γ_0 and Γ_1 *boundary maps* and the Hilbert space $\mathcal{G}$ *boundary space*. If T is an operator it is more convenient to define the boundary maps Γ_0 and Γ_1 just on $\operatorname{dom} T$ instead of on T. One can show that if $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is an ordinary boundary triple $A := \ker \Gamma_0$ is always selfadjoint, cf. for example Proposition 2.1 in [24]. Hence every ordinary boundary triple is also a generalized boundary triple. Note also that it was shown in [25, Lemma 6.1] that if $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a generalized boundary triple the range of $\Gamma := \binom{\Gamma_0}{\Gamma_1}$ is dense in $\mathcal{G} \times \mathcal{G}$ and its kernel coincides with S, i.e. $\ker \Gamma = \ker \Gamma_0 \cap \ker \Gamma_1 = S$.

Two important functions corresponding to a generalized boundary triple are the γ-field and the Weyl function. The following two lemmas collect some well known and important properties of these objects, cf. Lemma 6.2, Definition 6.2 and Equation (6.7) in [25].

Lemma 2.6. *Let S be a closed symmetric linear relation in the Hilbert $\mathcal{H}$ and let $(\mathcal{G}, \Gamma_0, \Gamma_1)$ be a generalized boundary triple for $\overline{T} = S^*$. Let $A := \ker \Gamma_0$, define for $\lambda \in \rho(A)$ the linear relation*

$$\hat{\mathcal{N}}_\lambda := \{\{u, \lambda u\} : u \in \ker(T - \lambda)\}$$

and consider the projection $\pi_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $\{u, u'\} \mapsto u$. Then the γ-field *defined by*

$$\gamma : \rho(A) \to \mathcal{L}(\mathcal{G}, \mathcal{H}), \qquad \lambda \mapsto \gamma(\lambda) := \pi_1 (\Gamma_0 \upharpoonright \hat{\mathcal{N}}_\lambda)^{-1},$$

is a holomorphic operator valued function which satisfies

$$\gamma(\lambda) - \gamma(\mu) = (\lambda - \mu)(A - \lambda)^{-1} \gamma(\mu)$$

for all $\lambda, \mu \in \rho(A)$. Moreover, the adjoint $\gamma(\lambda)^* \in \mathcal{L}(\mathcal{H}, \mathcal{G})$ of $\gamma(\lambda)$ for $\lambda \in \rho(A)$ satisfies

$$\gamma(\lambda)^* u = \Gamma_1 \{(A - \overline{\lambda})^{-1} u, u + \overline{\lambda}(A - \overline{\lambda})^{-1} u\}$$

for all $u \in \mathcal{H}$. If T is an operator the definition of $\gamma(\lambda)$ reads as $\gamma(\lambda) := (\Gamma_0 \upharpoonright \ker(T - \lambda))^{-1}$ and the identity for the adjoints can be simplified to $\gamma(\lambda)^* = \Gamma_1 (A - \overline{\lambda})^{-1}$.

Lemma 2.7. *Let S be a closed symmetric linear relation in the Hilbert $\mathcal{H}$ and let $(\mathcal{G},\Gamma_0,\Gamma_1)$ be a generalized boundary triple for $\overline{T}=S^*$. Let $A:=\ker\Gamma_0$ and define for $\lambda\in\rho(A)$ the linear relation*

$$\hat{\mathcal{N}_\lambda}:=\{\{u,\lambda u\}:u\in\ker(T-\lambda)\}.$$

Then the Weyl function *defined by*

$$M:\rho(A)\to\mathcal{L}(\mathcal{G}),\qquad \lambda\mapsto M(\lambda):=\Gamma_1(\Gamma_0\upharpoonright\hat{\mathcal{N}_\lambda})^{-1},$$

is a holomorphic operator valued function which satisfies

$$M(\lambda)-M(\mu)^*=(\lambda-\overline{\mu})\gamma(\mu)^*\gamma(\lambda)$$

for all $\lambda,\mu\in\rho(A)$. In particular $M(\lambda)=M(\overline{\lambda})^*$ for all $\lambda\in\rho(A)$. If T is an operator the definition of $M(\lambda)$ reads as $M(\lambda):=\Gamma_1\gamma(\lambda)$.

Analog results of Lemma 2.6 and Lemma 2.7 can be shown if the space $\mathcal{H}$ is a Krein space, cf. for example Section 2 in [24] or Section 2 in [6].

If $(\mathcal{G},\Gamma_0,\Gamma_1)$ is an ordinary boundary triple for S^* it is well-known that the mapping

$$\Theta\mapsto A_\Theta:=\left\{\{u,u'\}\in S^*:\Gamma\{u,u'\}\in\Theta\right\}$$

establishes a bijection between all selfadjoint linear relations Θ in $\mathcal{G}$ and all selfadjoint extensions of S. In the case that $(\mathcal{G},\Gamma_0,\Gamma_1)$ is just a generalized boundary triple this is no longer true. However, if we assume some additional assumptions we can still guarantee selfadjointness of A_Θ. The following theorem specifies a possible choice of these assumptions. The proof can be deduced for example easily from Theorem 2.8 in [9]. Nevertheless we will prove this theorem here because it will be essential for our further approach.

Theorem 2.8. *Let S be a closed symmetric linear relation in the Hilbert $\mathcal{H}$ and let $(\mathcal{G},\Gamma_0,\Gamma_1)$ be a generalized boundary triple for $\overline{T}=S^*$. Let $A:=\ker\Gamma_0$ and let Θ be a closed linear relation in $\mathcal{G}$. Define the linear relation*

$$A_\Theta:=\left\{\{u,u'\}\in T:\Gamma\{u,u'\}\in\Theta\right\}.$$

If $\lambda\in\rho(A)$ is chosen such that $[\Theta-M(\lambda)]^{-1}$ is an operator and $\operatorname{ran}\gamma(\overline{\lambda})^*$ *is contained in* $\operatorname{ran}[\Theta-M(\lambda)]$ *then $\lambda\in\rho(A_\Theta)$ and the identity*

$$(A_\Theta-\lambda)^{-1}=(A-\lambda)^{-1}+\gamma(\lambda)[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^* \tag{2.1}$$

holds. If we assume additionally that Θ is symmetric and $\lambda\in\mathbb{R}$ then A_Θ is selfadjoint in $\mathcal{H}$.

Remark 2.9. As already mentioned above in the case of an ordinary boundary triple stronger statements hold. But of course Theorem 2.1 is also true for an ordinary boundary triple, even in the case that S is a linear relation in a Krein space, cf. Theorem 2.1 in [24]. We will use Theorem 2.8 mainly in the case that λ is chosen such that $0\in\rho(\Theta-M(\lambda))$. Note that this implies that $[\Theta-M(\lambda)]^{-1}$ is an operator and $\operatorname{ran}[\Theta-M(\lambda)]=\mathcal{G}$.

Proof. At first we show that $(A_\Theta-\lambda)^{-1}$ is an operator. Let $v\in\ker(A_\Theta-\lambda)$, i.e.

$$\{v,0\}\in A_\Theta-\lambda=\big\{\{u,u'-\lambda u\}:\{u,u'\}\in A_\Theta\big\}.$$

Hence $\{v,\lambda v\}\in A_\Theta$, i.e. $\Gamma\{v,\lambda v\}\in\Theta$. Moreover $v\in\ker(T-\lambda)$. This implies $\{v,\lambda v\}\in\hat{\mathcal{N}}_\lambda$ and hence $\Gamma_1\{v,\lambda v\}=\Gamma_1(\Gamma_0\upharpoonright\hat{\mathcal{N}}_\lambda)^{-1}(\Gamma_0\upharpoonright\hat{\mathcal{N}}_\lambda)\{v,\lambda v\}=M(\lambda)\Gamma_0\{v,\lambda v\}$. Therefore

$$\begin{bmatrix}\Gamma_0\{v,\lambda v\}\\ 0\end{bmatrix}=\begin{bmatrix}\Gamma_0\{v,\lambda v\}\\ \Gamma_1\{v,\lambda v\}-M(\lambda)\Gamma_0\{v,\lambda v\}\end{bmatrix}\in\Theta-M(\lambda).$$

As $(\Theta-M(\lambda))^{-1}$ is an operator we conclude $\Gamma_0\{v,\lambda v\}=0$ and therefore $\{v,\lambda v\}\in A$. As $\lambda\in\rho(A)$ this implies $v=0$ and hence $\ker(A_\Theta-\lambda)=\{0\}$, i.e. $(A_\Theta-\lambda)^{-1}$ is an operator.

Next we show the identity (2.1). For this let $u\in\mathcal{H}$ be arbitrary. Due to $\lambda\in\rho(A)$ we have

$$\begin{aligned}\begin{bmatrix}u\\ (A-\lambda)^{-1}u\end{bmatrix}\in(A-\lambda)^{-1}\ &\Longrightarrow\ \begin{bmatrix}(A-\lambda)^{-1}u\\ u\end{bmatrix}\in(A-\lambda)\\ &\Longrightarrow\ \begin{bmatrix}(A-\lambda)^{-1}u\\ u+\lambda(A-\lambda)^{-1}u\end{bmatrix}\in(A-\lambda)+\lambda\subseteq A=\ker\Gamma_0.\end{aligned}$$

Moreover Lemma 2.6 implies

$$\gamma(\overline{\lambda})^*u=\Gamma_1\begin{bmatrix}(A-\lambda)^{-1}u\\ u+\lambda(A-\lambda)^{-1}u\end{bmatrix}.$$

Hence

$$\Gamma\begin{bmatrix}(A-\lambda)^{-1}u\\ u+\lambda(A-\lambda)^{-1}u\end{bmatrix}=\begin{bmatrix}0\\ \gamma(\overline{\lambda})^*u\end{bmatrix}. \tag{2.2}$$

As $\operatorname{ran}\gamma(\overline{\lambda})^*$ is contained in $\operatorname{ran}[\Theta-M(\lambda)]=\operatorname{dom}[\Theta-M(\lambda)]^{-1}$ and $[\Theta-M(\lambda)]^{-1}$ is an operator $[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u$ is well defined. Moreover $\gamma(\lambda):=\pi_1(\Gamma_0\upharpoonright\hat{\mathcal{N}}_\lambda)^{-1}$ implies

$$\pi_1(\Gamma_0\upharpoonright\hat{\mathcal{N}}_\lambda)^{-1}\Big([\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\Big)=\gamma(\lambda)[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\in\ker(T-\lambda)$$

and hence

$$(\Gamma_0\upharpoonright\hat{\mathcal{N}}_\lambda)^{-1}\Big([\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\Big)=\begin{bmatrix}\gamma(\lambda)[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\\ \lambda\gamma(\lambda)[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\end{bmatrix}\in\hat{\mathcal{N}}_\lambda.$$

Therefore

$$\Gamma_0\begin{bmatrix}\gamma(\lambda)[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\\ \lambda\gamma(\lambda)[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u\end{bmatrix}=[\Theta-M(\lambda)]^{-1}\gamma(\overline{\lambda})^*u$$

and

$$\Gamma_1 \begin{bmatrix} \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ \lambda\gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \end{bmatrix} = \Gamma_1(\Gamma_0 \upharpoonright \hat{\mathcal{N}}_\lambda)^{-1}\Big([\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u\Big)$$
$$= M(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u.$$

Hence

$$\Gamma \begin{bmatrix} \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ \lambda\gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \end{bmatrix} = \begin{bmatrix} [\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ M(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \end{bmatrix} \in M(\lambda). \tag{2.3}$$

Furthermore we have

$$\begin{bmatrix} [\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ \gamma(\overline{\lambda})^* u \end{bmatrix} \in \Theta - M(\lambda). \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) we observe

$$\Gamma \begin{bmatrix} (A-\lambda)^{-1}u + \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ u + \lambda(A-\lambda)^{-1}u + \lambda\gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \end{bmatrix}$$
$$= \begin{bmatrix} [\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ \gamma(\overline{\lambda})^* u + M(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \end{bmatrix} \in \Theta - M(\lambda) + M(\lambda) \subseteq \Theta.$$

Therefore

$$\begin{bmatrix} (A-\lambda)^{-1}u + \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ u + \lambda(A-\lambda)^{-1}u + \lambda\gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \end{bmatrix} \in A_\Theta$$

and hence

$$\begin{bmatrix} (A-\lambda)^{-1}u + \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u \\ u \end{bmatrix} \in (A_\Theta - \lambda).$$

Keeping in mind that $(A_\Theta - \lambda)^{-1}$ is an operator this implies

$$(A_\Theta - \lambda)^{-1}u = (A-\lambda)^{-1}u + \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^* u.$$

As $u \in \mathcal{H}$ was arbitrary this shows Krein's resolvent formula (2.1).

Next we show $\lambda \in \rho(A_\Theta)$. For this let $(\{v_n, v_n'\})_{n\in\mathbb{N}} \subseteq \Theta - M(\lambda)$ be a sequence which converges to some $\{v, v'\} \in \mathcal{G} \times \mathcal{G}$. For every $n \in \mathbb{N}$ there exists $\{u_n, u_n'\} \in \Theta$ such that $\{v_n, v_n'\} = \{u_n, u_n' - M(\lambda)u_n\}$. In particular $u_n = v_n \to v$ if $n \to \infty$. Hence

$$u_n' = v_n' + M(\lambda)u_n \xrightarrow{n\to\infty} v' + M(\lambda)v$$

because $M(\lambda) \in \mathcal{L}(\mathcal{G})$, cf. Lemma 2.7. As Θ is closed we get $\{v, v' + M(\lambda)v\} \in \Theta$. Hence $\{v, v'\} = \{v, v' + M(\lambda)v - M(\lambda)v\} \in \Theta - M(\lambda)$. Therefore $\Theta - M(\lambda)$ is closed and hence $[\Theta -$

$M(\lambda)]^{-1}$ is a closed operator. As $\gamma(\overline{\lambda})^* \in \mathcal{L}(\mathcal{H},\mathcal{G})$ and $\operatorname{ran}\gamma(\overline{\lambda})^*$ is included in $\operatorname{ran}[\Theta - M(\lambda)] = \operatorname{dom}[\Theta - M(\lambda)]^{-1}$ also $[\Theta - M(\lambda)]^{-1}\gamma(\overline{\lambda})^*$ is closed. Moreover it is defined on the whole space and hence bounded. Also the operator $(A-\lambda)^{-1}$ is bounded because $\lambda \in \rho(A)$. Hence Krein's resolvent formula (2.1) implies that $(A_\Theta - \lambda)^{-1}$ is bounded and therefore $(A_\Theta - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$. Hence $\lambda \in \rho(A_\Theta)$.

Next we show that the symmetry of Θ implies the symmetry of A_Θ. For this let $\hat{u} = \{u,u'\}, \hat{v} = \{v,v'\} \in A_\Theta$. Set $\hat{f} = \{f,f'\} := \Gamma\hat{u} = \{\Gamma_0\hat{u},\Gamma_1\hat{u}\}$ and $\hat{g} = \{g,g'\} := \Gamma\hat{v} = \{\Gamma_0\hat{v},\Gamma_1\hat{v}\}$. Note that due to the definition of A_Θ we have $\hat{f},\hat{g} \in \Theta$. As Θ is symmetric we get together with the abstract Green's identity

$$0 = \langle f',g\rangle_{\mathcal{H}} - \langle f,g'\rangle_{\mathcal{H}} = \langle \Gamma_1\hat{u},\Gamma_0\hat{v}\rangle_{\mathcal{G}} - \langle \Gamma_0\hat{u},\Gamma_1\hat{v}\rangle_{\mathcal{G}} = \langle u',v\rangle_{\mathcal{H}} - \langle u,v'\rangle_{\mathcal{H}}.$$

As this is true for all $\hat{v} = \{v,v'\} \in A_\Theta$ we get $\hat{u} = \{u,u'\} \in A_\Theta^*$. Hence $A_\Theta \subseteq A_\Theta^*$.

To show selfadjointness of A_Θ we can proceed for example analogously as in the proof of Theorem 4.2 (iii) in [27, Chapter III]. Let $\{u,u'\} \in A_\Theta^*$, i.e. $\{u,u'-\lambda u\} \in (A_\Theta^* - \lambda)$. As $\operatorname{ran}(A_\Theta - \lambda) = \operatorname{dom}(A_\Theta - \lambda)^{-1} = \mathcal{H}$ there exists $v \in \mathcal{H}$ such that $\{v,u'-\lambda u\} \in (A_\Theta - \lambda) \subseteq (A_\Theta^* - \lambda)$. Hence $\{u-v,0\} \in (A_\Theta^* - \lambda)$, i.e.

$$u - v \in \ker(A_\Theta^* - \lambda) = \big(\operatorname{ran}(A_\Theta - \lambda)\big)^\perp = \big(\operatorname{dom}(A_\Theta - \lambda)^{-1}\big)^\perp = \mathcal{H}^\perp = \{0\}.$$

Hence $u = v$ and $\{v,u'-\lambda v\} = \{v,u'-\lambda u\} \in (A_\Theta - \lambda)$ or $\{u,u'\} = \{v,u'\} \in A_\Theta$. This shows $A_\Theta^* \subseteq A_\Theta$ and with the symmetry of A_Θ we know that A_Θ is selfadjoint. □

The following lemma is a helpful tool to decide if a triple is a boundary triple. We omit the proof and refer to Theorem 2.3 and Remark 2.9 in [10].

Lemma 2.10. *Let $\mathcal{K}$ be a Krein space with inner product $[\![\cdot,\cdot]\!]_{\mathcal{K}}$. Let T be a linear relation in $\mathcal{K}$ and let $\Gamma = \binom{\Gamma_0}{\Gamma_1} : T \to \mathcal{G} \times \mathcal{G}$ be a linear mapping, which satisfies the following conditions:*

(i) Γ is surjective;

(ii) there exist $\lambda \in \mathbb{R}$ and a symmetric relation Θ in $\mathcal{G}$ such that $\operatorname{ran}(A_\Theta - \lambda) = \mathcal{H}$ holds for the linear relation $A_\Theta := \{\{u,u'\} \in T : \Gamma\{u,u'\} \in \Theta\}$;

(iii) the abstract Green's identity

$$[\![f',g]\!]_{\mathcal{K}} - [\![f,g']\!]_{\mathcal{K}} = [\![\Gamma_1\hat{f},\Gamma_0\hat{g}]\!]_{\mathcal{G}} - [\![\Gamma_0\hat{f},\Gamma_1\hat{g}]\!]_{\mathcal{G}}$$

holds for all $\hat{f} = \{f,f'\}$ and $\hat{g} = \{g,g'\} \in T$.

Then $S := \ker\Gamma$ is a closed symmetric linear relation in $\mathcal{K}$ and $S^+ = T$. Moreover $(\mathcal{G},\Gamma_0,\Gamma_1)$ is a boundary triple for S^+.

The following lemma is of the same flavor as the previous one and is a direct consequence of Theorem 2.3 in [9].

Lemma 2.11. *Let $\mathcal{H}$ be a Hilbert space space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Let T be a linear relation in $\mathcal{H}$ and let $\Gamma = \binom{\Gamma_0}{\Gamma_1} : T \to \mathcal{G}\times\mathcal{G}$ be a linear mapping, which satisfies the following conditions:*

(i) Γ_0 *is surjective and* $\operatorname{ran}\Gamma$ *is dense;*

(ii) $A := \ker\Gamma_0$ *is a selfadjoint linear relation in* $\mathcal{H}$*;*

(iii) the abstract Green's identity

$$\langle f',g\rangle_{\mathcal{H}} - \langle f,g'\rangle_{\mathcal{H}} = \langle\Gamma_1\hat{f},\Gamma_0\hat{g}\rangle_{\mathcal{G}} - \langle\Gamma_0\hat{f},\Gamma_1\hat{g}\rangle_{\mathcal{G}}$$

holds for all $\hat{f} = \{f,f'\}$ *and* $\hat{g} = \{g,g'\} \in T$.

Then $S := \ker\Gamma$ *is a closed symmetric linear relation in* $\mathcal{H}$ *and* $\overline{T} = S^*$. *Moreover* $(\mathcal{G},\Gamma_0,\Gamma_1)$ *is a generalized boundary triple for* S^*.

2.3 The Friedrichs extension

In this section we summarize some well-known facts about sesquilinear forms and the Friedrichs extension. For more details and proofs we refer to Chapter VI in [41].

Throughout this section $\mathcal{H}$ is a Hilbert space. For a symmetric sesquilinear form $\mathfrak{s}$ in $\mathcal{H}$ we define $\mathfrak{s}[u] := \mathfrak{s}[u,u]$ for $u \in \operatorname{dom}\mathfrak{s}$.

Definition 2.12. Let $\mathfrak{s}$ be a densely defined symmetric sesquilinear form in $\mathcal{H}$.

(i) $\mathfrak{s}$ is called *bounded from below by* $\gamma\in\mathbb{R}$ if $\mathfrak{s}[u,u] \geq \gamma\|u\|^2_{\mathcal{H}}$ holds for all $u \in \operatorname{dom}\mathfrak{s}$.

(ii) A sequence $(u_n)_n \subseteq \operatorname{dom}\mathfrak{s}$ is called $\mathfrak{s}$-*convergent* to $u\in\mathcal{H}$ if

$$\|u_n - u\|_{\mathcal{H}} \xrightarrow{n\to\infty} 0 \quad\text{and}\quad \mathfrak{s}[u_n-u_m,u_n-u_m] \xrightarrow{n,m\to\infty} 0.$$

In this case we write $u_n \xrightarrow{\mathfrak{s}} u$.

(iii) $\mathfrak{s}$ is called *closed* if $u_n \xrightarrow{\mathfrak{s}} u$ implies $u\in\operatorname{dom}\mathfrak{s}$ and $\mathfrak{s}[u_n-u,u_n-u] \xrightarrow{n\to\infty} 0$.

(iv) $\mathfrak{s}$ is called *closable* if there exists a closed symmetric sesquilinear form $\mathfrak{t}$ with $\operatorname{dom}\mathfrak{s}\subseteq\operatorname{dom}\mathfrak{t}$ and $\mathfrak{s}[u,v]=\mathfrak{t}[u,v]$ for all $u,v\in\operatorname{dom}\mathfrak{s}$.

(v) If $\mathfrak{s}$ is closable we define the *closure* $\overline{\mathfrak{s}}$ of $\mathfrak{s}$ by

$$\operatorname{dom}\overline{\mathfrak{s}} := \{u\in\mathcal{H} : \exists(u_n)_n\subseteq\operatorname{dom}\mathfrak{s} \text{ with } u_n\xrightarrow{\mathfrak{s}} u\},$$
$$\overline{\mathfrak{s}}[u,v] := \lim_{n\to\infty}\mathfrak{s}[u_n,v_n] \text{ for any sequences } (u_n)_n,(v_n)_n\subseteq\operatorname{dom}\mathfrak{s} \text{ with } u_n\xrightarrow{\mathfrak{s}} u, v_n\xrightarrow{\mathfrak{s}} v.$$

In this case $\overline{\mathfrak{s}}$ is the smallest (in the sense of intersections) closed extension of $\mathfrak{s}$.

(vi) Let $\mathfrak{s}$ be closed. A subspace $U \subseteq \operatorname{dom}\mathfrak{s}$ is called a *core* of $\mathfrak{s}$ if the closure of the restriction of $\mathfrak{s}$ to $U \times U$ equals $\mathfrak{s}$.

The following theorem is a special case of Theorem VI.2.1 in [41].

Theorem 2.13. *Let $\mathfrak{s}$ be a densely defined, closed symmetric sesquilinear form in $\mathcal{H}$ which is bounded from below by $\gamma \in \mathbb{R}$. Then there exists a unique selfadjoint operator $A \geq \gamma$ in $\mathcal{H}$ which satisfies the following items.*

(i) $\operatorname{dom}A \subseteq \operatorname{dom}\mathfrak{s}$ *and* $\langle Au, v\rangle_{\mathcal{H}} = \mathfrak{s}[u,v]$ *for all* $u \in \operatorname{dom}A$ *and* $v \in \operatorname{dom}\mathfrak{s}$.

(ii) $\operatorname{dom}A$ *is a core of* $\mathfrak{s}$.

(iii) Let $u \in \operatorname{dom}\mathfrak{s}$, $w \in \mathcal{H}$ *and* $\mathfrak{s}[u,v] = \langle w, v\rangle_{\mathcal{H}}$ *for all* v *in a core of* $\mathfrak{s}$. *Then* $u \in \operatorname{dom}A$ *and* $Au = w$.

The operator A is called the operator associated with $\mathfrak{s}$.

A proof for the following lemma can be found in [66, Satz 17.11].

Lemma 2.14. *Let S be a densely defined, closed symmetric operator in $\mathcal{H}$ which is bounded from below by $\gamma \in \mathbb{R}$. Then the symmetric sesquilinear form $\mathfrak{s}$ defined by*

$$\mathfrak{s}[u,v] := \langle Su, v\rangle_{\mathcal{H}}, \quad \operatorname{dom}\mathfrak{s} := \operatorname{dom}S, \tag{2.5}$$

is bounded from below by γ and closable. The operator associated with $\bar{\mathfrak{s}}$ will be denoted by $F(S)$ and is called the Friedrichs extension of S. Its domain satisfies $\operatorname{dom}F(S) = \operatorname{dom}\bar{\mathfrak{s}} \cap \operatorname{dom}S^*$.

An immediate consequence is the following corollary, cf. [41, Theorem VI.2.11].

Corollary 2.15. *Let S be a densely defined closed symmetric operator in $\mathcal{H}$ which is bounded from below by $\gamma \in \mathbb{R}$ and let $\mathfrak{s}$ be the corresponding sesquilinear form defined as in* (2.5). *Then the Friedrichs extension $F(S)$ of S is the only selfadjoint extension of S whose domain is contained in* $\operatorname{dom}\bar{\mathfrak{s}}$.

Proof. Let A be a selfadjoint extension of S with $\operatorname{dom}A \subseteq \operatorname{dom}\bar{\mathfrak{s}}$. In particular

$$\operatorname{dom}A \subseteq \operatorname{dom}\bar{\mathfrak{s}} \cap \operatorname{dom}S^* = \operatorname{dom}F(S),$$

cf. Lemma 2.14. As A and $F(S)$ are both restrictions of S^* it follows $A \subseteq F(S)$. Hence $A = F(S)$ because both operators are selfadjoint. □

In the last lemma of this section we investigate how the Friedrichs extension is influenced by bounded perturbations.

Lemma 2.16. *Let S be a densely defined closed symmetric operator in $\mathcal{H}$, bounded from below by $\gamma \in \mathbb{R}$ and let $B = B^* \in \mathcal{L}(\mathcal{H})$. Denote by $F(S)$ and $F(S+B)$ the Friedrichs extensions of S and $S+B$, respectively. Then $F(S)+B = F(S+B)$.*

Proof. Note that $S+B$ is bounded from below by $\gamma - \|B\|$, hence the Friedrichs extension $F(S+B)$ exists and is bounded from below by $\gamma - \|B\|$.

Denote by $\mathfrak{s}_S$ and $\mathfrak{s}_{S+B}$ the closable sesquilinear forms defined by S and $S+B$, respectively. Note that

$$\operatorname{dom}\mathfrak{s}_S = \operatorname{dom} S = \operatorname{dom}(S+B) = \operatorname{dom}\mathfrak{s}_{S+B}$$

because $B \in \mathcal{L}(\mathcal{H})$. Let now $(u_n)_n \subset \operatorname{dom}\mathfrak{s}_S$ with $u_n \xrightarrow{\mathfrak{s}_S} u \in \mathcal{H}$, i.e.

$$\|u_n - u\|_{\mathcal{H}} \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \mathfrak{s}_S[u_n - u_m] \xrightarrow{n,m\to\infty} 0.$$

This implies

$$\begin{aligned} \left|\mathfrak{s}_{S+B}[u_n - u_m]\right| &= \left|\langle (S+B)(u_n - u_m), u_n - u_m\rangle_{\mathcal{H}}\right| \\ &\le \left|\langle S(u_n - u_m), u_n - u_m\rangle_{\mathcal{H}}\right| + \left|\langle B(u_n - u_m), u_n - u_m\rangle_{\mathcal{H}}\right| \\ &\le \left|\mathfrak{s}_S[u_n - u_m]\right| + \|B\| \cdot \|u_n - u_m\|^2_{\mathcal{H}} \xrightarrow{n,m\to\infty} 0 \end{aligned}$$

and therefore $u_n \xrightarrow{\mathfrak{s}_{S+B}} u \in \mathcal{H}$. Analogously we observe that $u_n \xrightarrow{\mathfrak{s}_{S+B}} u \in \mathcal{H}$ implies $u_n \xrightarrow{\mathfrak{s}_S} u \in \mathcal{H}$. Hence $\operatorname{dom}\overline{\mathfrak{s}_S} = \operatorname{dom}\overline{\mathfrak{s}_{S+B}}$ and therefore

$$\operatorname{dom}\big(F(S)+B\big) = \operatorname{dom} F(S) \subseteq \operatorname{dom}\overline{\mathfrak{s}_S} = \operatorname{dom}\overline{\mathfrak{s}_{S+B}}.$$

Hence $F(S)+B$ is a selfadjoint operator whose domain is contained in $\operatorname{dom}\overline{\mathfrak{s}_{S+B}}$. Moreover $F(S)+B$ is an extension of $S+B$. According to Corollary 2.15 this means that $F(S)+B$ is the Friedrichs extension of $S+B$. □

2.4 Sobolev spaces

In this section we provide the definitions of Sobolev spaces on $\mathbb{R}^d$ and on manifolds in $\mathbb{R}^d$. Furthermore we define the trace operators and show some properties of Sobolev functions and their traces.

As usual we denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space and by $\mathscr{S}'(\mathbb{R}^d)$ its dual space, the space of tempered distributions. By $\mathscr{F}$ we denote the Fourier transform. For more details on the Schwartz space and the Fourier transform see for example Chapter V.3 in [56] and Chapter IX in [55].

Definition 2.17. The Sobolev space of order $s \in \mathbb{R}$ is defined by

$$H^s(\mathbb{R}^d) := \{u \in \mathscr{S}'(\mathbb{R}^d) : (1+|\cdot|^2)^{\frac{s}{2}}\mathscr{F}u \in L^2(\mathbb{R}^d)\}.$$

Equipped with the scalar product $\langle\cdot,\cdot\rangle_{H^s(\mathbb{R}^d)}$ defined by

$$\langle u,v\rangle_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \mathscr{F}u\overline{\mathscr{F}v}(1+|\cdot|^2)^s\,dx$$

$H^s(\mathbb{R}^d)$ becomes a Hilbert space. Note that $H^{-s}(\mathbb{R}^d)$ is the dual space of $H^s(\mathbb{R}^d)$ with the dual pairing $\langle\cdot,\cdot\rangle_{H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d)}$ defined by

$$\langle u,v\rangle_{H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \mathscr{F}u\overline{\mathscr{F}v}\,dx.$$

We will also make use of the dual pairings $(\cdot,\cdot)_{H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d)}$ defined by

$$(u,v)_{H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \mathscr{F}u\mathscr{F}v\,dx$$

which is bilinear instead of sesquilinear.

It is well-known that $(-\Delta-\lambda)^{-1}$ provides for $\lambda<0$ a bounded operator in $L^2(\mathbb{R}^d)$. The following lemma contains this observation as a special case.

Lemma 2.18. *Let $s\in\mathbb{R}$, $s\le r\le s+2$ and $\lambda<0$. Then for all $u\in H^s(\mathbb{R}^d)$ holds*

$$\|(-\Delta-\lambda)^{-1}u\|_{H^r(\mathbb{R}^d)} \le \frac{\min\{|\lambda|,1\}^{\frac{s-r}{2}}}{|\lambda|^{1+(s-r)/2}}\,\|u\|_{H^s(\mathbb{R}^d)}.$$

Here the derivatives of Δ have to be understood in a distributional sense.

Proof. Due to

$$\frac{\big||x|^2-\lambda\big|}{|x|^2+1} = \frac{|x|^2+|\lambda|}{|x|^2+1} \ge \left\{\begin{array}{ll} \frac{|\lambda|\cdot|x|^2+|\lambda|}{|x|^2+1} = |\lambda|, & \text{if } -1\le\lambda<0 \\ \frac{|x|^2+1}{|x|^2+1} = 1, & \text{if } \lambda\le -1. \end{array}\right\} = \min\{|\lambda|,1\}$$

we have

$$\begin{aligned}\frac{(|x|^2+1)^r}{(|x|^2-\lambda)^2} &= (|x|^2+1)^s\left(\frac{|x|^2+1}{|x|^2-\lambda}\right)^{r-s}\left(\frac{1}{|x|^2-\lambda}\right)^{2+s-r}\\ &\le (|x|^2+1)^s\left(\frac{1}{\min\{|\lambda|,1\}}\right)^{r-s}\left(\frac{1}{|\lambda|}\right)^{2+s-r}\end{aligned}$$

and hence

$$\|(-\Delta-\lambda)^{-1}u\|^2_{H^r(\mathbb{R}^d)} = \left\|(1+|x|^2)^{\frac{r}{2}}\mathcal{F}[(-\Delta-\lambda)^{-1}u]\right\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{(|x|^2+1)^r}{(|x|^2-\lambda)^2}|\mathcal{F}u|^2\,dx$$

$$\leq \frac{\min\{|\lambda|,1\}^{s-r}}{|\lambda|^{2+s-r}} \int_{\mathbb{R}^d} (|x|^2+1)^s|\mathcal{F}u|^2\,dx = \frac{\min\{|\lambda|,1\}^{s-r}}{|\lambda|^{2+s-r}}\,\|u\|^2_{H^s(\mathbb{R}^d)}$$

where we have used that differentiation becomes multiplication (up to a complex constant of absolute value 1) under Fourier transformation, cf. for example Satz VIII.5.12 in [69]. The result follows by taking the square roots. □

Following Definition 4.4 in [70] we define next Sobolev spaces on manifolds in $\mathbb{R}^d$.

Definition 2.19. Let $k \in \mathbb{N}$ and $\Sigma \subset \mathbb{R}^d$ be a compact C^k-manifold of codimension κ, i.e. there exists an index $m \in \mathbb{N}$, bounded open sets $\Omega_i \subseteq \mathbb{R}^{d-\kappa}$, relatively open sets $\Sigma_i \subseteq \Sigma$ and bijective functions $\sigma_i : \Omega_i \to \Sigma_i$ for $i \in \{1,\dots,m\}$, such that $\bigcup_{i=1}^m \Sigma_i = \Sigma$ and

$$\sigma_i^{-1}\circ\sigma_j \in C^k\left(\sigma_j^{-1}(\Sigma_i\cap\Sigma_j), \sigma_i(\Sigma_i\cap\Sigma_j)\right)$$

for all $i,j \in \{1,\dots,m\}$. Moreover let φ_i, $i \in \{1,\dots,m\}$, be a partition of unity subject to the cover Σ_i, $i \in \{1,\dots,m\}$. For $0 \leq s \leq k$ we define the Sobolev space $H^s(\Sigma)$ via

$$H^s(\Sigma) := \{f : \Sigma \to \mathbb{C} : (f\cdot\varphi_j)\circ\sigma_i \in H^s(\mathbb{R}^{d-\kappa})\}.$$

Here the function $(f\cdot\varphi_j)\circ\sigma_i$, which has compact support in Ω_i, is understood as its extension by zero to the whole $\mathbb{R}^{d-\kappa}$. A possible norm on $H^s(\Sigma)$ is given by

$$\|f\|^2_{H^s(\Sigma)} = \sum_{j=1}^m \|(f\cdot\varphi_j)\circ\sigma_j\|^2_{H^s(\mathbb{R}^{d-\kappa})}.$$

In particular we have $\|f\|_{H^r(\Sigma)} \leq \|f\|_{H^s(\Sigma)}$ for all $u \in H^s(\Sigma)$ and $r \leq s$. Note that these norms depend on the choice of the parametrizations σ_i and the partition of unity. However, each possible choice leads to an equivalent norm. For our further proceeding we mainly need the norm of $L^2(\Sigma) := H^0(\Sigma)$. Instead of the norms from above we will use the norm given by

$$\|f\|^2_{L^2(\Sigma)} = \int_\Sigma |f(x)|^2\,d\sigma(x),$$

where σ is the "surface" measure given by

$$\int_\Sigma f(x)\,d\sigma(x) := \sum_{j=1}^m \int_{\Omega_i} (f\cdot\varphi_j)\circ\sigma_j(s)\,\sqrt{\det\left([D\sigma_i(s)]^\top[D\sigma_i(s)]\right)}\,ds.$$

This definition has the advantage that it is independent from the choice of the parametrizations σ_i and the partition of unity. In the following we will assume without loss of generality that the maps σ_j are chosen such that $\|f\|_{L^2(\Sigma)} \leq \|f\|_{H^s(\Sigma)}$ holds for all s with $0 < s \leq k$ and all $u \in H^s(\Sigma)$.

If Σ is a manifold without boundary we can define $H^{-s}(\Sigma)$ as the dual space of $H^s(\Sigma)$. With the usual identification $L^2(\Sigma)$ becomes a subspace of $H^{-s}(\Sigma)$ and

$$\langle u, \varphi \rangle_{H^{-s}(\Sigma), H^s(\Sigma)} = \langle u, \varphi \rangle_{L^2(\Sigma)}$$

holds for all $u \in L^2(\Sigma)$ and $\varphi \in H^s(\Sigma)$. Analogously as for the Sobolev spaces $H^s(\mathbb{R}^d)$ we will also make use of the corresponding bilinear pairings $(\cdot,\cdot)_{H^{-s}(\Sigma), H^s(\Sigma)}$.

In the next lemma we define the trace operators. For a proof see for example Theorem 24.3 in [16] or Theorem 1 in [40, Chapter VII].

Lemma 2.20. *Let $\Sigma \subseteq \mathbb{R}^d$ be a compact C^k-manifold of codimension κ as in Definition 2.19 and $\frac{\kappa}{2} < s \leq k$. Then we can extend the map*

$$C_0^\infty(\mathbb{R}^d) \ni \varphi \mapsto \varphi|_\Sigma$$

uniquely to a continuous mapping $\operatorname{tr}_\Sigma^s : H^s(\mathbb{R}^d) \to H^{s-\frac{\kappa}{2}}(\Sigma)$, which we will call the trace operator and $\operatorname{tr}_\Sigma^s u$ the trace of u. The operator $\operatorname{tr}_\Sigma^s$ is surjective.

With the trace operator we can define now the distribution $h\delta_\Sigma$ for $h \in L^2(\Sigma)$, i.e. a δ-interaction on Σ with strength h. This will be one of the central objects of this thesis.

Lemma 2.21. *Let $\Sigma \subset \mathbb{R}^d$ be a compact C^k-manifold of codimension κ as in Definition 2.19 and let $s := \frac{\kappa}{2} + \varepsilon \leq k$ for some $\varepsilon > 0$. Define for $h \in L^2(\Sigma)$ the distribution $h\delta_\Sigma$ via*

$$\big(h\delta_\Sigma\big)(\varphi) := \langle h, \operatorname{tr}_\Sigma^s \varphi \rangle_{L^2(\Sigma)}, \qquad \varphi \in H^s(\mathbb{R}^d).$$

Then $h\delta_\Sigma \in H^{-s}(\mathbb{R}^d)$ and $\|h\delta_\Sigma\|_{H^{-s}(\mathbb{R}^d)} \leq \|\operatorname{tr}_\Sigma^s\| \cdot \|h\|_{L^2(\Sigma)}$. Moreover $h\delta_\Sigma \in H^{-\kappa/2}(\mathbb{R}^d)$ if and only if $h = 0$. In particular $h\delta_\Sigma = 0$ if and only if $h = 0$.

Proof. With Lemma 2.20 we obtain

$$\begin{aligned} \big|\big(h\delta_\Sigma\big)(\varphi)\big| = |\langle h, \operatorname{tr}_\Sigma^s \varphi \rangle_{L^2(\Sigma)}| &\leq \|h\|_{L^2(\Sigma)} \cdot \|\operatorname{tr}_\Sigma^s \varphi\|_{L^2(\Sigma)} \\ &\leq \|h\|_{L^2(\Sigma)} \cdot \|\operatorname{tr}_\Sigma^s \varphi\|_{H^\varepsilon(\Sigma)} \leq \|h\|_{L^2(\Sigma)} \cdot \|\operatorname{tr}_\Sigma^s\| \cdot \|\varphi\|_{H^s(\mathbb{R}^d)} \end{aligned}$$

and hence $h\delta_\Sigma \in H^{-s}(\mathbb{R}^d)$ with $\|h\delta_\Sigma\|_{H^{-s}(\mathbb{R}^d)} \leq \|\operatorname{tr}_\Sigma^s\| \cdot \|h\|_{L^2(\Sigma)}$. Furthermore we get $h\delta_\Sigma = 0$ if and only if $h \perp \operatorname{ran} \operatorname{tr}_\Sigma^s = H^\varepsilon(\Sigma)$, i.e. $h = 0$.

Next let $h \in L^2(\Sigma)$ with $h\delta_\Sigma \in H^{-\kappa/2}(\mathbb{R}^d)$. It is known, that for $1 < p < \infty$ and $\alpha > 0$ the $W^{\alpha,p}(\mathbb{R}^d)$-capacity of Σ defined by

$$\operatorname{Cap}(\Sigma, W^{\alpha,p}(\mathbb{R}^d)) := \inf\{\|u\|^2_{W^{\alpha,p}(\mathbb{R}^d)} : u \in \mathscr{S}(\mathbb{R}^d), u = 1 \text{ on } A \supset \Sigma, A \text{ open}\}$$

is 0, if and only if the codimension κ of Σ satisfies $\alpha p \leq \kappa$, cf. Corollary 3.3.4. and Corollary 5.1.15 in [2]. In our case this condition is satisfied for $p = 2$ and $\alpha = \frac{\kappa}{2}$ and therefore

$$0 = \operatorname{Cap}(\Sigma, H^{\kappa/2}(\mathbb{R}^d)) = \inf\{\|u\|^2_{H^{\kappa/2}(\mathbb{R}^d)} : u \in \mathscr{S}(\mathbb{R}^d), u = 1 \text{ on } A \supset \Sigma, A \text{ open}\}.$$

Hence there exists a sequence $(\varphi_n)_n \subset \mathscr{S}(\mathbb{R}^d)$ with $\|\varphi_n\|_{H^{\kappa/2}(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0$ and $\varphi_n = 1$ on Σ. Note that for $\psi \in C_0^\infty(\mathbb{R}^d)$ also $\|\psi\varphi_n\|_{H^{\kappa/2}(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0$. Hence, as $h\delta_\Sigma \in H^{-\kappa/2}(\mathbb{R}^d)$, we get

$$\big(h\delta_\Sigma\big)(\psi) = \langle h, \operatorname{tr}^s_\Sigma \psi\rangle_{L^2(\Sigma)} = \langle h, \operatorname{tr}^s_\Sigma \psi\varphi_n\rangle_{L^2(\Sigma)} = \big(h\delta_\Sigma\big)(\psi\varphi_n) \xrightarrow{n\to\infty} 0.$$

As $\psi \in C_0^\infty(\mathbb{R}^d)$ was arbitrary we conclude $h\delta_\Sigma = 0$ and therefore $h = 0$. □

Remark 2.22. A definition for the capacity can be found in [2, Def.2.7.1.], see also e.g. [2, Ch.2.2], [27, Ch.VIII.6] and [51, Ch.10.4.1] for definitions of slightly different concepts of capacity. The last part of the proof above mimics the proof of [27, Thm.VIII.6.3] and one can show without any additionally effort that Σ is (m, p)-polar for $m = \frac{\kappa}{2}$ and $p = 2$, i.e. $\{u \in H^{-\kappa/2}(\mathbb{R}^d) : \operatorname{supp} u \subseteq \Sigma\} = \{0\}$.

For the next lemma recall that a compact operator $K : \mathcal{H} \to \mathcal{G}$ belongs to the *Schatten-von Neumann class* of order $p > 0$ if the singular values $s_j(K)$ of K (counted with multiplicities) satisfy

$$\sum_{j=1}^{\infty} |s_j(K)|^p < \infty.$$

In this case we write $K \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$ or, if $\mathcal{H} = \mathcal{G}$, $K \in \mathfrak{S}_p(\mathcal{H})$.

Lemma 2.23. *Assume that $\Sigma \subseteq \mathbb{R}^d$ is a compact C^∞-manifold with codimension κ and $B \in \mathcal{L}(L^2(\mathbb{R}^d), H^r(\Sigma))$ with $\operatorname{ran} B \subseteq H^s(\Sigma)$, $s > r \geq 0$. Then $B \in \mathfrak{S}_p(L^2(\mathbb{R}^d), H^r(\Sigma))$ for $p > \frac{d-\kappa}{s-r}$ and the singular values of B satisfy $s_j(B) = O(j^{-\frac{s-r}{d-\kappa}})$ for $j \to \infty$.*

For the special case that Σ is the boundary of a compact C^∞-domain this lemma coincides with Lemma 3.4 in [11] and also the corresponding proof can be adopted.

Proof. Consider the operator

$$\Lambda := (I - \Delta^\Sigma_{\mathrm{LB}})^{\frac{s-r}{2}},$$

where $\Delta^\Sigma_{\mathrm{LB}}$ denotes the Laplace-Beltrami operator on Σ. The operator Λ provides an isomorphism between $H^s(\Sigma)$ and $H^r(\Sigma)$, cf. Corollary 5.3.2 in [4]. Hence $\Lambda^{-1} : H^r(\Sigma) \to H^s(\Sigma)$ is continuous, too. Furthermore $B : L^2(\mathbb{R}^d) \to H^r(\Sigma)$ is continuous and hence closed. As $\operatorname{ran} B \subseteq H^s(\Sigma)$ the operator

$$\tilde{B} : L^2(\mathbb{R}^d) \to H^s(\Sigma), \qquad u \mapsto Bu,$$

is well-defined. Next let $(u_n)_n \subseteq L^2(\mathbb{R}^d)$ with $u_n \xrightarrow{n\to\infty} u$ in $L^2(\mathbb{R}^d)$ and $\tilde{B}u_n \xrightarrow{n\to\infty} v$ in $H^s(\Sigma)$ for a certain $u \in L^2(\mathbb{R}^d)$ and a certain $v \in H^s(\mathbb{R}^d)$. Hence

$$\|Bu_n - v\|_{H^r(\Sigma)} \leq \|\tilde{B}u_n - v\|_{H^s(\Sigma)} \xrightarrow{n\to\infty} 0.$$

As B is closed it follows $\tilde{B}u = Bu = v$. Hence $\tilde{B}$ is closed too and therefore $\tilde{B} \in \mathcal{L}(L^2(\mathbb{R}^d), H^s(\Sigma))$. Hence we can write the operator B as

$$B = \Lambda^{-1}\Lambda\tilde{B},$$

where all operators on the right hand side are bounded. Denote by λ_j the j-the eigenvalue of $(I - \Delta_{\mathrm{LB}}^{\Sigma})^{\frac{1}{2}}$ in nondecreasing order and counted with multiplicities. As Σ is a C^∞-manifold we have

$$\lambda_j \sim c j^{\frac{1}{d-\kappa}}$$

for a certain constant $c > 0$, cf. (5.39) and the text below in [4]. Hence the eigenvalues μ_j of Λ^{-1} satisfy $\mu_j \sim C j^{-\frac{s-r}{d-\kappa}}$ for another constant $C > 0$. Keeping in mind that Λ is selfadjoint we get $s_j(\Lambda) \sim C j^{\frac{s-r}{d-\kappa}}$ and therefore

$$s_j(B) = s_j(\Lambda^{-1}\Lambda\tilde{B}) \leq s_j(\Lambda^{-1})\|\Lambda\tilde{B}\| \sim C\|\Lambda\tilde{B}\| j^{-\frac{s-r}{d-\kappa}}.$$

Hence $B \in \mathfrak{S}_p(L^2(\mathbb{R}^d), H^r(\Sigma))$ for $p > \frac{d-\kappa}{s-r}$. □

Remark 2.24. Note that in the proof of Lemma 2.23 the assumption that Σ is a compact C^∞-manifold was just used to specify the asymptotic decay of the eigenvalues of the operator $(I - \Delta_{\mathrm{LB}}^{\Sigma})^{\frac{1}{2}}$. But the behavior of these eigenvalues is also known for other geometries, e.g. for a closed C^2-curve. Therefore we get analogously as above the following variant of Lemma 2.23 :

Let Σ be a compact C^2-curve in $\mathbb{R}^3$ and $B \in \mathcal{L}(L^2(\mathbb{R}^3), H^r(\Sigma))$ with $\operatorname{ran} B \subseteq H^s(\Sigma)$, $2 \geq s > r \geq 0$. Then $B \in \mathfrak{S}_p(L^2(\mathbb{R}^3), H^r(\Sigma))$ for $p > \frac{1}{s-r}$ and the singular values of B satisfy $s_j(B) = O(j^{-(s-r)})$ for $j \to \infty$.

In the last lemma of this chapter we will use the symbol for the trace operator in a slightly different way than in Lemma 2.20. For a bounded C^∞-domain $\Omega \subseteq \mathbb{R}^d$ we denote by $\mathrm{tr}^1_{\partial\Omega} : H^1(\mathbb{R}^d) \to H^{1/2}(\partial\Omega)$ and $\mathrm{tr}^1_{\partial\Omega^c} : H^1(\mathbb{R}^d) \to H^{1/2}(\partial\Omega^c)$ the unique continuous extensions of the maps

$$C^\infty(\overline{\Omega}) \ni \varphi \mapsto \varphi|_{\partial\Omega} \qquad \text{and} \qquad C^\infty(\overline{\Omega^c}) \ni \varphi \mapsto \varphi|_{\partial\Omega^c},$$

respectively. For more details see for example Theorem 3.37 in [52]. Note that the boundaries $\partial\Omega$ and $\partial\Omega^c$ coincide.

Lemma 2.25. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded C^∞-domain. Let $u \in H^1(\Omega)$ and $v \in H^1(\Omega^c)$ such that $\mathrm{tr}^1_{\partial\Omega} u = \mathrm{tr}^1_{\partial\Omega^c} v$. Then $u \oplus v \in H^1(\mathbb{R}^d)$.*

Proof. As Ω is a C^∞-domain there exists a function $\tilde{u} \in H^1(\mathbb{R}^d)$ such that $\tilde{u}(x) = u(x)$ holds for almost every $x \in \Omega$, cf. Theorem 5.24 in [3]. Analogously there exists $\tilde{v} \in H^1(\mathbb{R}^d)$ with $\tilde{v}(x) = v(x)$ for almost every $x \in \Omega^c$. Define $\tilde{w} := \tilde{u} - \tilde{v} \in H^1(\mathbb{R}^d)$ and denote by w the restriction of $\tilde{w}$ to Ω. Due to $\mathrm{tr}^1_{\partial\Omega} u = \mathrm{tr}^1_{\partial\Omega^c} v$ we have $\mathrm{tr}^1_{\partial\Omega} w = 0$ and hence $w \in H^1_0(\Omega)$. Let $\hat{w}$ be the zero extension of w to $\mathbb{R}^d$. According to Theorem 5.29 in [3] $\hat{w}$ belongs to $H^1(\mathbb{R}^d)$ and hence also $\hat{w} + \tilde{v} \in H^1(\mathbb{R}^d)$. But for almost all $x \in \Omega$ we have

$$\hat{w}(x) + \tilde{v}(x) = w(x) + \tilde{v}(x) = \tilde{w}(x) + \tilde{v}(x) = \tilde{u}(x) - \tilde{v}(x) + \tilde{v}(x) = \tilde{u}(x) = u(x)$$

and for almost all $x \in \Omega^c$ we have

$$\hat{w}(x) + \tilde{v}(x) = 0 + \tilde{v}(x) = v(x).$$

Hence $u \oplus v = \hat{w} + \tilde{v} \in H^1(\mathbb{R}^d)$. □

3 SELFADJOINT OPERATORS WITH SINGULAR PERTURBATIONS

In this chapter we provide an approach for a rigorous definition of selfadjoint operators with singular perturbations which can be written formally as $\mathcal{A}_\vartheta = \tilde{A} - G\vartheta^{-1}G^*$. Depending on the range of G we have to distinguish between different cases.

In the first section we will fix the setting and introduce all relevant objects. The following sections are devoted to the different cases mentioned above.

Note that for the special case that G is a finite rank operator the following approach coincides with the one in [23].

3.1 A chain of Hilbert spaces

Let $A \geq 1$ be a selfadjoint operator in a Hilbert space $\mathcal{H}^0$. For $s \in \mathbb{N}$ set $\mathcal{H}^s := \operatorname{dom} A^{s/2}$, where the operator $A^{s/2}$ is defined via functional calculus. Together with the inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}^s} : \mathcal{H}^s \times \mathcal{H}^s \to \mathbb{C}, \qquad \langle u, v \rangle_{\mathcal{H}^s} := \langle A^{s/2}u, A^{s/2}v \rangle_{\mathcal{H}^0},$$

$\mathcal{H}^s$ becomes a Hilbert space. Set $\mathcal{H}^{-s} := (\mathcal{H}^s)'$. We will show in Lemma 3.1 that these spaces are contained into each other such that we obtain the following chain of Hilbert spaces:

$$\ldots \supseteq \mathcal{H}^{-2} \supseteq \mathcal{H}^{-1} \supseteq \mathcal{H}^0 \supseteq \mathcal{H}^1 \supseteq \mathcal{H}^2 \supseteq \ldots.$$

For $s \in \mathbb{N}$, $s \geq 2$, define the operator $A_s : \mathcal{H}^s \to \mathcal{H}^{s-2}$ via $A_s u = Au$ for $u \in \mathcal{H}^s$. The operator $A_1 : \mathcal{H}^1 \to \mathcal{H}^{-1}$ is defined by

$$\langle A_1 u, v \rangle_{\mathcal{H}^{-1},\mathcal{H}^1} := \langle A^{1/2}u, A^{1/2}v \rangle_{\mathcal{H}^0}, \quad v \in \mathcal{H}^1.$$

Furthermore define for $s \in \mathbb{N}_0$ the operators $A_{-s} : \mathcal{H}^{-s} \to \mathcal{H}^{-s-2}$ by

$$\langle A_{-s}u, v \rangle_{\mathcal{H}^{-s-2},\mathcal{H}^{s+2}} := \langle u, A_{s+2}v \rangle_{\mathcal{H}^{-s},\mathcal{H}^s}.$$

Lemma 3.1. *Let $s,t \in \mathbb{Z}$ with $s < t$. Then the following assertions hold.*

(i) The space $\mathcal{H}^t$ is dense in $\mathcal{H}^s$ and $\|u\|_{\mathcal{H}^s} \leq \|u\|_{\mathcal{H}^t}$ holds for all $u \in \mathcal{H}^t$.

(ii) The operator A_t satisfies $A_t u = A_s u$ for all $u \in \mathcal{H}^t$.

(iii) $A_s : \mathcal{H}^s \to \mathcal{H}^{s-2}$ is an isometric isomorphism.

Proof. (i)
Consider first the case $0 \le s < t$. The operators $A^{s/2}$ and $A^{t/2}$ defined via functional calculus are selfadjoint. In particular their domains are dense in $\mathcal{H}^0$. The inclusion $\mathcal{H}^s = \operatorname{dom} A^{s/2} \supseteq \operatorname{dom} A^{t/2} = \mathcal{H}^t$ follows with the spectral theorem. Moreover

$$\begin{aligned}\|u\|_{\mathcal{H}^s}^2 = \|A^{s/2}u\|_{\mathcal{H}^0}^2 &= \int_{\mathbb{R}} |x^{s/2}|^2 \, d\langle Eu, u\rangle = \int_1^\infty x^s \, d\langle Eu, u\rangle \\ &\le \int_1^\infty x^t \, d\langle Eu, u\rangle = \int_{\mathbb{R}} |x^{t/2}|^2 \, d\langle Eu, u\rangle = \|A^{t/2}u\|_{\mathcal{H}^0}^2 = \|u\|_{\mathcal{H}^t}^2\end{aligned}$$

and hence $\|u\|_{\mathcal{H}^s} \le \|u\|_{\mathcal{H}^t}$ for all $u \in \mathcal{H}^t$, cf. Theorem 5.9 in [59].

Let $u \in \mathcal{H}^s$ be arbitrary, hence $v := A^{s/2}u \in \mathcal{H}^0$. As $A^{(t-s)/2}$ is selfadjoint its domain is dense in $\mathcal{H}^0$. Let $(v_n)_n \subseteq \operatorname{dom} A^{(t-s)/2} = \mathcal{H}^{t-s}$ be a sequence which converges in $\mathcal{H}^0$ to v. Define $u_n := A^{-s/2}v_n \in \operatorname{dom} A^{t/2} = \mathcal{H}^t$ for each $n \in \mathbb{N}$. Hence

$$\|u_n - u\|_{\mathcal{H}^s} = \|A^{s/2}(u_n - u)\|_{\mathcal{H}^0} = \|v_n - v\|_{\mathcal{H}^0} \xrightarrow{n\to\infty} 0$$

and therefore $\mathcal{H}^t$ is dense in $\mathcal{H}^s$.

Next we show that $\mathcal{H}^{-s}$ is dense in $\mathcal{H}^{-t}$ for $0 \le s < t$. Denote by ι the continuous embedding $u \mapsto u$ from $\mathcal{H}^t$ to $\mathcal{H}^s$. Then $\iota' : \mathcal{H}^{-s} \to \mathcal{H}^{-t}$, $\psi \mapsto \psi|_{\mathcal{H}^t}$ is continuous too. As $\mathcal{H}^t$ is dense in $\mathcal{H}^s$ we get with Theorem 4.12 from [58] ($\perp$ denotes the annihilator in $\mathcal{H}^{-s}$)

$$\mathcal{H}^{-s} \supseteq \ker \iota' = (\operatorname{ran} \iota)_\perp = (\mathcal{H}^t)_\perp = \{\psi \in \mathcal{H}^{-s} : \langle \psi, v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = 0 \; \forall v \in \mathcal{H}^t\} = \{0\},$$

i.e. ι' is injective. Hence ι' is a continuous embedding from $\mathcal{H}^{-s}$ to $\mathcal{H}^{-t}$ and we can interpret $\mathcal{H}^{-s}$ as a subset of $\mathcal{H}^{-t}$. To see that $\mathcal{H}^{-s}$ is even dense in $\mathcal{H}^{-t}$ recall that both spaces are reflexive (because they are dual spaces of Hilbert spaces). Hence (with a suitable identification) $\iota = \iota''$ and in particular $\ker \iota'' = \ker \iota = \{0\}$. With Theorem 4.7 and Theorem 4.12 from [58] we get now

$$\begin{aligned}\overline{\mathcal{H}^{-s}}^{\mathcal{H}^{-t}} &= \overline{\operatorname{ran} \iota'}^{\mathcal{H}^{-s}} = {}_\perp((\operatorname{ran} \iota')_\perp) = {}_\perp(\ker \iota'') = {}_\perp\{0\} \\ &= \{\psi \in \mathcal{H}^{-t} : \langle \psi, v\rangle_{\mathcal{H}^{-t},\mathcal{H}^t} = 0 \text{ for all } v \in \{0\}\} = \mathcal{H}^{-t},\end{aligned}$$

i.e. $\mathcal{H}^{-s}$ is dense in $\mathcal{H}^{-t}$. Furthermore we have

$$\|\psi\|_{\mathcal{H}^{-t}} = \sup_{v \in B^t} \langle \psi, v\rangle_{\mathcal{H}^{-t},\mathcal{H}^t} = \sup_{v \in B^t} \langle \psi, v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} \le \sup_{v \in B^s} \langle \psi, v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \|\psi\|_{\mathcal{H}^{-s}},$$

for each $\psi \in \mathcal{H}^{-s}$, where the sets B^t and B^s are defined by

$$B^t := \{v \in \mathcal{H}^t : 0 < \|v\|_{\mathcal{H}^t} \le 1\} \subseteq \{v \in \mathcal{H}^s : 0 < \|v\|_{\mathcal{H}^s} \le 1\} =: B^s.$$

Next we show that $\mathcal{H}^t$ is dense in $\mathcal{H}^{-s}$ for arbitrary $s,t \geq 0$. Let $u \in \mathcal{H}^{-s}$ be arbitrary. As $\mathcal{H}^0$ is dense in $\mathcal{H}^{-s}$ there exists a sequence $(u_n)_n \subseteq \mathcal{H}^0$ with $\|u-u_n\|_{\mathcal{H}^{-s}} \leq \frac{1}{2n}$ for every $n \in \mathbb{N}$. As $\mathcal{H}^t$ is dense in $\mathcal{H}^0$ there exists a sequence $(v_n)_n \subseteq \mathcal{H}^t$ with $\|u_n - v_n\|_{\mathcal{H}^0} \leq \frac{1}{2n}$ for every $n \in \mathbb{N}$. Hence

$$\|u-v_n\|_{\mathcal{H}^{-s}} \leq \|u-u_n\|_{\mathcal{H}^{-s}} + \|u_n - v_n\|_{\mathcal{H}^{-s}} \leq \frac{1}{2n} + \|u_n - v_n\|_{\mathcal{H}^0} = \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

Moreover we have $\|u\|_{\mathcal{H}^{-s}} \leq \|u\|_{\mathcal{H}^0} \leq \|u\|_{\mathcal{H}^t}$ for all $u \in \mathcal{H}^t$.

(ii)

Next we show that $A_t u = A_s u$ holds for all $u \in \mathcal{H}^t$. For $2 \leq s < t$ this is obvious because by definition the action of both operators A_t and A_s is given by the action of A.

For $u \in \mathcal{H}^2$ we have $A_2 u \in \mathcal{H}^0 \subseteq \mathcal{H}^{-1}$ and hence

$$\langle A_2 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} = \langle A_2 u, v\rangle_{\mathcal{H}^0} = \langle Au, v\rangle_{\mathcal{H}^0} = \langle A^{1/2}u, A^{1/2}v\rangle_{\mathcal{H}^0} = \langle A_1 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1}$$

for all $v \in \mathcal{H}^1$. Hence $A_1 u = A_2 u$.

For $u \in \mathcal{H}^1$ we have $A_1 u \in \mathcal{H}^{-1} \subseteq \mathcal{H}^{-2}$ and hence

$$\begin{aligned}\langle A_1 u, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} &= \langle A_1 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} = \langle A^{1/2}u, A^{1/2}v\rangle_{\mathcal{H}^0}\\ &= \langle u, Av\rangle_{\mathcal{H}^0} = \langle u, A_2 v\rangle_{\mathcal{H}^0} = \langle A_0 u, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}\end{aligned}$$

for all $v \in \mathcal{H}^2$. Hence $A_0 u = A_1 u$.

For $u \in \mathcal{H}^{-s}$, $s \geq 0$, we have $A_{-s}u \in \mathcal{H}^{-s-2} \subseteq \mathcal{H}^{-s-3}$ and hence

$$\begin{aligned}\langle A_{-s}u, v\rangle_{\mathcal{H}^{-s-3},\mathcal{H}^{s+3}} &= \langle A_{-s}u, v\rangle_{\mathcal{H}^{-s-2},\mathcal{H}^{s+2}} = \langle u, A_{s+2}v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s}\\ &= \langle u, A_{s+3}v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \langle u, A_{s+3}v\rangle_{\mathcal{H}^{-s-1},\mathcal{H}^{s+1}} = \langle A_{-s-1}u, v\rangle_{\mathcal{H}^{s-3},\mathcal{H}^{s+3}}\end{aligned}$$

for all $v \in \mathcal{H}^{s+3}$. Hence $A_{-s-1}u = A_{-s}u$. The remaining cases follow by transitivity.

(iii)

It remains to show, that $A_s : \mathcal{H}^s \to \mathcal{H}^{s-2}$ is an isometric isomorphism. Consider at first the case $s \geq 2$. Then we have for all $u \in \mathcal{H}^s$

$$\|A_s u\|_{\mathcal{H}^{s-2}} = \|A^{\frac{s-2}{2}}Au\|_{\mathcal{H}^0} = \|A^{\frac{s}{2}}u\|_{\mathcal{H}^0} = \|u\|_{\mathcal{H}^s},$$

i.e. $A_s : \mathcal{H}^s \to \mathcal{H}^{s-2}$ is an isometry. Due to

$$\operatorname{ran} A_s = A\mathcal{H}^s = A \operatorname{dom} A^{\frac{s}{2}} = \operatorname{dom} A^{\frac{s}{2}-1} = \mathcal{H}^{s-2}$$

the operator $A_s : \mathcal{H}^s \to \mathcal{H}^{s-2}$ is even surjective and hence an isometric isomorphism.

Consider next the case $s = 1$. At first note that $\sigma(A^{1/2}) \subseteq [1,\infty[$, where the selfadjoint operator $A^{1/2}$ in $\mathcal{H}^0$ is defined via functional calculus. Hence $\operatorname{ran} A^{1/2} = \mathcal{H}^0$. Therefore

$$\begin{aligned}\|A_1 u\|_{\mathcal{H}^{-1}} &= \sup_{\substack{v\in\mathcal{H}^1\\ \|v\|_{\mathcal{H}^1}=1}} \langle A_1 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} = \sup_{\substack{v\in\operatorname{dom}A^{1/2}\\ \|A^{1/2}v\|_{\mathcal{H}^0}=1}} \langle A^{1/2}u, A^{1/2}v\rangle_{\mathcal{H}^0}\\ &= \sup_{\substack{w\in\mathcal{H}^0\\ \|w\|_{\mathcal{H}^0}=1}} \langle A^{1/2}u, w\rangle_{\mathcal{H}^0} = \|A^{1/2}u\|_{\mathcal{H}^0} = \|u\|_{\mathcal{H}^1}\end{aligned}$$

for all $u \in \mathcal{H}^1$. To show surjectivity let $\psi \in \mathcal{H}^{-1}$ be arbitrary. According to Riesz representation theorem there exists $u \in \mathcal{H}^1$ such that

$$\langle \psi, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} = \langle u, v\rangle_{\mathcal{H}^1} = \langle A^{1/2}u, A^{1/2}v\rangle_{\mathcal{H}^0} = \langle A_1 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1}$$

holds for all $v \in \mathcal{H}^1$, i.e. $A_1 u = \psi$. Hence A_1 is surjective and therefore an isometric isomorphism.

It remains to consider A_{-s} with $-s \le 0$. For this let $\psi \in \mathcal{H}^{-s}$ be arbitrary. We have already seen that $A_{s+2} : \mathcal{H}^{s+2} \to \mathcal{H}^s$ is surjective and isometric. Hence we get

$$\begin{aligned}\|A_{-s}\psi\|_{\mathcal{H}^{-s-2}} &= \sup_{\substack{v\in\mathcal{H}^{s+2}\\ \|v\|_{\mathcal{H}^{s+2}}=1}} \langle A_{-s}\psi, v\rangle_{\mathcal{H}^{-s-2},\mathcal{H}^{s+2}}\\ &= \sup_{\substack{v\in\mathcal{H}^{s+2}\\ \|v\|_{\mathcal{H}^{s+2}}=1}} \langle \psi, A_{s+2}v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \sup_{\substack{w\in\mathcal{H}^{s}\\ \|w\|_{\mathcal{H}^{s}}=1}} \langle \psi, w\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \|\psi\|_{\mathcal{H}^{-s}}.\end{aligned}$$

To show surjectivity let $\psi \in \mathcal{H}^{-s-2}$ be arbitrary. According to Riesz representation theorem there exists $u \in \mathcal{H}^{s+2}$ such that

$$\langle \psi, v\rangle_{\mathcal{H}^{-s-2},\mathcal{H}^{s+2}} = \langle u, v\rangle_{\mathcal{H}^{s+2}} = \langle A^{s/2+1}u, A^{s/2+1}v\rangle_{\mathcal{H}^0} = \langle A_{s+2}u, A_{s+2}v\rangle_{\mathcal{H}^s}.$$

holds for all $v \in \mathcal{H}^{s+2}$. Let $\varphi \in \mathcal{H}^{-s}$ the Riesz representation of $\langle A_{s+2}u, \cdot\rangle_{\mathcal{H}^s}$. Hence

$$\langle \psi, v\rangle_{\mathcal{H}^{-s-2},\mathcal{H}^{s+2}} = \langle A_{s+2}u, A_{s+2}v\rangle_{\mathcal{H}^s} = \langle \varphi, A_{s+2}v\rangle_{\mathcal{H}^{-s},\mathcal{H}^s} = \langle A_{-s}\varphi, v\rangle_{\mathcal{H}^{-s-2},\mathcal{H}^{s+2}}$$

for all $v \in \mathcal{H}^{s+2}$, i.e. $A_{-s}\varphi = \psi$. Hence A_{-s} is surjective and therefore an isometric isomorphism. □

Remark 3.2. For each $u \in \mathcal{H}^s = \operatorname{ran} A_{s+2}$ and all $j \in \mathbb{N}$ we have

$$A_{s+2}^{-j}u = A_{s+2}^{-1}\dots A_{s+2}^{-1}u = A_{s+2j}^{-1}A_{s+2(j-1)}^{-1}\cdots A_{s+4}^{-1}A_{s+2}^{-1}u \in \mathcal{H}^{s+2j}.$$

In particular elements in $\mathcal{H}^s$ with $s < 0$ can be "lifted up" to $\mathcal{H}^0$ by a repeated application of A_{s+2}^{-1}.

Example 3.3. An example for such a chain of Hilbert spaces are the Sobolev spaces $H^s(\mathbb{R}^d)$, $s \in \mathbb{Z}$, with the operator $A := -\Delta_{\text{free}} + 1$. Here $-\Delta_{\text{free}}$ is the free Laplace operator in $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. The norm $\|\cdot\|_{\mathcal{H}^s}$ generated by A is equivalent to the usual Sobolev norm $\|\cdot\|_{H^s(\mathbb{R}^d)}$.

Let $k \in \mathbb{N}$, $\mathcal{G}$ be another Hilbert space and $G : \mathcal{G} \to \mathcal{H}^{-k}$ an operator satisfying

$$G \in \mathcal{L}(\mathcal{G}, \mathcal{H}^{-k}), \quad \ker G = \{0\}, \quad \text{and} \quad \operatorname{ran} G \cap \mathcal{H}^{-k+1} = \{0\}. \tag{3.1}$$

Define the index j by

$$j := \left\lfloor \frac{k-1}{2} \right\rfloor = \begin{cases} \frac{k-1}{2} & \text{if } k \text{ is odd,} \\ \frac{k-2}{2} & \text{if } k \text{ is even.} \end{cases} \tag{3.2}$$

Hence $k - 2j = 1$ if k is odd and $k - 2j = 2$ if k is even. Furthermore we define

$$G_0 := A_{-k+2}^{-j} G : \mathcal{G} \to \mathcal{H}^{2j-k} = \begin{cases} \mathcal{H}^{-1} & \text{if } k \text{ is odd,} \\ \mathcal{H}^{-2} & \text{if } k \text{ is even.} \end{cases}$$

Note that G_0 as well as $G_0^* : \mathcal{H}^{k-2j} \to \mathcal{G}$ are both continuous.

Lemma 3.4. *The operator $S := A \restriction (\mathcal{H}^2 \cap \ker G_0^*)$ is a closed symmetric operator in $\mathcal{H}^0$ whose adjoint (linear relation) S^* contains the operator*

$$Tu := A_0 u - G_0 h, \qquad \operatorname{dom} T := \{u \in \mathcal{H}^0 : \exists h \in \mathcal{G} \text{ with } A_0 u - G_0 h \in \mathcal{H}^0\}. \tag{3.3}$$

If k is odd then $\operatorname{dom} T \subseteq \mathcal{H}^1$. Furthermore the map $\Gamma_0 : \operatorname{dom} T \to \mathcal{G}$, $u \mapsto h$ with h as in (3.3)*, is surjective and* $\ker \Gamma_0 = \mathcal{H}^2$*. In particular $A \subseteq T$.*

Proof. Let $(u_n)_n$ be a sequence in $\operatorname{dom} S$ with $u_n \xrightarrow{n\to\infty} u$ and $Su_n \xrightarrow{n\to\infty} v$ in $\mathcal{H}^0$. Because A is closed and $Su_n = Au_n$ we get $u \in \operatorname{dom} A$ and $Au = v$. Hence

$$\|u_n - u\|_{\mathcal{H}^{k-2j}} \le \|u_n - u\|_{\mathcal{H}^2} = \|Au_n - Au\|_{\mathcal{H}^0} = \|Su_n - v\|_{\mathcal{H}^0} \xrightarrow{n\to\infty} 0.$$

As $G_0^* : \mathcal{H}^{k-2j} \to \mathcal{G}$ is continuous $\ker G_0^*$ is closed in $\mathcal{H}^{k-2j}$ and therefore $u \in \ker G_0^*$. Hence $u \in \operatorname{dom} S$ with $Su = v$, i.e. S is closed. The fact that S is symmetric follows directly from the selfadjointness of A.

Next we show that T is a well defined operator. For this we have to show that the element h appearing in (3.3) is unique: Let $h_1, h_2 \in \mathcal{G}$ with $A_0 u - G_0 h_1 = v_1 \in \mathcal{H}^0$ and $A_0 u - G_0 h_2 = v_2 \in \mathcal{H}^0$. It follows

$$\mathcal{H}^{2j-k+1} \supseteq \mathcal{H}^0 \ni v_1 - v_2 = A_0 u - G_0 h_1 - (A_0 u - G_0 h_2) = G_0(h_2 - h_1) \in \operatorname{ran} G_0.$$

As $\operatorname{ran} G \cap \mathcal{H}^{-k+1} = \{0\}$ we get due to Lemma 3.1 $\operatorname{ran} G_0 \cap \mathcal{H}^{2j-k+1} = \{0\}$ and hence $G_0(h_2 - h_1) = v_1 - v_2 = 0$. Due to $\ker G_0 = \ker A_{-k+2}^{-j} G = \{0\}$ this implies $h_2 = h_1$.

Moreover $T \subseteq S^*$ because for all $u \in \operatorname{dom} T$ and all $v \in \operatorname{dom} S = \mathcal{H}^2 \cap \ker G_0^*$ holds

$$\begin{aligned}\langle Tu, v\rangle_{\mathcal{H}^0} &= \langle A_0 u - G_0 h, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}\\ &= \langle A_0 u, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} - \langle G_0 h, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} = \langle u, A_2 v\rangle_{\mathcal{H}^0,\mathcal{H}^0} - \langle h, G_0^* v\rangle_{\mathcal{G}} = \langle u, Sv\rangle_{\mathcal{H}^0}.\end{aligned}$$

If $u \in \mathcal{H}^2$ then $Tu = Au \in \mathcal{H}^0$ with $h = 0$. Hence $\mathcal{H}^2 \subseteq \ker \Gamma_0$ and $A \subseteq T$. On the other hand, if $u \in \ker \Gamma_0$ we have $h = 0$ and therefore $A_0 u \in \mathcal{H}^0$. Hence $u \in \mathcal{H}^2$.

If k is odd then $\operatorname{ran} G_0 \subseteq \mathcal{H}^{-1}$. Let $u \in \operatorname{dom} T$ and $v := Tu$. Hence $A_0 u = v + G_0 h \in \mathcal{H}^{-1}$ and therefore $u \in \mathcal{H}^1$, see Lemma 3.1.

It remains to show that Γ_0 is surjective. Let $h \in \mathcal{G}$. Then we have $G_0 h \in \mathcal{H}^{2j-k}$. As $A_{2j-k+2} : \mathcal{H}^{2j-k+2} \to \mathcal{H}^{2j-k}$ is surjective, see Lemma 3.1, there exists $u \in \mathcal{H}^{2j-k+2} \subseteq \mathcal{H}^0$ with $A_{2j-k+2} u = G_0 h$ and hence $A_0 u - G_0 h = 0 \in \mathcal{H}^0$. This means $u \in \operatorname{dom} T$ with $\Gamma_0 u = h$. □

For the following recall that $\operatorname{dom} T$ can be written as $\operatorname{dom} T = \mathcal{H}^2 \dotplus \ker T$, cf. Lemma 2.1, because the selfadjoint operator A is contained in T. This means that every $u \in \operatorname{dom} T$ can be written uniquely as $u = u_c + u_s$ with $u_c \in \mathcal{H}^2$ and $u_s \in \ker T$. Moreover note that

- if k is odd, then $\operatorname{dom} G_0^* = \mathcal{H}^1 \supseteq \operatorname{dom} T$, see Lemma 3.4.
- if k is even, then $\operatorname{dom} G_0^* = \mathcal{H}^2 \ni u_c$.

This implies that the map Γ_1 in the following theorem is well defined.

Theorem 3.5. *The triple* $(\mathcal{G}, \Gamma_0, \Gamma_1)$ *with the boundary maps*

$$\begin{aligned}&\Gamma_0 : \operatorname{dom} T \to \mathcal{G}, \qquad u \mapsto h \qquad \text{with } u \text{ as in (3.3)},\\ &\Gamma_1 : \operatorname{dom} T \to \mathcal{G}, \qquad u \mapsto \begin{cases} G_0^* u & \text{if } k \text{ is odd,}\\ G_0^* u_c & \text{if } k \text{ is even,}\end{cases}\end{aligned}$$

is a generalized boundary triple for $\overline{T} = S^*$.

Proof. At first we show that $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ has dense range. For this define the space

$$\mathcal{G}^+ := \operatorname{ran}(\Gamma_1 \restriction \ker \Gamma_0) = \operatorname{ran}(G_0^* \restriction \mathcal{H}^2) = \operatorname{ran}(G^* A_k^{-j} \restriction \mathcal{H}^2) = \operatorname{ran}(G^* \restriction \mathcal{H}^{2j+2}).$$

It was shown in [25, Lemma 6.1] (for an arbitrary generalized boundary triple) that $\mathcal{G}^+$ is dense in $\mathcal{G}$. Indeed, $\mathcal{H}^{2j+2}$ is dense in $\mathcal{H}^k$ (if k is even these spaces even coincide) and therefore

$$\begin{aligned}(\mathcal{G}^+)^\perp &= \{h \in \mathcal{G} : \langle h, g\rangle_{\mathcal{G}} = 0 \;\forall\, g \in \mathcal{G}^+\} = \{h \in \mathcal{G} : \langle h, G^* u\rangle_{\mathcal{G}} = 0 \;\forall\, u \in \mathcal{H}^{2j+2}\}\\ &= \{h \in \mathcal{G} : \langle Gh, u\rangle_{\mathcal{H}^{-k},\mathcal{H}^k} = 0 \;\forall\, u \in \mathcal{H}^{2j+2}\} = \{h \in \mathcal{G} : Gh = 0\} = \ker G = \{0\}.\end{aligned}$$

Next let $(h,k) \in \mathcal{G} \times \mathcal{G}$ be arbitrary. As Γ_0 is surjective, see Lemma 3.4, there exists $u \in \operatorname{dom} T$ with $\Gamma_0 u = h$. Moreover there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \ker \Gamma_0$ such that $\{\Gamma_1 u_n\}_{n\in\mathbb{N}}$ converges to $k - \Gamma_1 u$ because $\operatorname{ran}(\Gamma_1 \restriction \ker \Gamma_0) = \mathcal{G}^+$ is dense. It follows

$$\Gamma(u+u_n) = \begin{bmatrix} \Gamma_0(u+u_m) \\ \Gamma_1(u+u_n) \end{bmatrix} = \begin{bmatrix} \Gamma_0 u \\ \Gamma_1 u + \Gamma_1 u_n \end{bmatrix} \xrightarrow{n\to\infty} \begin{bmatrix} h \\ \Gamma_1 u + k - \Gamma_1 u \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

and hence $\operatorname{ran}\Gamma$ is dense in $\mathcal{G} \times \mathcal{G}$. Keeping in mind that $A = T \restriction \ker \Gamma_0$ is selfadjoint it remains to show that the abstract Green's identity holds, cf. Lemma 2.11.

We will first consider the case that k is odd. Let $u, v \in \operatorname{dom} T$ be arbitrary. Hence

$$\begin{aligned} \langle Tu, v\rangle_{\mathcal{H}^0} - \langle u, Tv\rangle_{\mathcal{H}^0} &= \langle A_0 u - G_0 h, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} - \langle u, A_0 v - G_0 k\rangle_{\mathcal{H}^1,\mathcal{H}^{-1}} \\ &= \langle A_0 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} - \langle G_0 h, v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} - \langle u, A_0 v\rangle_{\mathcal{H}^1,\mathcal{H}^{-1}} + \langle u, G_0 k\rangle_{\mathcal{H}^1,\mathcal{H}^{-1}} \\ &= -\langle h, G_0^* v\rangle_{\mathcal{G}} + \langle G_0^* u, k\rangle_{\mathcal{G}} = \langle \Gamma_1 u, \Gamma_0 v\rangle_{\mathcal{G}} - \langle \Gamma_0 u, \Gamma_1 v\rangle_{\mathcal{G}}. \end{aligned}$$

Consider now the case that k is even. Let $u, v \in \operatorname{dom} T$ be arbitrary. Recall that u can be written as $u = u_c + u_s$ with $u_c \in \mathcal{H}^2 = \ker \Gamma_0$ and $u_s \in \ker T$, cf. Lemma 2.1. Hence $\Gamma_0 u = \Gamma_0 u_s$,

$$\begin{aligned} A_0 u_s &= A_0 u_s - G_0 \Gamma_0 u_s + G_0 \Gamma_0 u = T u_s + G_0 \Gamma_0 u = G_0 \Gamma_0 u, \\ Tu &= T u_c + T u_s = A_0 u_c - G_0 \Gamma_0 u_c = A u_c, \end{aligned}$$

and analogous results hold for v. Therefore

$$\begin{aligned} \langle Tu, v\rangle_{\mathcal{H}^0} - \langle u, Tv\rangle_{\mathcal{H}^0} &= \langle A u_c, v_c + v_s\rangle_{\mathcal{H}^0} - \langle u_c + u_s, A v_c\rangle_{\mathcal{H}^0} \\ &= \langle A u_c, v_s\rangle_{\mathcal{H}^0} - \langle u_s, A v_c\rangle_{\mathcal{H}^0} \\ &= \langle u_c, A_0 v_s\rangle_{\mathcal{H}^2,\mathcal{H}^{-2}} - \langle A_0 u_s, v_c\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} \\ &= \langle u_c, G_0 \Gamma_0 v\rangle_{\mathcal{H}^2,\mathcal{H}^{-2}} - \langle G_0 \Gamma_0 u, v_c\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} \\ &= \langle G_0^* u_c, \Gamma_0 v\rangle_{\mathcal{G}} - \langle \Gamma_0 u, G_0^* v_c\rangle_{\mathcal{G}} = \langle \Gamma_1 u, \Gamma_0 v\rangle_{\mathcal{G}} - \langle \Gamma_0 u, \Gamma_1 v\rangle_{\mathcal{G}}. \end{aligned}$$ □

Our next aim is to characterize S^*. For this we will use again the space

$$\mathcal{G}^+ := \operatorname{ran}(\Gamma_1 \restriction \ker \Gamma_0) = \operatorname{ran}(G_0^* \restriction \mathcal{H}^2) = \operatorname{ran}(G^* A_k^{-j} \restriction \mathcal{H}^2) = \operatorname{ran}(G^* \restriction \mathcal{H}^{2j+2}).$$

We have already seen in the proof of Theorem 3.5 that $\mathcal{G}^+$ is dense in $\mathcal{G}$. Hence there exists a norm $\|\cdot\|_{\mathcal{G}^+}$ such that $(\mathcal{G}^+, \|\cdot\|_{\mathcal{G}^+})$ becomes a Hilbert space which is continuously embedded into $\mathcal{G}$, see Proposition 2.9 and 2.10 in [14]. Consider the Gelfand triple $\mathcal{G}^+ \subseteq \mathcal{G} \subseteq \mathcal{G}^-$, where $\mathcal{G}^-$ denotes the dual space of $\mathcal{G}^+$. Let

$$\begin{aligned} &\iota_- : \mathcal{G}^- \to \mathcal{G} \text{ be an isometric isomorphism and} \\ &\iota_+ := (\iota_-^{-1})^* : \mathcal{G}^+ \to \mathcal{G}. \end{aligned} \tag{3.4}$$

Then ι_+ is an isometric isomorphism too and for all $u \in \mathcal{G}^+$ and $v \in \mathcal{G}^-$ holds

$$\langle u, v\rangle_{\mathcal{G}^+,\mathcal{G}^-} = \langle u, \iota_-^{-1}\iota_- v\rangle_{\mathcal{G}^+,\mathcal{G}^-} = \langle (\iota_-^{-1})^* u, \iota_- v\rangle_{\mathcal{G}} = \langle \iota_+ u, \iota_- v\rangle_{\mathcal{G}}.$$

We are now able to prove the following lemma, which gives a representation of S^* and is a special case of Theorem 2.12 in [14]. For this we have to extend the operator $G : \mathcal{G} \to \mathcal{H}^{-k}$ to $\mathcal{G}^-$, which is done with the operator $(G^\circledast)^*$ appearing in the next theorem.

Theorem 3.6. *Consider the operator $G^\circledast : \mathcal{H}^{2j+2} \to \mathcal{G}^+ = \operatorname{ran}(G^* \restriction \mathcal{H}^{2j+2})$, $u \mapsto G^* u$ and assume that* $\operatorname{ran}(G^\circledast)^* \cap \mathcal{H}^{-k+1} = \{0\}$ *holds. Then S is densely defined and S^* satisfies*

$$S^* u = A_0 u - A_{-2j}^{-j} (G^\circledast)^* h, \;\; \operatorname{dom} S^* = \{u \in \mathcal{H}^0 : \exists h \in \mathcal{G}^- \text{ with } A_0 u - A_{-2j}^{-j} (G^\circledast)^* h \in \mathcal{H}^0\}.$$

In particular S^* is an operator. An ordinary boundary triple for S^* is given by $(\mathcal{G}, \hat\Gamma_0, \hat\Gamma_1)$, where the mappings $\hat\Gamma_0, \hat\Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ are given by

$$\hat\Gamma_0 u = \iota_- h, \qquad \hat\Gamma_1 u = G^\circledast A_{2j+2}^{-j} u_c, \qquad u = u_c + u_s \in \mathcal{H}^2 \dotplus \ker S^* = \operatorname{dom} S^*.$$

Proof. We define the operator $\hat S$ by

$$\hat S u = A_0 u - A_{-2j}^{-j} (G^\circledast)^* h, \quad \operatorname{dom} \hat S = \{u \in \mathcal{H}^0 : \exists h \in \mathcal{G}^- \text{ with } A_0 u - A_{-2j}^{-j} (G^\circledast)^* h \in \mathcal{H}^0\},$$

and show $\hat S = S^*$. The fact, that $\hat S$ is an operator, can be seen analogously as for the operator T: Let $h_1, h_2 \in \mathcal{G}^-$ with $A_0 u - A_{-2j}^{-j} (G^\circledast)^* h_1 = v_1 \in \mathcal{H}^0$ and $A_0 u - A_{-2j}^{-j} (G^\circledast)^* h_2 = v_2 \in \mathcal{H}^0$. It follows

$$\mathcal{H}^{2j-k+1} \supseteq \mathcal{H}^0 \ni v_1 - v_2 = A_{-2j}^{-j} (G^\circledast)^* (h_2 - h_1).$$

Due to Lemma 3.1 it follows $(G^\circledast)^*(h_2 - h_1) \in \mathcal{H}^{-k+1}$. As $\operatorname{ran}(G^\circledast)^* \cap \mathcal{H}^{-k+1} = \{0\}$ by assumption this implies $h_2 = h_1$.

For $v \in \operatorname{dom} S = \operatorname{dom} A \cap \ker G_0^*$ and $u \in \operatorname{dom} \hat S$ we get

$$\begin{aligned} \langle \hat S u, v\rangle_{\mathcal{H}^0} &= \langle A_0 u - A_{-2j}^{-j} (G^\circledast)^* h, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} = \langle A_0 u, v\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} - \langle (G^\circledast)^* h, A_2^{-j} v\rangle_{\mathcal{H}^{-2j-2},\mathcal{H}^{2j+2}} \\ &= \langle u, Av\rangle_{\mathcal{H}^0} - \langle h, G^\circledast A^{-j} v\rangle_{\mathcal{G}^-,\mathcal{G}^+} = \langle u, Sv\rangle_{\mathcal{H}^0} - \langle h, G^* A_2^{-j} v\rangle_{\mathcal{G}^-,\mathcal{G}^+} = \langle u, Sv\rangle_{\mathcal{H}^0}. \end{aligned}$$

Hence $\hat S \subseteq S^*$. For the other inclusion let ι_- and ι_+ as in (3.4). Recall that every $u \in \operatorname{dom} \hat S$ can be written as $u = u_c + u_s$ with $u_c \in \mathcal{H}^2$ and $u_s \in \ker \hat S$, cf. Lemma 2.1. Define now

$$\begin{aligned} \hat\Gamma_0 &: \operatorname{dom} \hat S \to \mathcal{G}, & u &\mapsto \iota_- h, \\ \hat\Gamma_1 &: \operatorname{dom} \hat S \to \mathcal{G}, & u &\mapsto \iota_+ G^\circledast A_{2j+2}^{-j} u_c. \end{aligned}$$

We will show next that $(\mathcal{G}, \hat\Gamma_0, \hat\Gamma_1)$ is an ordinary boundary triple for S^*. At first note that $\hat S \restriction \ker \hat\Gamma_0 = A$ because

$$\ker \hat\Gamma_0 = \{u \in \mathcal{H}^0 : A_0 u \in \mathcal{H}^0\} = \mathcal{H}^2 = \operatorname{dom} A.$$

Hence $\operatorname{ran}(\hat{S} \upharpoonright \ker \hat{\Gamma}_0) = \operatorname{ran} A = \mathcal{H}$ because $A \geq 1$. The kernel of $\hat{\Gamma} := \binom{\hat{\Gamma}_0}{\hat{\Gamma}_1}$ is given by

$$\ker \hat{\Gamma} = \{u \in \ker \hat{\Gamma}_0 : \hat{\Gamma}_1 u = 0\} = \{u \in \mathcal{H}^2 : G^{\circledast} A_{2j+2}^{-j} u = 0\} = \{u \in \mathcal{H}^2 : G_0^* u = 0\}$$

and hence $\hat{S} \upharpoonright \ker \hat{\Gamma} = S$, cf. the definition of S in Lemma 3.4.

Note that $G^{\circledast}$ is surjective, cf. the definition of the space $\mathcal{G}^+$. Moreover A_{2j+2}^{-j} is an isomorphism between $\mathcal{H}^2$ and $\mathcal{H}^{2j+2}$, cf. Lemma 3.1, and ι_+ is an isomorphism between $\mathcal{G}^+$ and $\mathcal{G}$. Hence $\hat{\Gamma}_1$ is surjective too.

Let $h, k \in \mathcal{G}$ be arbitrary. Hence there exists $u \in \mathcal{H}^0$ such that $A_0 u = A_{-2}^{-j} (G^{\circledast})^* \iota_-^{-1} h \in \mathcal{H}^{-2}$ and therefore $A_0 u - A_{-2}^{-j} (G^{\circledast})^* \iota_-^{-1} h = 0 \in \mathcal{H}^0$. This means $u \in \operatorname{dom} \hat{S}$ and $\hat{\Gamma}_0 u = \iota_- \iota_-^{-1} h = h$. As $\hat{\Gamma}_1$ is surjective there exists $v \in \operatorname{dom} \hat{S}$ with $\hat{\Gamma}_1 v = k - \hat{\Gamma}_1 u$. Without loss of generality we can assume $v \in \mathcal{H}^2$ because for the action of $\hat{\Gamma}_1$ just the $\mathcal{H}^2$-part of v is important. Hence it follows due to $\mathcal{H}^2 \subseteq \ker \hat{\Gamma}_0$

$$\begin{bmatrix} \hat{\Gamma}_0(u+v) \\ \hat{\Gamma}_1(u+v) \end{bmatrix} = \begin{bmatrix} h \\ \hat{\Gamma}_1 u + \hat{\Gamma}_1 v \end{bmatrix} = \begin{bmatrix} h \\ \hat{\Gamma}_1 u + (k - \hat{\Gamma}_1 u) \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e. $\hat{\Gamma} = \binom{\hat{\Gamma}_0}{\hat{\Gamma}_1} : \operatorname{dom} \hat{S} \to \mathcal{G} \times \mathcal{G}$ is surjective. It remains to show the abstract Green's identity, cf. Lemma 2.10. For $u, v \in \operatorname{dom} \hat{S}$ we have

$$\begin{aligned} \langle \hat{S}u, v \rangle_{\mathcal{H}^0} &= \langle \hat{S}(u_c + u_s), v_c + v_s \rangle_{\mathcal{H}^0} = \langle Au_c, v_c \rangle_{\mathcal{H}^0} + \langle Au_c, v_s \rangle_{\mathcal{H}^0} \quad \text{and} \\ \langle u, \hat{S}v \rangle_{\mathcal{H}^0} &= \langle u_c + u_s, \hat{S}(v_c + v_s) \rangle_{\mathcal{H}^0} = \langle u_c, Av_c \rangle_{\mathcal{H}^0} + \langle u_s, Av_c \rangle_{\mathcal{H}^0}. \end{aligned}$$

Note that $0 = \hat{S}u_s = A_0 u_s - A_{-2}^{-j} (G^{\circledast})^* h_u$ implies $A_0 u_s = A_{-2}^{-j} (G^{\circledast})^* h_u$. Analogously we get $A_0 v_s = A_{-2}^{-j} (G^{\circledast})^* h_v$. Hence

$$\begin{aligned} \langle \hat{S}u, v \rangle_{\mathcal{H}^0} - \langle u, \hat{S}v \rangle_{\mathcal{H}^0} &= \langle Au_c, v_s \rangle_{\mathcal{H}^0} - \langle u_c, Av_s \rangle_{\mathcal{H}^0} = \langle u_c, A_0 v_s \rangle_{\mathcal{H}^2, \mathcal{H}^{-2}} - \langle A_0 u_s, v_c \rangle_{\mathcal{H}^{-2}, \mathcal{H}^2} \\ &= \langle u_c, A_{-2}^{-j} (G^{\circledast})^* h_v \rangle_{\mathcal{H}^2, \mathcal{H}^{-2}} - \langle A_{-2}^{-j} (G^{\circledast})^* h_u, v_c \rangle_{\mathcal{H}^{-2}, \mathcal{H}^2} \\ &= \langle G^{\circledast} A_{2j+2}^{-j} u_c, h_v \rangle_{\mathcal{G}^+, \mathcal{G}^-} - \langle h_u, G^{\circledast} A_{2j+2}^{-j} v_c \rangle_{\mathcal{G}^-, \mathcal{G}^+} \\ &= \langle \iota_+ G^{\circledast} A_{2j+2}^{-j} u_c, \iota_- h_v \rangle_{\mathcal{G}} - \langle \iota_- h_u, \iota_+ G^{\circledast} A_{2j+2}^{-j} v_c \rangle_{\mathcal{G}} = \langle \hat{\Gamma}_1 u, \hat{\Gamma}_0 v \rangle_{\mathcal{G}} - \langle \hat{\Gamma}_0 u, \hat{\Gamma}_1 v \rangle_{\mathcal{G}}. \end{aligned}$$

Hence $(\mathcal{G}, \hat{\Gamma}_0, \hat{\Gamma}_1)$ is an ordinary boundary triple for S^* and $S^* = \hat{S}$, cf. Lemma 2.10. In particular S^* is an operator and S is densely defined. □

As $(\mathcal{G}, \hat{\Gamma}_0, \hat{\Gamma}_1)$ is an ordinary boundary triple for S^* the operator $S^* \upharpoonright \ker \hat{\Gamma}_1$ is always selfadjoint. For a generalized boundary triple this is no longer the case, as we can see in the following lemma.

Lemma 3.7. *Assume that k is even,* $\operatorname{dom} T \neq \operatorname{dom} S^*$ *and* $\operatorname{ran}(G^{\circledast})^* \cap \mathcal{H}^{-k+1} = \{0\}$ *with $G^{\circledast}$ defined as in Theorem 3.6. Then the operator $A_{\Gamma_1} := T \upharpoonright \ker \Gamma_1$ is essentially selfadjoint, but not selfadjoint.*

Proof. Due to Lemma 2.1 we get

$$\mathcal{H}^2 \dot{+} \ker T = \operatorname{dom} T \subsetneq \operatorname{dom} S^* = \mathcal{H}^2 \dot{+} \ker S^*$$

and hence $\ker T \subsetneq \ker S^*$. Let $\left(\mathcal{G}, \hat{\Gamma}_0, \hat{\Gamma}_1\right)$ be the ordinary boundary triple for S^* from Theorem 3.6 and define the operator $B := S^* \restriction \operatorname{dom} B$ with

$$\operatorname{dom} B := \ker \hat{\Gamma}_1 \supseteq \ker \Gamma_1 = \operatorname{dom} A_{\Gamma_1}.$$

As $(\mathcal{G}, \hat{\Gamma}_0, \hat{\Gamma}_1)$ is an ordinary boundary triple B is a selfadjoint extension of A_{Γ_1}. Furthermore we have $\ker T \subseteq \ker \Gamma_1$, because $\operatorname{dom} T = \mathcal{H}^2 \dot{+} \ker T$ and hence

$$u \in \ker T \implies u_c = 0 \implies \Gamma_1 u = G_0^* u_c = 0 \implies u \in \ker \Gamma_1,$$

cf. the definition of Γ_1 in Theorem 3.5 for the case that k is even. Using $\ker T \subseteq \ker \Gamma_1$ we get

$$\operatorname{dom} A_{\Gamma_1} = \operatorname{dom} T \cap \ker \Gamma_1 = (\mathcal{H}^2 \dot{+} \ker T) \cap \ker \Gamma_1 = \mathcal{H}^2 \cap \ker \Gamma_1 \dot{+} \ker T = \operatorname{dom} S \dot{+} \ker T.$$

Analogously as above we get $\ker S^* \subseteq \ker \hat{\Gamma}_1$ and hence $\operatorname{dom} B = \operatorname{dom} S \dot{+} \ker S^*$. Together with $\ker T \subsetneq \ker S^*$ this implies

$$\operatorname{dom} A_{\Gamma_1} = \operatorname{dom} S \dot{+} \ker T \subsetneq \operatorname{dom} S \dot{+} \ker S^* = \operatorname{dom} B$$

and therefore $A_{\Gamma_1} \subsetneq B$. It remains to show $\overline{A_{\Gamma_1}} \supseteq B$. For this recall that if M and N are two closed subspaces of a Hilbert space, then the following are equivalent:

- $M + N$ is closed and $M \cap N = \{0\}$.
- There exists $\rho > 0$ such that $\rho \sqrt{\|f\|^2 + \|g\|^2} \leq \|f + g\|$ for all $f \in M$ and $g \in N$.

With $\operatorname{dom} S^* = \mathcal{H}^2 \dot{+} \ker S^*$ and $A = S^* \restriction \mathcal{H}^2$ we get in $\mathcal{H} \times \mathcal{H}$ the decomposition

$$\begin{aligned} S^* &= \left\{ \{u, S^* u\} : u \in \operatorname{dom} S^* \right\} \\ &= \left\{ \{u_c + u_s, S^*(u_c + u_s)\} : u_c \in \mathcal{H}^2, u_s \in \ker S^* \right\} \\ &= \left\{ \{u_c, S^* u_c\} + \{u_s, S^* u_s\} : u_c \in \mathcal{H}^2, u_s \in \ker S^* \right\} \\ &= \left\{ \{u_c, A u_c\} + \{u_s, 0\} : u_c \in \operatorname{dom} A, u_s \in \ker S^* \right\} = A \dot{+} \hat{\mathcal{N}}_0(S^*), \end{aligned}$$

where the subspace $\hat{\mathcal{N}}_0(S^*)$ is defined by$\hat{\mathcal{N}}_0(S^*) = \{\{u_s, 0\} : u_s \in \ker S^*\}$. Note that also the sum above is a direct sum (in $\mathcal{H} \times \mathcal{H}$) because the sum in the decomposition of $\operatorname{dom} S^*$ is also a direct sum (in $\mathcal{H}$). Analogously we can decompose T into $T = A \dot{+} \hat{\mathcal{N}}_0(T)$. As $S^* = A \dot{+} \hat{\mathcal{N}}_0(S^*)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$ there exists $\rho > 0$ such that

$$\rho \sqrt{\|\hat{f}\|^2 + \|\hat{g}\|^2} \leq \|\hat{f} + \hat{g}\| \tag{3.5}$$

for all $\hat{f} \in A$ and $\hat{g} \in \hat{\mathcal{N}}_0(S^*)$, cf. the statement mentioned above. As $\overline{\hat{\mathcal{N}}_0(T)}$ is a subset of the closed set $\hat{\mathcal{N}}_0(S^*)$ the estimate (3.5) holds also for all $\hat{g} \in \overline{\hat{\mathcal{N}}_0(T)}$. It follows that $A \dotplus \overline{\hat{\mathcal{N}}_0(T)}$ is closed. Hence we have

$$A \dotplus \hat{\mathcal{N}}_0(T) \subseteq A \dotplus \overline{\hat{\mathcal{N}}_0(T)} \quad \Longrightarrow \quad \overline{A \dotplus \hat{\mathcal{N}}_0(T)} \subseteq A \dotplus \overline{\hat{\mathcal{N}}_0(T)}.$$

Using this we get

$$A \dotplus \hat{\mathcal{N}}_0(S^*) = S^* = \overline{T} = \overline{A \dotplus \hat{\mathcal{N}}_0(T)} \subseteq A \dotplus \overline{\hat{\mathcal{N}}_0(T)} \subseteq A \dotplus \hat{\mathcal{N}}_0(S^*).$$

In particular $\hat{\mathcal{N}}_0(S^*) = \overline{\hat{\mathcal{N}}_0(T)}$ and therefore $\ker S^* = \overline{\ker T}$. Let now $u \in \operatorname{dom} B = \operatorname{dom} S \dotplus \ker S^*$. Hence $u = u_c + u_s$ with $u_c \in \operatorname{dom} S$ and $u_s \in \ker S^*$. Choose a sequence $(u_s^{(n)})_n \subseteq \ker T$ with $u_s^{(n)} \xrightarrow{n\to\infty} u_s$. Then

$$\begin{aligned} \operatorname{dom} A_{\Gamma_1} &= \operatorname{dom} S \dotplus \ker T \ni u_c + u_s^{(n)} \xrightarrow{n\to\infty} u_c + u_s = u, \\ A_{\Gamma_1}(u_c + u_s^{(n)}) &= Su_c + 0 = Su_c + S^* u_s = Bu. \end{aligned}$$

Hence $u \in \operatorname{dom} \overline{A_{\Gamma_1}}$ with $\overline{A_{\Gamma_1}} u = Bu$. Therefore $B \subseteq \overline{A_{\Gamma_1}}$. Together with $A_{\Gamma_1} \subsetneq B$ this implies $A_{\Gamma_1} \neq \overline{A_{\Gamma_1}} = B = B^*$. □

3.2 Singular perturbation with $k = 1$

In this section we consider singular perturbations of the selfadjoint operator A of the form

$$A_\vartheta = A_0 - G\vartheta^{-1}G^*$$

for the case that G maps into $\mathcal{H}^{-1}$, i.e. $k = 1$. The mathematical rigorous definition of this operator is done with the generalized boundary triple from Theorem 3.5. Note that $G_0 = G$ because $k = 1$ implies $j = 0$, cf. (3.2) on page 29. Hence the operator T from (3.3) in Lemma 3.4 is given by

$$Tu := A_0 u - Gh, \qquad \operatorname{dom} T := \{u \in \mathcal{H}^0 : \exists h \in \mathcal{G} \text{ with } A_0 u - Gh \in \mathcal{H}^0\},$$

and the boundary maps of the generalized boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ are given by

$$\begin{aligned} \Gamma_0 &: \operatorname{dom} T \to \mathcal{G}, \qquad u \mapsto h, \\ \Gamma_1 &: \operatorname{dom} T \to \mathcal{G}, \qquad u \mapsto G^* u. \end{aligned}$$

For a symmetric linear relation ϑ in $\mathcal{G}$ with, e.g., $0 \in \rho(\vartheta)$ the operator A_ϑ is defined by

$$A_\vartheta u := Tu, \qquad \operatorname{dom} A_\vartheta := \{u \in \operatorname{dom} T : \Gamma u \in \vartheta\} = \{u \in \operatorname{dom} T : \{h, G^* u\} \in \vartheta\}.$$

As ϑ^{-1} is an operator the "abstract boundary condition" $\{h, G^*u\} \in \vartheta$ in the definition of $\operatorname{dom} A_\vartheta$ can also be written as $h = \vartheta^{-1}G^*u$. Hence the action of A_ϑ is given by

$$A_\vartheta u = Tu = A_0 u - Gh = A_0 u - G\vartheta^{-1}G^*u,$$

which is exactly the desired action. The advantage of defining A_ϑ with the help of $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is that one is now able to apply the whole machinery of generalized boundary triples to analyze the operator A_ϑ.

In the following example we will define Schrödinger operators with δ-interactions on the boundary of a C^∞-domain as singular perturbations of the Laplace operator. In order to get the same setting as in [12] we assume that the boundary is C^∞-smooth, although much weaker assumptions are possible.

Example 3.8. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded C^∞-domain with boundary Σ. Define in $L^2(\mathbb{R}^d)$ the selfadjoint operator

$$Au := (-\Delta + 1)u, \qquad \operatorname{dom} A = H^2(\mathbb{R}^d).$$

As already mentioned in Example 3.3 the chain of Hilbert spaces induced by A coincides with the Sobolev spaces $H^s(\mathbb{R}^d)$, $s \in \mathbb{Z}$. For $h \in L^2(\Sigma)$ and $\varphi \in H^1(\mathbb{R}^d)$ define

$$\big(h\delta_\Sigma\big)\varphi := (h, \operatorname{tr}^1_\Sigma \varphi)_{L^2(\Sigma)}.$$

As Σ is a manifold of codimension 1 we get with Lemma 2.21 (for $\varepsilon = \frac{1}{2}$) $h\delta_\Sigma \in H^{-1}(\mathbb{R}^d)$. Moreover the operator

$$G : L^2(\Sigma) \to H^{-1}(\mathbb{R}^d), \qquad h \mapsto h\delta_\Sigma,$$

is continuous with $\|G\| \le \|\operatorname{tr}^1_\Sigma\|$, injective and satisfies $\operatorname{ran} G \cap L^2(\mathbb{R}^d) = \{0\}$. Hence G satisfies all required conditions in (3.1) on page 29 for $\mathcal{G} = L^2(\Sigma)$ and $k = 1$. The operators S and T and the boundary maps from Lemma 3.4 and Theorem 3.5 are given by

$$\begin{aligned} Su &= (-\Delta + 1)u, & \operatorname{dom} S &= \{u \in H^2(\mathbb{R}^d) : \operatorname{tr}^2_\Sigma u = 0\}, \\ Tu &= (-\Delta + 1)u - h\delta_\Sigma, & \operatorname{dom} T &= \{u \in L^2(\mathbb{R}^d) : \exists h \in L^2(\Sigma) \\ & & &\qquad \text{with } (-\Delta + 1)u - h\delta_\Sigma \in L^2(\mathbb{R}^d)\}, \end{aligned}$$

and

$$\begin{aligned} \Gamma_0 &: \operatorname{dom} T \to L^2(\Sigma), \qquad u \mapsto h, \\ \Gamma_1 &: \operatorname{dom} T \to L^2(\Sigma), \qquad u \mapsto \operatorname{tr}^1_\Sigma u. \end{aligned} \tag{3.6}$$

Note that Γ_1 is well defined because $\operatorname{dom} T \subseteq H^1(\mathbb{R}^d)$, cf. Lemma 3.4, and that

$$\langle Gh, u\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} = \langle h\delta_\Sigma, u\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} = \int_\Sigma h \cdot \overline{\operatorname{tr}^1_\Sigma u}\, ds = \langle h, \operatorname{tr}^1_\Sigma u\rangle_{L^2(\Sigma)}$$

holds for all $u \in H^1(\mathbb{R}^d)$ and $h \in L^2(\Sigma)$, i.e. $G^* u = \operatorname{tr}^1_\Sigma u$. Hence if we assume that the parameter ϑ is in $\mathbb{R} \setminus \{0\}$ (it is also possible to allow a function ϑ on Σ with $\vartheta^{-1} \in L^\infty(\Sigma)$) the operator A_ϑ is given by

$$\begin{aligned} A_\vartheta u = Tu = (-\Delta + 1)u - h\delta_\Sigma &= (-\Delta+1)u - \vartheta^{-1} \operatorname{tr}^1_\Sigma u \cdot \delta_\Sigma, \\ \operatorname{dom} A_\vartheta = \{u \in \operatorname{dom} T : \vartheta \Gamma_0 u = \Gamma_1 u\} &= \{u \in \operatorname{dom} T : \vartheta h = \operatorname{tr}^1_\Sigma u\}. \end{aligned}$$

In particular the action of A_ϑ coincides (up to the constant $+1$) for $\vartheta = \alpha$ with the one given in (1.1) on page 1, which was our first formal definition of a Schrödinger operator with δ-interaction of strength $\frac{1}{\alpha}$ on Σ.

A consequence of the next Lemma is that the operators A_ϑ constructed above in Example 3.8 coincide with those which are known in the literature as Schrödinger operators with δ-interactions on manifolds of codimension 1.

Lemma 3.9. *The generalized boundary triple* $(L^2(\Sigma), \Gamma_0, \Gamma_1)$ *with the boundary maps* Γ_0 *and* Γ_1 *from* (3.6) *coincides with the one given in Proposition 3.2 in [12].*

Proof. Note that every $u \in \operatorname{dom} T$ can be written as $u = u_i \oplus u_e$ with $u_i \in L^2(\Omega)$ and $u_e \in L^2(\Omega^c)$. Next consider the operator

$$\begin{aligned} \tilde{T} u &= (-\Delta+1)u_i \oplus (-\Delta+1)u_i, \\ \operatorname{dom} \tilde{T} &= \{u = u_i \oplus u_e \in H^{3/2}_\Delta(\Omega) \oplus H^{3/2}_\Delta(\Omega^c) : \operatorname{tr}^1_{\partial\Omega} u_i = \operatorname{tr}^1_{\partial\Omega^c} u_e\} \end{aligned}$$

with $\operatorname{tr}^1_{\partial\Omega}$ and $\operatorname{tr}^1_{\partial\Omega^c}$ as defined in the text before Lemma 2.25 and

$$\begin{aligned} H^{3/2}_\Delta(\Omega) &:= \{u_i \in H^{3/2}(\Omega) : \Delta u_i \in L^2(\Omega)\} \quad \text{and} \\ H^{3/2}_\Delta(\Omega^c) &:= \{u_e \in H^{3/2}(\Omega^c) : \Delta u_e \in L^2(\Omega^c)\}. \end{aligned}$$

Note that $\operatorname{tr}^1_{\partial\Omega} u_i = \operatorname{tr}^1_{\partial\Omega^c} u_e$ implies $u \in H^1(\mathbb{R}^d)$, cf. Lemma 2.25. We define now the boundary maps $\tilde{\Gamma}_0, \tilde{\Gamma}_1 : \operatorname{dom} \tilde{T} \to L^2(\Sigma)$ by

$$\begin{aligned} \tilde{\Gamma}_0 u &:= \partial_{\nu_e} u_e|_\Sigma + \partial_{\nu_i} u_i|_\Sigma \quad \text{and} \\ \tilde{\Gamma}_1 u &:= \operatorname{tr}^1_\Sigma u, \end{aligned}$$

where ∂_{ν_e} and ∂_{ν_i} denote the normal derivatives with the normal vector ν_e and ν_i pointing outwards the domains Ω^c and Ω, respectively (i.e. they point in opposite directions). According to Proposition 3.2 in [12] the triple $(L^2(\Sigma), \tilde{\Gamma}_0, \tilde{\Gamma}_1)$ is a generalized boundary triple for the closure of $\tilde{T}$. Hence we get with Greens identity for every $\varphi \in H^2(\mathbb{R}^d) \subseteq \ker \tilde{\Gamma}_0$ and $u \in \operatorname{dom} \tilde{T}$

$$\begin{aligned} \langle (-\Delta+1)u, \varphi \rangle_{H^{-2}(\mathbb{R}^d), H^2(\mathbb{R}^d)} &= \langle u, (-\Delta+1)\varphi \rangle_{L^2(\mathbb{R}^d)} = \langle u, \tilde{T}\varphi \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \tilde{T} u, \varphi \rangle_{L^2(\mathbb{R}^d)} - \langle \tilde{\Gamma}_1 u, \tilde{\Gamma}_0 \varphi \rangle_{L^2(\Sigma)} + \langle \tilde{\Gamma}_0 u, \tilde{\Gamma}_1 \varphi \rangle_{L^2(\Sigma)} \\ &= \langle \tilde{T} u, \varphi \rangle_{L^2(\mathbb{R}^d)} + \langle \tilde{\Gamma}_0 u, \operatorname{tr}^1_\Sigma u \rangle_{L^2(\Sigma)} \\ &= \langle \tilde{T} u, \varphi \rangle_{L^2(\mathbb{R}^d)} + \langle (\tilde{\Gamma}_0 u)\delta_\Sigma, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)}. \end{aligned}$$

Therefore we get

$$\langle(-\Delta+1)u-(\tilde{\Gamma}_0 u)\delta_\Sigma,\varphi\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)}=\langle\tilde{T}u,\varphi\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)}.$$

As this identity holds for all $\varphi\in H^2(\mathbb{R}^d)$ we get $(-\Delta+1)u-(\tilde{\Gamma}_0 u)\delta_\Sigma=\tilde{T}u\in L^2(\mathbb{R}^d)$ and $u\in\operatorname{dom}T$. Hence $\tilde{T}\subseteq T$, $\tilde{\Gamma}_0\subseteq\Gamma_0$ and $\tilde{\Gamma}_1\subseteq\Gamma_1$. In particular we get $\ker\tilde{T}\subseteq\ker T$. As $\tilde{\Gamma}_0$ maps $\ker\tilde{T}$ isomorphically to $L^2(\Sigma)$, $\tilde{\Gamma}_0$ is a restriction of Γ_0 and Γ_0 maps $\ker T$ isomorphically to $L^2(\Sigma)$ both kernels coincide. Hence

$$\operatorname{dom}\tilde{T}=H^2(\mathbb{R}^d)\dotplus\ker\tilde{T}=H^2(\mathbb{R}^d)\dotplus\ker T=\operatorname{dom}T.$$

Therefore $\tilde{T}=T$ and the triples $(L^2(\Sigma),\tilde{\Gamma}_0,\tilde{\Gamma}_1)$ and $(L^2(\Sigma),\Gamma_0,\Gamma_1)$ coincide, i.e. our approach for δ-interactions on hypersurfaces coincides with the approach presented in [12]. □

Recall that the Schrödinger operators constructed in [12] with a generalized boundary triple can be constructed alternatively with a semi-bounded sesquilinear form, cf. Proposition 3.7 in [12] for more details.

At the end of this section we will provide an explicit representation of the operator S^* which will be used later on in Section 4.2.

Theorem 3.10. *The adjoint operator of S from Example 3.8 is given by*

$$\begin{aligned}S^*u&=(-\Delta+1)u-h\delta_\Sigma,\\ \operatorname{dom}S^*&=\{u\in L^2(\mathbb{R}^d):\exists h\in H^{-3/2}(\Sigma)\text{ with }(-\Delta+1)u-h\delta_\Sigma\in L^2(\mathbb{R}^d)\}\end{aligned}$$

where the distribution $h\delta_\Sigma\in H^{-2}(\mathbb{R}^d)$ for $h\in H^{-3/2}(\Sigma)$ is given by

$$\big(h\delta_\Sigma\big)(\varphi):=(h,\operatorname{tr}^2_\Sigma\varphi)_{H^{-3/2}(\Sigma),H^{3/2}(\Sigma)}. \tag{3.7}$$

Proof. As $k=1$ we have $j=0$. Hence the space $\mathcal{G}^+$ defined in the proof of Theorem 3.5 and its dual space $\mathcal{G}^-$ are given by

$$\mathcal{G}^+=\operatorname{ran}\big(G^*\upharpoonright H^2(\mathbb{R}^d)\big)=\operatorname{ran}\big(\operatorname{tr}^1_\Sigma\upharpoonright H^2(\mathbb{R}^d)\big)=H^{3/2}(\Sigma)\qquad\text{and}\qquad\mathcal{G}^-=H^{-3/2}(\Sigma),$$

cf. Lemma 2.20. As $G^*u=\operatorname{tr}^1_\Sigma u$ for all $u\in H^1(\mathbb{R}^d)$ the operator $G^{\circledast}$ defined in Theorem 3.6 is given by $G^{\circledast}=\operatorname{tr}^2_\Sigma:H^2(\mathbb{R}^d)\to H^{3/2}(\Sigma)$ and due to

$$\begin{aligned}\langle(G^{\circledast})^*h,\varphi\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)}&=\langle h,G^{\circledast}\varphi\rangle_{H^{-3/2}(\Sigma),H^{3/2}(\Sigma)}\\&=\langle h,\operatorname{tr}^2_\Sigma\varphi\rangle_{H^{-3/2}(\Sigma),H^{3/2}(\Sigma)}=\langle h\delta_\Sigma,\varphi\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)}\end{aligned}$$

for all $h\in H^{-3/2}(\Sigma)$ and $\varphi\in H^2(\mathbb{R}^d)$ the operator $(G^{\circledast})^*:H^{-3/2}(\Sigma)\to H^{-2}(\mathbb{R}^d)$ satisfies $(G^{\circledast})^*h=h\delta_\Sigma$ with $h\delta_\Sigma$ defined as in (3.7). The facts that $h\delta_\Sigma$ belongs to $H^{-2}(\mathbb{R}^d)$ and that $\operatorname{ran}(G^{\circledast})^*\cap L^2(\mathbb{R}^d)=\{0\}$ holds can be seen analogously as in Lemma 2.21:

With Lemma 2.20 we obtain for $h \in H^{-3/2}(\Sigma)$

$$\begin{aligned}\big|\big(h\delta_\Sigma\big)(\varphi)\big| &= |(h, \operatorname{tr}^2_\Sigma \varphi)_{H^{-3/2}(\Sigma),H^{3/2}(\Sigma)}| \\ &\le \|h\|_{H^{-3/2}(\Sigma)} \cdot \|\operatorname{tr}^2_\Sigma \varphi\|_{H^{3/2}(\Sigma)} \le \|h\|_{H^{-3/2}(\Sigma)} \cdot \|\operatorname{tr}^2_\Sigma\| \cdot \|\varphi\|_{H^2(\Sigma)}\end{aligned}$$

and hence $h\delta_\Sigma \in H^{-2}(\mathbb{R}^d)$ with $\|h\delta_\Sigma\|_{H^{-2}(\mathbb{R}^d)} \le \|\operatorname{tr}^2_\Sigma\| \cdot \|h\|_{H^{-3/2}(\Sigma)}$. Furthermore there exists a sequence $(\varphi_n)_n \subset \mathscr{S}(\mathbb{R}^d)$ with $\|\varphi_n\|_{L^2(\mathbb{R}^d)} \le \|\varphi_n\|_{H^{1/2}(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0$ and $\varphi_n = 1$ on Σ. Hence we get for every $\psi \in C_0^\infty(\mathbb{R}^d)$ and every $h\delta_\Sigma \in \operatorname{ran}(G^{\circledast})^* \cap L^2(\mathbb{R}^d)$

$$\begin{aligned}\big(h\delta_\Sigma\big)(\psi) &= (h, \operatorname{tr}^2_\Sigma \psi)_{H^{-3/2}(\Sigma),H^{3/2}(\Sigma)} \\ &= (h, \operatorname{tr}^2_\Sigma \psi\varphi_n)_{H^{-3/2}(\Sigma),H^{3/2}(\Sigma)} = \big(h\delta_\Sigma\big)(\psi\varphi_n) = (h\delta_\Sigma, \psi\varphi_n)_{L^2(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0,\end{aligned}$$

i.e. $h\delta_\Sigma = 0$ and therefore $h = 0$. The representation of S^* follows now with Theorem 3.6. □

3.3 Singular perturbation with $k = 2$

If G maps into $\mathcal{H}^{-2}$ it is not possible to define a selfadjoint operator associated to

$$\mathcal{A}_\vartheta = A_0 - G\vartheta^{-1}G^*$$

except for the case that the perturbation $G\vartheta^{-1}G^*$ is absent. The reason is that the domain of A_ϑ would be to small for selfadjointness. To see this note that $k = 2$ implies $j = 0$, cf. (3.2). Hence $G_0 = G$ and the operator T from (3.3) in Lemma 3.4 is again given by

$$Tu := A_0u - Gh, \qquad \operatorname{dom} T := \{u \in \mathcal{H}^0 : \exists h \in \mathcal{G} \text{ with } A_0u - Gh \in \mathcal{H}^0\}.$$

Obviously a realization A_ϑ of $\mathcal{A}_\vartheta$ in $\mathcal{H}^0$ has to be a restriction of T with $h = \vartheta^{-1}G^*u$. On the other hand $\operatorname{dom} G^* = \mathcal{H}^k = \mathcal{H}^2$ and hence

$$\operatorname{dom} A_\vartheta \subseteq \mathcal{H}^2 \cap \operatorname{dom} T \subseteq \mathcal{H}^2.$$

But $u \in \mathcal{H}^2$ implies $A_0u \in \mathcal{H}^0$ and hence (due to the uniqueness of h)

$$A_0u - G\vartheta^{-1}G^*u = A_\vartheta u = Tu = A_0u = Au.$$

So A_ϑ is either A itself or a restriction of A. In the second case A_ϑ is only symmetric because A is already selfadjoint.

A way to get nevertheless selfadjoint perturbations of A which are at least quite similar to our original aim is to use a regularization trick. Recall that we can decompose $\operatorname{dom} T$ into $\operatorname{dom} T = \mathcal{H}^2 \dot{+} \ker T$, cf. Lemma 2.1. With the (nonorthogonal) projection P defined by

$$P : \operatorname{dom} T \to \mathcal{H}^2, \qquad u = u_c + u_s \mapsto u_c,$$

we modify the expression $\mathcal{A}_\vartheta$ slightly to

$$\tilde{\mathcal{A}}_\vartheta = A_0 + G\vartheta^{-1}G^*P.$$

The boundary maps of the generalized boundary triple are given in the case $k = 2$ by

$$\begin{aligned}\Gamma_0 &: \operatorname{dom} T \to \mathcal{G}, \qquad u \mapsto h,\\ \Gamma_1 &: \operatorname{dom} T \to \mathcal{G}, \qquad u \mapsto G^* u_c.\end{aligned}$$

For a symmetric linear relation ϑ in $\mathcal{G}$ with $0 \in \rho(\vartheta)$ the operator A_ϑ is now defined by

$$A_\vartheta u := Tu, \qquad \operatorname{dom} A_\vartheta := \{u \in \operatorname{dom} T : \Gamma u \in \vartheta\} = \{u \in \operatorname{dom} T : \{h, G^* u_c\} \in \vartheta\}.$$

Hence the action of A_ϑ coincides with the action of the expression $\tilde{\mathcal{A}}_\vartheta$:

$$A_\vartheta u = Tu = A_0 u - Gh = A_0 u - G\vartheta^{-1}G^* u_c.$$

An example for singular perturbations of selfadjoint operators with $k = 2$ are again Schrödinger operators with δ-interactions, but now supported on manifolds of codimension 2 or 3. We will investigate this example in detail in Chapter 4.

3.4 The supersingular case $k > 2$

If $k > 2$ the task to give a meaning to the expression $\mathcal{A}_\vartheta = A_0 - G\vartheta^{-1}G^*$ is more challenging. The reason is that it is not possible to give any meaningful sense to the expression $\mathcal{A}_\vartheta$ as an operator in the Hilbert space $\mathcal{H}^0$ except for the case that it is a restriction of A. Indeed, if $v := \mathcal{A}_\vartheta u = A_0 u - G\vartheta^{-1}G^* u$ would belong to $\mathcal{H}^0$ for some $u \in \mathcal{H}^0$ this would imply

$$\operatorname{ran} G \ni G\vartheta^{-1}G^* u = A_0 u - v \in \mathcal{H}^{-2} \subseteq \mathcal{H}^{-k+1}.$$

But G is assumed to be injective with $\operatorname{ran} G \cap \mathcal{H}^{-k+1} = \{0\}$. This means $\vartheta^{-1}G^* u = 0$ and hence $A_0 u \in \mathcal{H}^0$ or, equivalently, $u \in \mathcal{H}^2$. Therefore every realization of $\mathcal{A}_\vartheta$ must be a restriction of A if we are limited to the space $\mathcal{H}^0$.

Hence if one wants to construct a selfadjoint realization A_ϑ of the expression $\mathcal{A}_\vartheta$ it is necessary to extend the space, i.e. we consider a space which contains $\mathcal{H}^0$. Of course one could consider the space $\mathcal{H}^{-k}$, but this space is much larger than necessary. Therefore we will consider a smaller space, just large enough for our purpose. In order to do spectral analysis this space should be chosen in such a way that $\operatorname{ran}(A_\vartheta - \lambda)^{-1}$ is contained for all suitable λ. Inspired by the formal calculation

$$\begin{aligned}(A_\vartheta - \lambda)^{-1} &= (A - \lambda)^{-1}\left[A_\vartheta - \lambda + G\vartheta^{-1}G^*\right](A_\vartheta - \lambda)^{-1}\\ &= (A - \lambda)^{-1}\left[I + G\vartheta^{-1}G^*(A_\vartheta - \lambda)^{-1}\right]\\ &= (A - \lambda)^{-1} + (A - \lambda)^{-1}G\vartheta^{-1}G^*(A_\vartheta - \lambda)^{-1}\end{aligned}$$

a possible choice might be $\mathcal{H}^0 + (A-\lambda)^{-1} \operatorname{ran} G$, but this space is λ-dependent. Using

$$\begin{aligned} I - \lambda^j A^{-j} &= (I - \lambda A^{-1}) \left(I + \lambda A^{-1} + \ldots + \lambda^{j-1} A^{-(j-1)} \right) \\ &= (A - \lambda) \left(A^{-1} + \lambda A^{-2} + \ldots + \lambda^{j-1} A^{-j} \right) \end{aligned}$$

we can write the resolvent $(A-\lambda)^{-1}$ as

$$\begin{aligned} (A-\lambda)^{-1} &= (A-\lambda)^{-1} \left[(A-\lambda) \left(A^{-1} + \lambda A^{-2} + \ldots + \lambda^{j-1} A^{-j} \right) + \lambda^j A^{-j} \right] \\ &= \left(A^{-1} + \lambda A^{-2} + \ldots + \lambda^{j-1} A^{-j} \right) + \lambda^j (A-\lambda)^{-1} A^{-j}. \end{aligned}$$

Hence

$$(A_\vartheta - \lambda)^{-1} = (A-\lambda)^{-1} + \left(A^{-1} + \ldots + \lambda^{j-1} A^{-j} + \lambda^j (A-\lambda)^{-1} A^{-j} \right) G \vartheta^{-1} G^* (A_\vartheta - \lambda)^{-1}.$$

Keeping in mind that $\operatorname{ran}(A-\lambda)^{-1} A^{-j} G \subseteq \mathcal{H}^{-k+2j+2} \subseteq \mathcal{H}^0$ we get

$$\operatorname{ran}(A_\vartheta - \lambda)^{-1} \subseteq \mathcal{H}^0 + A^{-1} \operatorname{ran} G + \ldots + A^{-j} \operatorname{ran} G. \tag{3.8}$$

(Recall that this calculation is just a formal calculation. In particular $A^{-1} \operatorname{ran} G$ is not well defined because $\operatorname{ran} G \subseteq \mathcal{H}^{-k}$ and $\operatorname{dom} A^{-1} \subseteq \mathcal{H}^0$. This problem can be resolved by replacing A^{-1} by A^{-1}_{-k+2}. In order to keep notation simple we will omit the index $-k+2$ as it will be clear from the context which operator is meant.) In particular the right hand side of (3.8) is independent of λ. For technical reasons it is better to consider the space $\tilde{\mathcal{K}} := \mathcal{H}^0 \dotplus \sum_{l=1}^{2j} A^{-l} \operatorname{ran} G$, which we will call in the following the *extension space*. Note that $A^{-l} G$ provides an isomorphism between $\mathcal{G}$ and $A^{-l} \operatorname{ran} G$. Hence the space $\mathcal{H}^0 \dotplus \sum_{l=1}^{2j} A^{-l} \operatorname{ran} G$ is isomorphic to the space $\mathcal{K} := \mathcal{H}^0 \times \mathcal{G}^j \times \mathcal{G}^j$, which we will call in the following the *model space*. In the next subsection we will equip $\mathcal{K}$ with an inner product such that it becomes a Krein space and construct a boundary triple which allows us to define a certain linear relation H_Θ. Afterwards, in Subsection 3.4.2, we motivate by an example how this linear relation H_Θ can be seen as a selfadjoint realization of the formal expression $\mathcal{A}_\vartheta$.

3.4.1 A boundary triple in the model space $\mathcal{K}$

Consider the spaces $\mathfrak{h} := \mathcal{G}^j \times \mathcal{G}^j$ and $\mathcal{K} := \mathcal{H}^0 \times \mathfrak{h} = \mathcal{H}^0 \times \mathcal{G}^j \times \mathcal{G}^j$. We write the elements of $\mathfrak{h}$ and $\mathcal{K}$ as

$$\begin{bmatrix} f \\ f' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u \\ f \\ f' \end{bmatrix}$$

with $f, f' \in \mathcal{G}^j$ and $u \in \mathcal{H}^0$, respectively. Sometimes it is more convenient to write these elements as row vectors, e.g. as

$$(u; f; f') = (u; f_1, \ldots, f_j; f'_1, \ldots, f'_j)$$

with $u \in \mathcal{H}^0$ and $f, f' \in \mathcal{G}^j$ or $f_1, \dots, f_j, f'_1, \dots, f'_j \in \mathcal{G}$. Here a comma is used to separate the different entries from $\mathcal{G}$, whereas a semicolon is used to distinguish the entries from $\mathcal{H}^0$ and $\mathcal{G}^j$. We equip the space $\mathfrak{h}$ with the inner product $[\![\cdot,\cdot]\!]_{\mathfrak{h}}$ defined by

$$\left[\!\!\left[\begin{bmatrix} f \\ f' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right]\!\!\right]_{\mathfrak{h}} := \sum_{l=1}^{j} \left(\langle f_l, g'_{j-l+1} \rangle_{\mathcal{G}} + \langle f'_l, g_{j-l+1} \rangle_{\mathcal{G}} \right)$$

$$= \left\langle B \begin{bmatrix} f \\ f' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle_{\mathcal{G}^j \times \mathcal{G}^j}, \qquad \text{with} \quad B := \begin{bmatrix} & & I_{\mathcal{G}} \\ & \iddots & \\ I_{\mathcal{G}} & & \end{bmatrix} \in \mathcal{L}(\mathcal{G}^j \times \mathcal{G}^j),$$

where $I_{\mathcal{G}}$ denotes the identity in $\mathcal{G}$. Furthermore, we equip the space $\mathcal{K}$ with the inner product $[\![\cdot,\cdot]\!]_{\mathcal{K}}$ defined by

$$\left[\!\!\left[\begin{bmatrix} u \\ f \\ f' \end{bmatrix}, \begin{bmatrix} v \\ g \\ g' \end{bmatrix} \right]\!\!\right]_{\mathcal{K}} := \langle u, v \rangle_{\mathcal{H}^0} + \left[\!\!\left[\begin{bmatrix} f \\ f' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right]\!\!\right]_{\mathfrak{h}}.$$

In this way $(\mathfrak{h}, [\![\cdot,\cdot]\!]_{\mathfrak{h}})$ and $(\mathcal{K}, [\![\cdot,\cdot]\!]_{\mathcal{K}})$ become Krein spaces.

Recall the definition of the operator T in equation (3.3) in Lemma 3.4 and the boundary maps Γ_0 and Γ_1 from Theorem 3.5. With the help of these objects we define the linear relation $\tilde{T}$ in $\mathcal{K}$ (i.e. a linear subspace of $\mathcal{K} \times \mathcal{K}$) by

$$\tilde{T} := \left\{ \left\{ \begin{bmatrix} u \\ f \\ g \end{bmatrix}, \begin{bmatrix} Tu \\ f' \\ g' \end{bmatrix} \right\} : \begin{array}{c} u \in \operatorname{dom} T, \\ f, f', g, g' \in \mathcal{G}^j \end{array}, \begin{array}{c} f'_l = f_{l+1} \\ g'_l = g_{l+1} \end{array} \text{ for } 1 \le l < j, \begin{array}{c} f_1 = \Gamma_1 u \\ g'_j = \Gamma_0 u \end{array} \right\}.$$

Note that there are no restrictions concerning the element f'_j. Hence $f'_j \in \mathcal{G}$ is arbitrary and therefore $\operatorname{mul} \tilde{T} = \operatorname{span}\{(0;0,\dots,0,f'_j;0,\dots,0) : f'_j \in \mathcal{G}\} \neq \{0\}$. Due to its important role and to distinguish it from the other components we will denote the component f'_j in the following by φ.

Define now the boundary mappings $\tilde{\Gamma}_0 : \tilde{T} \to \mathcal{G}$ and $\tilde{\Gamma}_1 : \tilde{T} \to \mathcal{G}$ by

$$\tilde{\Gamma}_0 \Big\{ (u; \Gamma_1 u, f_2, \dots, f_j; g_1, \dots, g_j), (Tu; f_2, \dots, f_j, \varphi; g_2, \dots, g_j, \Gamma_0 u) \Big\} = g_1,$$
$$\tilde{\Gamma}_1 \Big\{ (u; \Gamma_1 u, f_2, \dots, f_j; g_1, \dots, g_j), (Tu; f_2, \dots, f_j, \varphi; g_2, \dots, g_j, \Gamma_0 u) \Big\} = \varphi.$$

As usual we define $\tilde{\Gamma} := \binom{\tilde{\Gamma}_0}{\tilde{\Gamma}_1} : \tilde{T} \to \mathcal{G} \times \mathcal{G}$. For the next theorem we need the γ-field $\gamma(\lambda)$ and the Weyl function $M(\lambda)$ of the generalized boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ from Theorem 3.5.

Theorem 3.11. *$\tilde{S} := \ker \tilde{\Gamma}$ is a closed symmetric relation in $\mathcal{K}$ with $\tilde{S}^+ = \tilde{T}$ and $(\mathcal{G}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$ is an ordinary boundary triple for $\tilde{S}^+$. The linear relation $H_0 := \ker \tilde{\Gamma}_0$ is selfadjoint in $\mathcal{K}$,*

$\rho(H_0) = \rho(A)$, $\sigma_p(H_0) = \sigma_p(A)$ and $\sigma_c(H_0) = \sigma_c(A)$. Moreover the matrix representation

$$(H_0 - \lambda)^{-1} = \left[\begin{array}{c|c:c|c} (A-\lambda)^{-1} & 0 \;\cdots\; 0 & 0 & \gamma(\lambda)\begin{bmatrix}\lambda^{j-1} & \cdots & 1\end{bmatrix} \\ \hline \gamma(\bar{\lambda})^* \begin{bmatrix} 1 \\ \vdots \\ \lambda^{j-1} \end{bmatrix} & \begin{array}{c} 0 \;\cdots\; 0 \\ \hline J_\lambda \end{array} & \begin{array}{c} 0 \\ \hline 0 \\ \vdots \\ 0 \end{array} & M(\lambda)\Lambda_\lambda \\ \hline \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} 0 \;\cdots\; 0 \\ \hline J_\lambda \end{array} & \begin{array}{c} 0 \\ \hline 0 \\ \vdots \\ 0 \end{array} & 0 \end{array}\right]$$

holds for all $\lambda \in \rho(H_0)$ with the matrices $J_\lambda \in \mathcal{G}^{(j-1)\times(j-1)}$ and $\Lambda_\lambda \in \mathcal{G}^{j\times j}$ defined by

$$J_\lambda := \begin{bmatrix} 1 & & & \\ -\lambda & \ddots & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ \lambda & \ddots & & \\ \vdots & \ddots & \ddots & \\ \lambda^{j-2} & \cdots & \lambda & 1 \end{bmatrix} \quad \textit{and} \quad \Lambda_\lambda := \begin{bmatrix} \lambda^{j-1} & \cdots & \lambda & 1 \\ \lambda^{j} & \cdots & \lambda^2 & \lambda \\ \vdots & & \vdots & \vdots \\ \lambda^{2j-2} & \cdots & \lambda^{j} & \lambda^{j-1} \end{bmatrix}.$$

Proof. At first we will show that $\tilde{S} := \ker \tilde{\Gamma}$ is a closed symmetric relation in $\mathcal{K}$ with $\tilde{S}^+ = \tilde{T}$ and that $(\mathcal{G}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$ is an ordinary boundary triple for $\tilde{S}^+$. According to Lemma 2.10 it suffices to prove the following items:

(i) $\operatorname{ran} \tilde{\Gamma} = \mathcal{G} \times \mathcal{G}$.

(ii) There exists $\lambda \in \mathbb{R}$ such that $\operatorname{ran}(H_0 - \lambda) = \mathcal{K}$, i.e. for every $V \in \mathcal{K}$ there exist $U, U' \in \mathcal{K}$ with $\{U, U'\} \in \tilde{T}$, $\tilde{\Gamma}_0\{U, U'\} = 0$ and $U' - \lambda U = V$.

(iii) For all $\{U, U'\}, \{V, V'\} \in \tilde{T}$ holds

$$[\![U', V]\!]_{\mathcal{K}} - [\![U, V']\!]_{\mathcal{K}} = \langle \tilde{\Gamma}_1\{U, U'\}, \tilde{\Gamma}_0\{V, V'\}\rangle_{\mathcal{G}} - \langle \tilde{\Gamma}_1\{U, U'\}, \tilde{\Gamma}_0\{V, V'\}\rangle_{\mathcal{G}}.$$

Let $\lambda \in \rho(A)$. Let $V = (v; h; k) \in \mathcal{K} = \mathcal{H}^0 \times \mathcal{G}^j \times \mathcal{G}^j$ be arbitrary. Set $g_1 := 0$ and

$$\begin{bmatrix} g_2 \\ \vdots \\ \vdots \\ g_j \end{bmatrix} := J_\lambda \begin{bmatrix} k_1 \\ \vdots \\ \vdots \\ k_{j-1} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -\lambda & \ddots & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}^{-1} \begin{bmatrix} k_1 \\ \vdots \\ \vdots \\ k_{j-1} \end{bmatrix}.$$

In particular we have $g_j = \sum_{r=1}^{j-1} \lambda^{j-r-1} k_r$. Moreover define

$$u := (A-\lambda)^{-1}v + \gamma(\lambda)(k_j + \lambda g_j) = (A-\lambda)^{-1}v + \gamma(\lambda)\sum_{r=1}^{j} \lambda^{j-r} k_r \in \operatorname{dom} T,$$

$$f_1 := \Gamma_1 u = \Gamma_1 (A-\lambda)^{-1} v + \Gamma_1 \gamma(\lambda) \sum_{r=1}^{j} \lambda^{j-r} k_r = \gamma(\overline{\lambda})^* v + M(\lambda) \sum_{r=1}^{j} \lambda^{j-r} k_r,$$

$$\begin{bmatrix} f_2 \\ \vdots \\ \vdots \\ f_j \end{bmatrix} := J_\lambda \left(\begin{bmatrix} h_1 \\ \vdots \\ \vdots \\ h_{j-1} \end{bmatrix} + \begin{bmatrix} \lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & & & \\ -\lambda & \ddots & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} h_1 \\ \vdots \\ \vdots \\ h_{j-1} \end{bmatrix} + \begin{bmatrix} \lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

and $\varphi := h_j + \lambda f_j$. Hence if we set $U := (u; f; g)$ and

$$U' = (u'; f'; g') := (Tu; f_2, \ldots, f_j, \varphi; g_2, \ldots, g_j, \Gamma_0 u)$$

we obtain

$$\{U, U'\} = \Big\{ (u; f_1, \ldots, f_j; g_1, \ldots, g_j), (Tu; f_2, \ldots, f_j, \varphi; g_2, \ldots, g_j, \Gamma_0 u) \Big\} \in \tilde{T}$$

according to the definition of the linear relation $\tilde{T}$. Moreover we get $\tilde{\Gamma}_0\{U, U'\} = g_1 = 0$ and

$$u' - \lambda u = (T-\lambda)u = (T-\lambda)(A-\lambda)^{-1}v + (T-\lambda)\gamma(\lambda)(k_j + \lambda g_j) = v$$

because $\operatorname{ran}\gamma(\lambda) \subseteq \ker(T-\lambda)$. Due to $\varphi = h_j + \lambda f_j$ we obtain further

$$f' - \lambda f = \begin{bmatrix} f_2 \\ \vdots \\ f_j \\ \varphi \end{bmatrix} + \begin{bmatrix} -\lambda f_1 \\ \vdots \\ \vdots \\ -\lambda f_j \end{bmatrix} = \begin{bmatrix} f_2 - \lambda f_1 \\ \vdots \\ f_j - \lambda f_{j-1} \\ h_j \end{bmatrix}.$$

As

$$\begin{aligned} \begin{bmatrix} f_2 - \lambda f_1 \\ \vdots \\ \vdots \\ f_j - \lambda f_{j-1} \end{bmatrix} &= \begin{bmatrix} -\lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} f_2 \\ f_3 - \lambda f_2 \\ \vdots \\ f_j - \lambda f_{j-1} \end{bmatrix} \\ &= \begin{bmatrix} -\lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & & & \\ -\lambda & \ddots & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \begin{bmatrix} f_2 \\ \vdots \\ \vdots \\ f_j \end{bmatrix} = \begin{bmatrix} -\lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \left(\begin{bmatrix} h_1 \\ \vdots \\ \vdots \\ h_{j-1} \end{bmatrix} + \begin{bmatrix} \lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = \begin{bmatrix} h_1 \\ \vdots \\ \vdots \\ h_{j-1} \end{bmatrix} \end{aligned}$$

we conclude $f' - \lambda f = h$. Moreover we get with $\Gamma_0 u = \Gamma_0 (A-\lambda)^{-1} v + \Gamma_0 \gamma(\lambda)(k_j + \lambda g_j) = k_j + \lambda g_j$

$$g' - \lambda g = \begin{bmatrix} g_2 \\ \vdots \\ g_j \\ \Gamma_0 u \end{bmatrix} - \lambda \begin{bmatrix} g_1 \\ \vdots \\ \vdots \\ g_j \end{bmatrix} = \begin{bmatrix} g_2 - \lambda g_1 \\ \vdots \\ g_j - \lambda g_{j-1} \\ k_j \end{bmatrix}.$$

Due to $g_1 = 0$ and

$$\begin{bmatrix} g_2 - \lambda g_1 \\ \vdots \\ \vdots \\ g_j - \lambda g_{j-1} \end{bmatrix} = \begin{bmatrix} g_2 \\ g_3 - \lambda g_2 \\ \vdots \\ g_j - \lambda g_{j-1} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -\lambda & \ddots & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \begin{bmatrix} g_2 \\ g_3 \\ \vdots \\ g_j \end{bmatrix} = \begin{bmatrix} k_1 \\ \vdots \\ \vdots \\ k_{j-1} \end{bmatrix}$$

we conclude $g' - \lambda g = k$. Together with $u' - \lambda u = v$ and $f' - \lambda f = h$ it follows

$$U' - \lambda U = (u' - \lambda u; f' - \lambda f; g' - \lambda g) = (v; h; k) = V.$$

Hence we have shown that for $\lambda \in \rho(A)$ and $V \in \mathcal{K}$ there exists $U, U' \in \mathcal{K}$ with $\{U, U'\} \in \tilde{T}$, $\tilde{\Gamma}_0 \{U, U'\} = 0$ and $U' - \lambda U = V$. As A is a selfadjoint operator semi-bounded from below the intersection $\rho(A) \cap \mathbb{R}$ is nonempty. Hence item (ii) is satisfied.

In particular we have shown $\{U, V\} = \{U, U' - \lambda U\} \in H_0 - \lambda$, i.e. $\{V, U\} \in (H_0 - \lambda)^{-1}$. We will show later $\rho(A) = \rho(H_0)$. Hence $(H_0 - \lambda)^{-1}$ is an operator and $(H_0 - \lambda)^{-1} V = U$ holds. Note that

$$U = \begin{bmatrix} u \\ f_1 \\ f_2 \\ \vdots \\ f_j \\ g_1 \\ g_2 \\ \vdots \\ g_j \end{bmatrix} = \begin{bmatrix} (A-\lambda)^{-1} v + \gamma(\lambda) \sum_{r=1}^{j} \lambda^{j-r} k_r \\ \gamma(\overline{\lambda})^* v + M(\lambda) \sum_{r=1}^{j} \lambda^{j-r} k_r \\ J_\lambda \left(\begin{bmatrix} h_1 \\ \vdots \\ \vdots \\ h_{j-1} \end{bmatrix} + \begin{bmatrix} \lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \\ 0 \\ J_\lambda \begin{bmatrix} k_1 \\ \vdots \\ \vdots \\ k_{j-1} \end{bmatrix} \end{bmatrix}. \tag{3.9}$$

Due to

$$
\begin{aligned}
J_\lambda \begin{bmatrix} \lambda f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & & & \\ \lambda & \ddots & & \\ \vdots & \ddots & \ddots & \\ \lambda^{j-2} & \cdots & \lambda & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left(\gamma(\overline{\lambda})^* v + M(\lambda) \sum_{r=1}^{j} \lambda^{j-r} k_r \right) \\
&= \begin{bmatrix} \lambda \\ \vdots \\ \vdots \\ \lambda^{j-1} \end{bmatrix} \left(\gamma(\overline{\lambda})^* v + M(\lambda) \begin{bmatrix} \lambda^{j-1} & \cdots & \lambda & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ \vdots \\ k_j \end{bmatrix} \right) \\
&= \begin{bmatrix} \lambda \cdot \gamma(\overline{\lambda})^* v \\ \vdots \\ \lambda^{j-1} \gamma(\overline{\lambda})^* v \end{bmatrix} + M(\lambda) \begin{bmatrix} \lambda^j & \cdots & \lambda^2 & \lambda \\ \vdots & & \vdots & \vdots \\ \lambda^{2j-2} & \cdots & \lambda^j & \lambda^{j-1} \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ \vdots \\ k_j \end{bmatrix}
\end{aligned}
$$

equation (3.9) can also be written as

$$
U = \left[\begin{array}{c|cccc|ccc}
(A-\lambda)^{-1} & 0 & \cdots & 0 & 0 & \lambda^{j-1}\gamma(\lambda) & \cdots & 1\cdot\gamma(\lambda) \\ \hline
1\cdot\gamma(\bar{\lambda})^* & 0 & \cdots & 0 & 0 & & & \\ \cline{2-5}
\lambda\cdot\gamma(\bar{\lambda})^* & & & & 0 & & & \\
\vdots & & J_\lambda & & \vdots & & M(\lambda)\Lambda_\lambda & \\
\lambda^{j-1}\cdot\gamma(\bar{\lambda})^* & & & & 0 & & & \\ \hline
0 & 0 & \cdots & 0 & 0 & & & \\ \cline{2-5}
0 & & & & 0 & & & \\
\vdots & & J_\lambda & & \vdots & & 0 & \\
0 & & & & 0 & & &
\end{array} \right]
\left[\begin{array}{c} v \\ \hline h_1 \\ h_2 \\ \vdots \\ h_j \\ \hline k_1 \\ k_2 \\ \vdots \\ k_j \end{array} \right]. \tag{3.10}
$$

Together with $(H_0 - \lambda)^{-1} V = U$ equation (3.10) shows the matrix representation of the resolvent.

To show item (i) it suffices to note that for arbitrary $g_1 \in \mathcal{G}$ and $\varphi \in \mathcal{G}$ the element

$$
\Big\{ (0;0,\ldots,0;g_1,0\ldots,0),(0;0,\ldots,0,\varphi;0\ldots,0) \Big\}
$$

belongs to $\tilde{T}$ and that

$$
\tilde{\Gamma} \Big\{ (0;0,\ldots,0;g_1,0\ldots,0),(0;0,\ldots,0,\varphi;0\ldots,0) \Big\} = \begin{bmatrix} g_1 \\ \varphi \end{bmatrix}.
$$

To show item (iii) let $\{U,U'\},\{V,V'\} \in T$ be arbitrary. Then

$$\begin{aligned}
[\![U',V]\!]_{\mathcal{K}} &= \Big[\!\!\Big[\big(Tu;f_2,\dots,f_j,\varphi;g_2,\dots,g_j,\Gamma_0 u\big),\big(v;\Gamma_1 v,h_2,\dots,h_j;k_1,\dots,k_j\big)\Big]\!\!\Big]_{\mathcal{K}} \\
&= \langle Tu,v\rangle_{\mathcal{H}^0} + \langle f_2,k_j\rangle_{\mathcal{G}} + \ldots + \langle f_j,k_2\rangle_{\mathcal{G}} + \langle \varphi,k_1\rangle_{\mathcal{G}} \\
&\qquad\qquad + \langle g_2,h_j\rangle_{\mathcal{G}} + \ldots + \langle g_j,h_2\rangle_{\mathcal{G}} + \langle \Gamma_0 u,\Gamma_1 v\rangle_{\mathcal{G}} \\
&= \langle Tu,v\rangle_{\mathcal{H}^0} + \langle \Gamma_0 u,\Gamma_1 v\rangle_{\mathcal{G}} + \langle \varphi,k_1\rangle_{\mathcal{G}} \\
&\quad + \langle f_2,k_j\rangle_{\mathcal{G}} + \ldots + \langle f_j,k_2\rangle_{\mathcal{G}} + \langle g_2,h_j\rangle_{\mathcal{G}} + \ldots + \langle g_j,h_2\rangle_{\mathcal{G}}
\end{aligned}$$

and analogously

$$\begin{aligned}
[\![U,V']\!]_{\mathcal{K}} &= \Big[\!\!\Big[\big(u;\Gamma_1 u,f_2,\dots,f_j;g_1,\dots,g_j\big),\big(Tv;h_2,\dots,h_j,\psi;k_2,\dots,k_j,\Gamma_0 v\big)\Big]\!\!\Big]_{\mathcal{K}} \\
&= \langle u,Tv\rangle_{\mathcal{H}^0} + \langle \Gamma_1 u,\Gamma_0 v\rangle_{\mathcal{G}} + \langle f_2,k_j\rangle_{\mathcal{G}} + \ldots + \langle f_j,k_2\rangle_{\mathcal{G}} \\
&\qquad\qquad + \langle g_1,\psi\rangle_{\mathcal{G}} + \langle g_2,h_j\rangle_{\mathcal{G}} + \ldots + \langle g_j,h_2\rangle_{\mathcal{G}} \\
&= \langle u,Tv\rangle_{\mathcal{H}^0} + \langle \Gamma_1 u,\Gamma_0 v\rangle_{\mathcal{G}} + \langle g_1,\psi\rangle_{\mathcal{G}} \\
&\quad + \langle f_2,k_j\rangle_{\mathcal{G}} + \ldots + \langle f_j,k_2\rangle_{\mathcal{G}} + \langle g_2,h_j\rangle_{\mathcal{G}} + \ldots + \langle g_j,h_2\rangle_{\mathcal{G}}.
\end{aligned}$$

Using Green's identity for the triple $(\mathcal{G},\Gamma_0,\Gamma_1)$ and the definition of $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ we get

$$\begin{aligned}
&[\![U',V]\!]_{\mathcal{K}} - [\![U,V']\!]_{\mathcal{K}} \\
&\quad = \langle Tu,v\rangle_{\mathcal{H}^0} + \langle \Gamma_0 u,\Gamma_1 v\rangle_{\mathcal{G}} + \langle \varphi,k_1\rangle_{\mathcal{G}} - \langle u,Tv\rangle_{\mathcal{H}^0} - \langle \Gamma_1 u,\Gamma_0 v\rangle_{\mathcal{G}} - \langle g_1,\psi\rangle_{\mathcal{G}} \\
&\quad = \langle \varphi,k_1\rangle_{\mathcal{G}} - \langle g_1,\psi\rangle_{\mathcal{G}} = \langle \tilde{\Gamma}_1\{U,U'\},\tilde{\Gamma}_0\{V,V'\}\rangle_{\mathcal{G}} - \langle \tilde{\Gamma}_0\{U,U'\},\tilde{\Gamma}_1\{V,V'\}\rangle_{\mathcal{G}}.
\end{aligned}$$

As (i), (ii) and (iii) are satisfied we know due to Lemma 2.10 that $\tilde{S} := \ker\tilde{\Gamma}$ is a closed symmetric relation in $\mathcal{K}$ with $\tilde{S}^+ = \tilde{T}$ and that $(\mathcal{G},\tilde{\Gamma}_0,\tilde{\Gamma}_1)$ is an ordinary boundary triple for $\tilde{S}^+$. Hence it follows immediately that $H_0 := \ker\tilde{\Gamma}_0$ is a selfadjoint linear relation in $\mathcal{K}$, cf. for example Proposition 2.1 in [24] (see also the text before Theorem 2.8).

Next we compare the spectra of A and H_0. At first let $\lambda \in \sigma_p(A)$ and let u be a corresponding eigenvector. In particular $u \in \operatorname{dom} A = \ker\Gamma_0$. According to the definition of $\tilde{T}$ and $\tilde{\Gamma}_0$ we get

$$\Big\{(u;\Gamma_1 u,\lambda\Gamma_1 u,\dots,\lambda^{j-1}\Gamma_1 u;0,\dots,0),(Tu;\lambda\Gamma_1 u,\dots,\lambda^j\Gamma_1 u;0,\dots,0,\Gamma_0 u)\Big\} \in \ker\tilde{\Gamma}_0 = H_0.$$

Due to $u \in \operatorname{dom} A = \ker\Gamma_0$ and $Au = \lambda u$ we get

$$\begin{bmatrix} Tu \\ \lambda\Gamma_1 u \\ \vdots \\ \lambda^{j-1}\Gamma_1 u \\ \lambda^j\Gamma_1 u \\ 0 \\ \vdots \\ 0 \\ \Gamma_0 u \end{bmatrix} = \begin{bmatrix} Au \\ \lambda\cdot\Gamma_1 u \\ \vdots \\ \lambda\cdot\lambda^{j-2}\Gamma_1 u \\ \lambda\cdot\lambda^{j-1}\Gamma_1 u \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} u \\ \Gamma_1 u \\ \vdots \\ \lambda^{j-2}\Gamma_1 u \\ \lambda^{j-1}\Gamma_1 u \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

i.e. $0 \neq (u;\Gamma_1 u,\lambda\Gamma_1 u,\dots,\lambda^{j-1}\Gamma_1 u;0,\dots,0) \in \ker(H_0-\lambda)$ and therefore $\lambda \in \sigma_p(H_0)$. On the other hand, if $\lambda \in \sigma_p(H_0)$ there exists $(u;f;g) \in \mathcal{K}\setminus\{0\}$ with $(u;f;g) \in \ker(H_0-\lambda)$. Therefore $u \in \operatorname{dom} T$, $g_1 = 0$ and

$$\begin{bmatrix} Tu \\ f_2 \\ \vdots \\ f_j \\ \varphi \\ g_2 \\ \vdots \\ g_j \\ \Gamma_0 u \end{bmatrix} = \lambda \begin{bmatrix} u \\ \Gamma_1 u \\ f_2 \\ \vdots \\ f_j \\ 0 \\ g_2 \\ \vdots \\ g_j \end{bmatrix}.$$

Hence $\Gamma_0 u = \lambda g_j = \lambda^2 g_{j-1} = \lambda^{j-1} g_2 = \lambda^j \cdot 0 = 0$, i.e. $u \in \operatorname{dom} A$ and therefore $Au = Tu = \lambda u$, i.e. $u \in \ker(A-\lambda)$. Note that $u = 0$ would imply $f_2 = \lambda\Gamma_1 u = 0$, hence $f_3 = \lambda f_2 = 0$ etc. such that we would finally get $(u;f;g) = 0$, which is a contradiction. Hence u is an eigenvector of A, i.e. $\lambda \in \sigma_p(A)$. Therefore

$$\sigma_p(A) = \sigma_p(H_0). \tag{3.11}$$

In item (i) we have shown that $\operatorname{ran}(H_0-\lambda) = \mathcal{K}$ holds for all $\lambda \in \rho(A)$. Moreover for all $\lambda \in \rho(A)$ holds $\ker(H_0-\lambda) = \{0\}$, cf. (3.11). Hence $(H_0-\lambda)^{-1}$ is a closed operator defined on the whole space $\mathcal{K}$ and the closed graph theorem implies $(H_0-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, i.e. $\lambda \in \rho(H_0)$. As this is true for all $\lambda \in \rho(A)$ we conclude

$$\rho(A) \subseteq \rho(H_0).$$

If $\lambda \in \rho(H_0)$ then $(H_0-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$ and $\operatorname{ran}(H_0-\lambda) = \mathcal{K}$. Hence for a given $v \in \mathcal{H}^0$ exists $\{U,U'\} \in H_0$ such that $U' - \lambda U = (v;0;0)$, i.e. there exist $f_2,\dots,f_j,\varphi,g_2,\dots,g_j \in \mathcal{G}$ and $u \in \operatorname{dom} T$ with

$$\begin{bmatrix} Tu \\ f_2 \\ \vdots \\ f_j \\ \varphi \\ g_2 \\ \vdots \\ g_j \\ \Gamma_0 u \end{bmatrix} - \lambda \begin{bmatrix} u \\ \Gamma_1 u \\ f_2 \\ \vdots \\ f_j \\ 0 \\ g_2 \\ \vdots \\ g_j \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $\Gamma_0 u = \lambda g_j = \lambda^2 g_{j-1} = \lambda^{j-1} g_2 = \lambda^j \cdot 0 = 0$, i.e. $u \in \operatorname{dom} A$ and therefore $(A-\lambda)u = (T-\lambda)u = v$. As $v \in \mathcal{H}^0$ was arbitrary we get $\operatorname{ran}(A-\lambda) = \mathcal{H}^0$. Moreover $\ker(A-\lambda) = \{0\}$, cf. (3.11), and $(A-\lambda)^{-1}$ is closed. Hence $(A-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$ and therefore $\lambda \in \rho(A)$. As this is true for all $\lambda \in \rho(H_0)$ we conclude $\rho(H_0) \subseteq \rho(A)$ and hence

$$\rho(A) = \rho(H_0). \tag{3.12}$$

If $\lambda \in \sigma_c(A)$ we conclude from (3.12) and (3.11) that $\lambda \in \sigma(H_0) \setminus \sigma_p(H_0)$. Furthermore we know $\lambda \in \mathbb{R}$ because A is selfadjoint in the Hilbert space $\mathcal{H}^0$. If $\lambda \in \sigma_r(H_0)$ then $\overline{\operatorname{ran}(H_0-\lambda)} \neq \mathcal{K}$ and hence $(\operatorname{ran}(H_0-\lambda))^{\perp} \neq \{0\}$, where $\perp$ denotes the orthogonal complement with respect to the Hilbert space structure of $\mathcal{K}$. Moreover, we have

$$\begin{aligned}\{0\} &= \ker(H_0-\lambda) = \ker(H_0^+ - \overline{\lambda}) = \ker(H_0-\lambda)^+ \\ &= \ker\big(\mathcal{J}(H_0-\lambda)^*\mathcal{J}\big) = \ker(H_0-\lambda)^* = \big(\operatorname{ran}(H_0-\lambda)\big)^{\perp} \neq \{0\},\end{aligned}$$

where $(H_0-\lambda)^*$ denotes the Hilbert space adjoint of $H_0-\lambda$ and $\mathcal{J}$ a fundamental symmetry of $\mathcal{K}$. Obviously, this is a contradiction and hence $\lambda \notin \sigma_r(H_0)$. Therefore $\lambda \in \sigma_c(H_0)$ and hence $\sigma_c(A) \subseteq \sigma_c(H_0)$. As $\mathbb{C} = \rho(A) \dot\cup \sigma_p(A) \dot\cup \sigma_c(A)$ we conclude with (3.12) and (3.11)

$$\sigma_c(A) = \sigma_c(H_0).$$

□

In the next theorem we investigate the connection between the γ-fields and Weyl functions of $(\mathcal{G}, \tilde\Gamma_0, \tilde\Gamma_1)$ and $(\mathcal{G}, \Gamma_0, \Gamma_1)$.

Theorem 3.12. *The γ-field of the ordinary boundary triple $(\mathcal{G}, \tilde\Gamma_0, \tilde\Gamma_1)$ for $\tilde S^+$ is given by*

$$\begin{aligned}\tilde\gamma : \rho(H_0) &\to \mathcal{L}(\mathcal{G},\mathcal{K}), \\ \lambda &\mapsto \tilde\gamma(\lambda),\end{aligned} \qquad \textit{with} \qquad \tilde\gamma(\lambda)g_1 = \begin{bmatrix} \lambda^j \gamma(\lambda) g_1 \\ \hline \lambda^j M(\lambda) g_1 \\ \vdots \\ \lambda^{2j-1} M(\lambda) g_1 \\ \hline g_1 \\ \vdots \\ \lambda^{j-1} g_1 \end{bmatrix}, \qquad g_1 \in \mathcal{G}.$$

For each $\lambda \in \rho(H_0)$ the Krein space adjoint $\tilde\gamma(\lambda)^+ \in \mathcal{L}(\mathcal{K},\mathcal{G})$ of $\tilde\gamma(\lambda)$ satisfies

$$\tilde\gamma(\lambda)^+(v;h;k) = \overline{\lambda}^j \gamma(\lambda)^* v + \left(\sum_{l=1}^{j} \overline{\lambda}^{j-l} h_l\right) + \overline{\lambda}^j M(\lambda)^* \left(\sum_{l=1}^{j} \overline{\lambda}^{j-l} k_l\right)$$

for all $(v;h;k) \in \mathcal{K}$. Moreover the corresponding Weyl function $\tilde M : \rho(H_0) \to \mathcal{L}(\mathcal{G})$ satisfies $\tilde M(\lambda) = \lambda^{2j} M(\lambda)$ for all $\lambda \in \rho(H_0)$.

Proof. Recall $\tilde\gamma(\lambda) = \pi_1(\tilde\Gamma_0 \upharpoonright \hat{\mathcal{N}}_\lambda)^{-1}$ with $\hat{\mathcal{N}}_\lambda = \{\{U, \lambda U\} : U \in \ker(\tilde S^+ - \lambda)\}$ for $\lambda \in \rho(H_0)$. Note that $U \in \ker(\tilde S^+ - \lambda)$ implies $\{U, \lambda U\} \in \tilde S^+ = \tilde T$ and hence

$$(\lambda u; \lambda\Gamma_1 u, \lambda f_2, \dots, \lambda f_j; \lambda g_1, \dots, \lambda g_j) = \lambda U = U' = (Tu; f_2, \dots, f_j, \varphi; g_2, \dots, g_j, \Gamma_0 u).$$

In particular we have $\lambda g_1 = g_2, \ldots, \lambda g_{j-1} = g_j, \lambda g_j = \Gamma_0 u$ and hence $\Gamma_0 u = \lambda^j g_1$. Moreover we have $\lambda \Gamma_1 u = f_2, \ldots, \lambda f_{j-1} = f_j$. Furthermore $u \in \ker(T-\lambda)$ and hence $\Gamma_1 u = M(\lambda)\Gamma_0 u = \lambda^j M(\lambda) g_1$. Due to $\tilde{\Gamma}_0\{U, \lambda U\} = g_1$ we get therefore

$$\tilde{\gamma}(\lambda) g_1 = \pi_1\{U, \lambda U\} = U = \begin{bmatrix} u \\ \Gamma_1 u \\ f_2 \\ \vdots \\ f_j \\ g_1 \\ \vdots \\ g_{j-1} \\ g_j \end{bmatrix} = \begin{bmatrix} u \\ \Gamma_1 u \\ \lambda \Gamma_1 u \\ \vdots \\ \lambda^{j-1}\Gamma_1 u \\ g_1 \\ \lambda g_1 \\ \vdots \\ \lambda^{j-1} g_1 \end{bmatrix} = \begin{bmatrix} u \\ \lambda^j M(\lambda) g_1 \\ \lambda^{j+1} M(\lambda) g_1 \\ \vdots \\ \lambda^{2j-1} M(\lambda) g_1 \\ g_1 \\ \lambda g_1 \\ \vdots \\ \lambda^{j-1} g_1 \end{bmatrix}.$$

Due to $\tilde{M}(\lambda) = \tilde{\Gamma}_1(\tilde{\Gamma}_0 \upharpoonright \hat{\mathcal{N}}_\lambda)^{-1}$ we observe analogously

$$\tilde{M}(\lambda) g_1 = \tilde{\Gamma}_1\{U, \lambda U\} = \varphi = \lambda f_j = \lambda \cdot \lambda^{2j-1} M(\lambda) g_1 = \lambda^{2j} M(\lambda) g_1$$

for all $g_1 \in \mathcal{G}$.

To show the representation of $\tilde{\gamma}(\lambda)^+$ let $g_1 \in \mathcal{G}$ and $(v;h;k) \in \mathcal{K}$ be arbitrary. Hence

$$\begin{aligned}
[\![g_1, \tilde{\gamma}(\lambda)^+(v;h;k)]\!]_{\mathcal{G}} = [\![\tilde{\gamma}(\lambda) g_1, (v;h;k)]\!]_{\mathcal{K}} &= \left[\!\!\left[\begin{bmatrix} \lambda^j \gamma(\lambda) g_1 \\ \lambda^j M(\lambda) g_1 \\ \vdots \\ \lambda^{2j-1} M(\lambda) g_1 \\ g_1 \\ \vdots \\ \lambda^{j-1} g_1 \end{bmatrix}, \begin{bmatrix} v \\ h_1 \\ \vdots \\ h_j \\ k_1 \\ \vdots \\ k_j \end{bmatrix} \right]\!\!\right]_{\mathcal{K}} \\
&= \langle \lambda^j \gamma(\lambda) g_1, v\rangle_{\mathcal{H}^0} + \langle \lambda^j M(\lambda) g_1, k_j\rangle_{\mathcal{G}} + \ldots + \langle \lambda^{2j-1} M(\lambda) g_1, k_1\rangle_{\mathcal{G}} \\
&\qquad + \langle g_1, h_j\rangle_{\mathcal{G}} + \ldots + \langle \lambda^{j-1} g_1, h_1\rangle_{\mathcal{G}} \\
&= \langle g_1, \overline{\lambda}^j \gamma(\lambda)^* v\rangle_{\mathcal{G}} + \langle g_1, \overline{\lambda}^j M(\lambda)^* k_j\rangle_{\mathcal{G}} + \ldots + \langle g_1, \overline{\lambda}^{2j-1} M(\lambda)^* k_1\rangle_{\mathcal{G}} \\
&\qquad + \langle g_1, h_j\rangle_{\mathcal{G}} + \ldots + \langle g_1, \overline{\lambda}^{j-1} h_1\rangle_{\mathcal{G}} \\
&= \left\langle g_1, \overline{\lambda}^j \gamma(\lambda)^* v + \left(\sum_{l=1}^{j} \overline{\lambda}^{j-l} h_l\right) + \overline{\lambda}^j M(\lambda)^* \left(\sum_{l=1}^{j} \overline{\lambda}^{j-l} k_l\right)\right\rangle_{\mathcal{G}}.
\end{aligned}$$

As $g_1 \in \mathcal{G}$ and $(v;h;k) \in \mathcal{K}$ are arbitrary the desired representation of $\tilde{\gamma}(\lambda)^+$ follows. □

As in Lemma 3.7 we define for the following theorem the operator $A_{\Gamma_1} := T \upharpoonright \ker \Gamma_1$.

Theorem 3.13. *Let Θ be a closed linear relation in $\mathcal{G}\times\mathcal{G}$, $\lambda\in\rho(A)=\rho(H_0)$ such that $0\in\rho[\Theta-\tilde{M}(\lambda)]$ and let $H_\Theta := \{\{U,U'\}\in S^+ : \{\tilde{\Gamma}_0\{U,U'\},\tilde{\Gamma}_1\{U,U'\}\}\in\Theta\}$. Then $\lambda\in\rho(H_\Theta)$ and*

$$(H_\Theta-\lambda)^{-1}=(H_0-\lambda)^{-1}+\tilde{\gamma}(\lambda)\big[\Theta-\tilde{M}(\lambda)\big]^{-1}\tilde{\gamma}(\overline{\lambda})^{+}. \tag{3.13}$$

Furthermore, if we define the operators

$$P_{\mathcal{H}^0}:\mathcal{K}\to\mathcal{H}^0,\quad (u;f;g)\mapsto u \qquad \text{and} \qquad E_{\mathcal{H}^0}:\mathcal{H}^0\to\mathcal{K},\quad v\mapsto(v;0;0),$$

we get the formula

$$P_{\mathcal{H}^0}(H_\Theta-\lambda)^{-1}E_{\mathcal{H}^0}=(A-\lambda)^{-1}+\lambda^{2j}\gamma(\lambda)\big[\Theta-\lambda^{2j}M(\lambda)\big]^{-1}\gamma(\overline{\lambda})^{*}. \tag{3.14}$$

In particular $P_{\mathcal{H}^0}H_\Theta^{-1}E_{\mathcal{H}^0}=A^{-1}$. Moreover the identity $P_{\mathcal{H}^0}H_\Theta E_{\mathcal{H}^0}=A_{\Gamma_1}$ holds.

Proof. From Theorem 2.8 and Remark 2.9 we observe $\lambda\in\rho(H_\Theta)$ and the formula (3.13). From the matrix representation in Theorem 3.11 we know $P_{\mathcal{H}^0}(H_0-\lambda)^{-1}E_{\mathcal{H}^0}=(A-\lambda)^{-1}$. Furthermore we observe from the representations in Theorem 3.12 the identities $P_{\mathcal{H}^0}\tilde{\gamma}(\lambda)=\lambda^j\gamma(\lambda)$ and $\tilde{\gamma}(\overline{\lambda})^{+}E_{\mathcal{H}^0}=\lambda^j\gamma(\overline{\lambda})^{*}$ as well as $\tilde{M}(\lambda)=\lambda^{2j}M(\lambda)$. Hence (3.13) implies

$$\begin{aligned}P_{\mathcal{H}^0}(H_\Theta-\lambda)^{-1}E_{\mathcal{H}^0}&=P_{\mathcal{H}^0}(H_0-\lambda)^{-1}E_{\mathcal{H}^0}+P_{\mathcal{H}^0}\tilde{\gamma}(\lambda)\Big[\Theta-\tilde{M}(\lambda)\Big]^{-1}\tilde{\gamma}(\overline{\lambda})^{+}E_{\mathcal{H}^0}\\&=(A-\lambda)^{-1}+\lambda^j\gamma(\lambda)\Big[\Theta-\lambda^{2j}M(\lambda)\Big]^{-1}\lambda^j\gamma(\overline{\lambda})^{*},\end{aligned}$$

which yields (3.14). For the special case $\lambda=0$ we get $P_{\mathcal{H}^0}H_\Theta^{-1}E_{\mathcal{H}^0}=A^{-1}$. To show the last statement note that the linear relation H_Θ is given by

$$\begin{aligned}H_\Theta&=\big\{\{U,U'\}\in S^+:\big\{\tilde{\Gamma}_0\{U,U'\},\tilde{\Gamma}_1\{U,U'\}\big\}\in\Theta\big\}\\&=\left\{\begin{bmatrix}(u;\Gamma_1u,f_2,\dots,f_j;g_1,\dots,g_j)\\(Tu;f_2,\dots,f_j,\varphi;g_2,\dots,g_j,\Gamma_0u)\end{bmatrix}:u\in\operatorname{dom}T,\{g_1,\varphi\}\in\Theta\right\}.\end{aligned} \tag{3.15}$$

Written as a linear relation the operator $E_{\mathcal{H}^0}$ has the representation

$$E_{\mathcal{H}^0}=\big\{\big(v,(v;0;0)\big):v\in\mathcal{H}^0\big\}. \tag{3.16}$$

According to the definition of multiplication of linear relations (3.15) and (3.16) imply

$$\begin{aligned}H_\Theta E_{\mathcal{H}^0}=\Big\{\Big\{u,(Tu;f_2,\dots,f_j,\varphi;g_2,\dots,g_j,\Gamma_0u)\Big\}:&\\(u,0,0)=(u;\Gamma_1u,f_2,\dots,f_j;g_1,\dots,g_j),u\in\operatorname{dom}T,\{g_1,\varphi\}\in\Theta\Big\}&\\=\Big\{\Big\{u,(Tu;0,\dots,0,\varphi;0,\dots,0,\Gamma_0u)\Big\}:u\in\operatorname{dom}T,\Gamma_1u=0,\{0,\varphi\}\in\Theta\Big\}&.\end{aligned}$$

Note that $u\in\operatorname{dom}T$ and $\Gamma_1u=0$ implies $u\in\operatorname{dom}A_{\Gamma_1}$. Hence we get

$$P_{\mathcal{H}^0}H_\Theta E_{\mathcal{H}^0}=\big\{\{u,Tu\}:u\in\operatorname{dom}T,\Gamma_1u=0\big\}=T\restriction\ker\Gamma_1=A_{\Gamma_1}.$$

□

As a corollary of the previous theorem we get the following observation about Schatten-von Neumann classes.

Corollary 3.14. *Let Θ be a closed linear relation in $\mathcal{G}\times\mathcal{G}$, $\lambda\in\rho(A)=\rho(H_0)$ such that $0\in\rho[\Theta-\tilde{M}(\lambda)]$ and let $H_\Theta:=\{\{U,U'\}\in S^+:\{\tilde{\Gamma}_0\{U,U'\},\tilde{\Gamma}_1\{U,U'\}\}\in\Theta\}$. Furthermore assume that $G^*:\mathcal{H}^k\to\mathcal{G}$ is a compact operator in $\mathfrak{S}_p(\mathcal{H}^k,\mathcal{G})$ for some $p>0$. Then*

$$P_{\mathcal{H}^0}(H_\Theta-\lambda)^{-1}E_{\mathcal{H}^0}-(A-\lambda)^{-1}\in\mathfrak{S}_{\frac{p}{2}}(\mathcal{H}).$$

Proof. With equation (3.14) from Theorem 3.13 we observe

$$P_{\mathcal{H}^0}(H_\Theta-\lambda)^{-1}E_{\mathcal{H}^0}=(A-\lambda)^{-1}+\lambda^{2j}\gamma(\lambda)\left[\Theta-\lambda^{2j}M(\lambda)\right]^{-1}\gamma(\overline{\lambda})^*.$$

Note that $\operatorname{ran}(A-\overline{\lambda})^{-1}=\mathcal{H}^2$ because $\lambda\in\rho(A)$. Hence we get with Lemma 2.6

$$\gamma(\overline{\lambda})^*=\Gamma_1(A-\overline{\lambda})^{-1}=G_0^*(A-\overline{\lambda})^{-1}=G^*A_{k-2j+2}^{-j}(A-\overline{\lambda})^{-1}.$$

As $A_{k-2j+2}^{-j}(A-\overline{\lambda})^{-1}:\mathcal{H}^0\to\mathcal{H}^{2j+2}\subseteq\mathcal{H}^k$ is continuous and $G^*\in\mathfrak{S}_p(\mathcal{H}^k,\mathcal{G})$ we get $\gamma(\overline{\lambda})^*\in\mathfrak{S}_{\frac{p}{2}}(\mathcal{H}^0,\mathcal{G})$, cf. Corollary 2.2 in Chapter II of [37]. This implies $\gamma(\overline{\lambda})\in\mathfrak{S}_{\frac{p}{2}}(\mathcal{G},\mathcal{H}^0)$. As

$$\left[\Theta-\lambda^{2j}M(\lambda)\right]^{-1}=\left[\Theta-\tilde{M}(\lambda)\right]^{-1}\in\mathcal{L}(\mathcal{G})$$

because $0\in\rho[\Theta-\tilde{M}(\lambda)]$ we observe again with Corollary 2.2 in Chapter II of [37]

$$\lambda^{2j}\gamma(\lambda)\left[\Theta-\lambda^{2j}M(\lambda)\right]^{-1}\gamma(\overline{\lambda})^*\in\mathfrak{S}_{\frac{p}{2}}(\mathcal{H}).$$

From this the claimed result follows. □

3.4.2 Connection of the model space $\mathcal{K}$ with the extension space

We have already motivated at the beginning of Section 3.4 that $\tilde{\mathcal{K}}=\mathcal{H}^0\dot{+}\sum_{l=1}^{2j}A^{-l}\operatorname{ran}G$ would be a suitable space for an operator associated to the formal expression $\mathcal{A}_\vartheta$ (as in Section 3.4 we write in the following just A^{-l} instead of A_{-1}^{-l}). In this subsection we motivate how an inner product on a subspace of $\tilde{\mathcal{K}}$ should be defined and how the linear relation H_Θ corresponds to $\mathcal{A}_\vartheta$. In order to avoid extensive calculations we just consider here the case $k=3$, the other cases are similar, but more technical.

At first we note that $k=3$ implies $j=1$, hence the extension space $\tilde{\mathcal{K}}$ is given by

$$\tilde{\mathcal{K}}=\mathcal{H}^0\dot{+}A^{-1}\operatorname{ran}G\dot{+}A^{-2}\operatorname{ran}G.$$

Define the subspace

$$\hat{\mathcal{K}}:=\mathcal{H}^3\dot{+}A^{-1}\operatorname{ran}G\dot{+}A^{-2}\operatorname{ran}G.$$

Let $u+A^{-1}Gf+A^{-2}Gg$ and $v+A^{-1}Gh+A^{-2}Gk$ be two elements in $\hat{\mathcal{K}}$. Assume that $[\![\cdot,\cdot]\!]_{\hat{\mathcal{K}}}$ is an inner product on $\hat{\mathcal{K}}$ which is compatible with the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}^0}$ and the dual pairings $\langle\cdot,\cdot\rangle_{\mathcal{H}^{-1},\mathcal{H}^1}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}^{-2},\mathcal{H}^2}$. Then

$$\begin{aligned}
&[\![u+A^{-1}Gf+A^{-2}Gg, v+A^{-1}Gh+A^{-2}Gk]\!]_{\hat{\mathcal{K}}}\\
&\qquad = [\![u,v]\!]_{\hat{\mathcal{K}}} + [\![u,A^{-1}Gh]\!]_{\hat{\mathcal{K}}} + [\![u,A^{-2}Gk]\!]_{\hat{\mathcal{K}}}\\
&\qquad + [\![A^{-1}Gf,v]\!]_{\hat{\mathcal{K}}} + [\![A^{-1}Gf,A^{-1}Gh]\!]_{\hat{\mathcal{K}}} + [\![A^{-1}Gf,A^{-2}Gk]\!]_{\hat{\mathcal{K}}}\\
&\qquad + [\![A^{-2}Gg,v]\!]_{\hat{\mathcal{K}}} + [\![A^{-2}Gg,A^{-1}Gh]\!]_{\hat{\mathcal{K}}} + [\![A^{-2}Gg,A^{-2}Gk]\!]_{\hat{\mathcal{K}}}\\
&\qquad = \langle u,v\rangle_{\mathcal{H}^0} + \langle u,A^{-1}Gh\rangle_{\mathcal{H}^1,\mathcal{H}^{-1}} + \langle u,A^{-2}Gk\rangle_{\mathcal{H}^0}\\
&\qquad + \langle A^{-1}Gf,v\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} + [\![A^{-1}Gf,A^{-1}Gh]\!]_{\hat{\mathcal{K}}} + \langle A^{-1}Gf,A^{-2}Gk\rangle_{\mathcal{H}^{-1},\mathcal{H}^1}\\
&\qquad + \langle A^{-2}Gg,v\rangle_{\mathcal{H}^0} + \langle A^{-2}Gg,A^{-1}Gh\rangle_{\mathcal{H}^1,\mathcal{H}^{-1}} + \langle A^{-2}Gg,A^{-2}Gk\rangle_{\mathcal{H}^0}.
\end{aligned}$$

Note that there is no chance to give a meaning to $[\![A^{-1}Gf,A^{-1}Gh]\!]_{\hat{\mathcal{K}}}$ such that it is compatible with $\langle\cdot,\cdot\rangle_{\mathcal{H}^0}$ and the corresponding dual pairings because $A^{-1}Gf$ and $A^{-1}Gh$ belong both to $\mathcal{H}^{-1}\setminus\mathcal{H}^0$ for $f,h\neq 0$. Therefore we will set it equal to 0 in the following. Hence we get with the adjoint $G^*:\mathcal{H}^3\to\mathcal{G}$ of $\mathcal{G}$

$$\begin{aligned}
&[\![u+A^{-1}Gf+A^{-2}Gg, v+A^{-1}Gh+A^{-2}Gk]\!]_{\hat{\mathcal{K}}}\\
&\qquad = \langle u,v\rangle_{\mathcal{H}^0} + \langle G^*A^{-1}u,h\rangle_{\mathcal{G}} + \langle G^*A^{-2}u,k\rangle_{\mathcal{G}}\\
&\qquad + \langle f,G^*A^{-1}v\rangle_{\mathcal{G}} + 0 + \langle G^*A^{-3}Gf,k\rangle_{\mathcal{G}}\\
&\qquad + \langle g,G^*A^{-2}v\rangle_{\mathcal{G}} + \langle g,G^*A^{-3}Gh\rangle_{\mathcal{G}} + \langle A^{-2}Gg,A^{-2}Gk\rangle_{\mathcal{H}^0}.
\end{aligned}$$

Regrouping the equation above we get

$$\begin{aligned}
&[\![u+A^{-1}Gf+A^{-2}Gg, v+A^{-1}Gh+A^{-2}Gk]\!]_{\hat{\mathcal{K}}}\\
&\qquad = \langle u,v\rangle_{\mathcal{H}^0} + \langle G^*A^{-1}u,h\rangle_{\mathcal{G}} \qquad + \langle f,G^*A^{-1}v\rangle_{\mathcal{G}}\\
&\qquad\quad + \langle G^*A^{-2}u,k\rangle_{\mathcal{G}} \qquad + \langle g,G^*A^{-2}v\rangle_{\mathcal{G}}\\
&\qquad\quad + \langle G^*A^{-3}Gf,k\rangle_{\mathcal{G}} \qquad + \langle g,G^*A^{-3}Gh\rangle_{\mathcal{G}} + \langle A^{-2}Gg,A^{-2}Gk\rangle_{\mathcal{H}^0},
\end{aligned} \tag{3.17}$$

which we will use as our definition for $[\![\cdot,\cdot]\!]_{\hat{\mathcal{K}}}$.

Next we observe with a formal calculation

$$\begin{aligned}
\mathcal{A}_\vartheta\big(u+A^{-1}Gf+A^{-2}Gg\big) &= (A-G\vartheta^{-1}G^*)\big(u+A^{-1}Gf+A^{-2}Gg\big)\\
&= Au+Gf+A^{-1}Gg-G\vartheta^{-1}G^*u-G\vartheta^{-1}G^*\left(A^{-1}Gf+A^{-2}Gg\right)\\
&= Au+A^{-1}Gg+G\big(f-\vartheta^{-1}G^*u\big)-G\vartheta^{-1}G^*\left(A^{-1}Gf+A^{-2}Gg\right).
\end{aligned}$$

Note that due to $\operatorname{dom} G^* = \mathcal{H}^3$ it is not possible to give a reasonable meaning to the last summand $G\vartheta^{-1}G^*(A^{-1}Gf+A^{-2}Gg)$. Therefore we will ignore it in this consideration. Due to $\operatorname{ran} G \subseteq$

$\mathcal{H}^{-3}$ and $\operatorname{ran} G \cap \mathcal{H}^{-2} = \{0\}$ the expression $G(f - \vartheta^{-1}G^*u)$ just belongs to $\hat{\mathcal{K}}$ if $f = \vartheta^{-1}G^*u$. Hence a reasonable realization A_ϑ of $\mathcal{A}_\vartheta$ is given by

$$A_\vartheta(u + A^{-1}Gf + A^{-2}Gg) := Au + A^{-1}Gg,$$
$$\operatorname{dom} A_\vartheta := \{u + A^{-1}Gf + A^{-2}Gg \in \hat{\mathcal{K}} : f = \vartheta^{-1}G^*u\}.$$

Lemma 3.15. *The operator $V : \hat{\mathcal{K}} \to \mathcal{K}$ defined by*

$$u + A^{-1}Gf + A^{-2}Gg \mapsto \begin{bmatrix} u + A^{-2}Gg \\ G^*A^{-1}u + G^*A^{-3}Gg \\ f \end{bmatrix} = \begin{bmatrix} u + A^{-2}Gg \\ G_0^*(u + A^{-2}Gg) \\ f \end{bmatrix}$$

is isometric. Moreover, if ϑ is a closed operator in $\mathcal{G}$ with $0 \in \rho(\vartheta)$ and H_ϑ is defined as in Theorem 3.13, then $H_\vartheta V x = V A_\vartheta x$ holds for all $x \in \operatorname{dom} A_\vartheta$.

Proof. Let $x = u + A^{-1}Gf + A^{-2}Gg$ and $y = v + A^{-1}Gh + A^{-2}Gk$ be two elements in $\hat{\mathcal{K}}$. Then

$$\begin{aligned}
[\![Vx, Vy]\!]_{\mathcal{K}} &= \left[\!\!\left[\begin{bmatrix} u + A^{-2}Gg \\ G_0^*(u + A^{-2}Gg) \\ f \end{bmatrix}, \begin{bmatrix} v + A^{-2}Gk \\ G_0^*(v + A^{-2}Gk) \\ h \end{bmatrix} \right]\!\!\right]_{\mathcal{K}} \\
&= \langle u + A^{-2}Gg, v + A^{-2}Gk \rangle_{\mathcal{H}^0} + \left[\!\!\left[\begin{bmatrix} G_0^*(u + A^{-2}Gg) \\ f \end{bmatrix}, \begin{bmatrix} G_0^*(v + A^{-2}Gk) \\ h \end{bmatrix} \right]\!\!\right]_{\mathfrak{h}} \\
&= \langle u + A^{-2}Gg, v + A^{-2}Gk \rangle_{\mathcal{H}^0} + \left\langle \begin{bmatrix} f \\ G_0^*(u + A^{-2}Gg) \end{bmatrix}, \begin{bmatrix} G_0^*(v + A^{-2}Gk) \\ h \end{bmatrix} \right\rangle_{\mathcal{G} \times \mathcal{G}}.
\end{aligned}$$

Using $G_0^* = G^*A^{-1}$ and the definition of $[\![\cdot,\cdot]\!]_{\hat{\mathcal{K}}}$ in (3.17) we get

$$\begin{aligned}
[\![Vx, Vy]\!]_{\mathcal{K}} &= \langle u, v \rangle_{\mathcal{H}^0} + \langle u, A^{-2}Gk \rangle_{\mathcal{H}^0} + \langle A^{-2}Gg, v \rangle_{\mathcal{H}^0} + \langle A^{-2}Gg, A^{-2}Gk \rangle_{\mathcal{H}^0} \\
&\quad + \langle f, G^*A^{-1}v \rangle_{\mathcal{G}} + \langle f, G^*A^{-3}Gk \rangle_{\mathcal{G}} + \langle G^*A^{-1}u, h \rangle_{\mathcal{G}} + \langle G^*A^{-3}Gg, h \rangle_{\mathcal{G}} = [\![x, y]\!]_{\hat{\mathcal{K}}}.
\end{aligned}$$

This shows that $V : \hat{\mathcal{K}} \to \mathcal{K}$ is an isometric operator. To show the second statement let $x = u + A^{-1}Gf + A^{-2}Gg \in \operatorname{dom} A_\vartheta$, i.e. $f = \vartheta^{-1}G^*u$. Note that $u + A^{-2}Gg \in \mathcal{H}^1$ and

$$A_0(u + A^{-2}Gg) - G_0 g = A_0 u + A^{-1}Gg - A^{-1}Gg = Au \in \mathcal{H}^0.$$

This means $u + A^{-2}Gg \in \operatorname{dom} T$, $T(u + A^{-2}Gg) = Au$ and $\Gamma_0(u + A^{-2}Gg) = g$. Hence

$$Vx = \begin{bmatrix} u + A^{-2}Gg \\ G_0^*(u + A^{-2}Gg) \\ f \end{bmatrix} = \begin{bmatrix} u + A^{-2}Gg \\ \Gamma_1(u + A^{-2}Gg) \\ \vartheta^{-1}G^*u \end{bmatrix} \in \operatorname{dom} \tilde{T}$$

and therefore

$$\{U, U'\} := \left\{ \begin{bmatrix} u + A^{-2}Gg \\ \Gamma_1(u + A^{-2}Gg) \\ \vartheta^{-1}G^*u \end{bmatrix}, \begin{bmatrix} T(u + A^{-2}Gg) \\ G^*u \\ \Gamma_0(u + A^{-2}Gg) \end{bmatrix} \right\} \in \tilde{T}.$$

Moreover $\vartheta\tilde{\Gamma}_0\{U,U'\} = \vartheta\vartheta^{-1}G^*u = G^*u = \tilde{\Gamma}_1\{U,U'\}$, i.e. $\{U,U'\} \in \operatorname{dom} H_\vartheta$. Hence

$$\begin{aligned} H_\vartheta V x &= \begin{bmatrix} T(u+A^{-2}Gg) \\ G^*u \\ \Gamma_0(u+A^{-2}Gg) \end{bmatrix} = \begin{bmatrix} Au \\ G^*u \\ g \end{bmatrix} = \begin{bmatrix} Au + A^{-2}G0 \\ G^*A^{-1}Au + G^*A^{-3}G0 \\ g \end{bmatrix} \\ &= V(Au + A^{-1}Gg + A^{-2}G0) = V(Au + A^{-1}Gg) = VA_\vartheta x. \end{aligned}$$ □

3.4.3 An example for supersingular perturbations

Let $\Sigma \subset \mathbb{R}^d$ be a C^∞-manifold of codimension 4. As in Example 3.8 define in $L^2(\mathbb{R}^d)$ the selfadjoint operator

$$Au := (-\Delta + 1)u, \qquad \operatorname{dom} A = H^2(\mathbb{R}^d).$$

For $h \in L^2(\Sigma)$ define $h\delta_\Sigma$ via

$$\big(h\delta_\Sigma\big)(\varphi) := (h, \operatorname{tr}^3_\Sigma \varphi)_{L^2(\Sigma)}, \qquad \varphi \in H^3(\mathbb{R}^d).$$

According to Lemma 2.21 (with $\varepsilon = 1$) the distribution $h\delta_\Sigma$ belongs to $H^{-3}(\mathbb{R}^d)$ and

$$G : L^2(\Sigma) \to H^{-3}(\mathbb{R}^d), \quad h \mapsto h\delta_\Sigma,$$

is a bounded, injective operator which satisfies $\operatorname{ran} G \cap H^{-2}(\mathbb{R}^d) = \{0\}$. In particular G satisfies all conditions required in (3.1) on page 29 for $k = 3$. Hence the index j from (3.2) is given by $j = 1$ and

$$G_0 := A_{-1}^{-1}G = (-\Delta + 1)^{-1}G$$

is a continuous operator from $L^2(\Sigma)$ to $H^{-1}(\mathbb{R}^d)$. Note that the operator $G_0^* : H^1(\mathbb{R}^d) \to L^2(\Sigma)$ is given by $G_0^* u = \operatorname{tr}^3_\Sigma(-\Delta+1)^{-1}u$ because

$$\begin{aligned} \langle h, G_0^* u\rangle_{L^2(\Sigma)} &= \langle G_0 h, u\rangle_{H^{-1}(\mathbb{R}^d),H^1(\mathbb{R}^d)} = \langle (-\Delta+1)^{-1}Gh, u\rangle_{H^{-1}(\mathbb{R}^d),H^1(\mathbb{R}^d)} \\ &= \langle Gh, (-\Delta+1)^{-1}u\rangle_{H^{-3}(\mathbb{R}^d),H^3(\mathbb{R}^d)} = \langle h, \operatorname{tr}^3_\Sigma(-\Delta+1)^{-1}u\rangle_{L^2(\Sigma)} \end{aligned}$$

holds for all $u \in H^1(\mathbb{R}^d)$ and all $h \in L^2(\Sigma)$. Hence the operators S and T defined in Lemma 3.4 are given by

$$Su = (-\Delta+1)u, \qquad \operatorname{dom} S = \{u \in H^2(\mathbb{R}^d) : \operatorname{tr}^3_\Sigma(-\Delta+1)^{-1}u = 0\}$$

and

$$\begin{aligned} Tu &= (-\Delta+1)u - (-\Delta+1)^{-1}h\delta_\Sigma \\ \operatorname{dom} T &= \{u \in L^2(\mathbb{R}^d) : \exists h \in L^2(\Sigma) \text{ with } (-\Delta+1)u - (-\Delta+1)^{-1}h\delta_\Sigma \in L^2(\mathbb{R}^d)\} \end{aligned}$$

The corresponding generalized boundary triple from Theorem 3.5 is $(L^2(\Sigma),\Gamma_0,\Gamma_1)$ with

$$\begin{aligned} \Gamma_0 &: \operatorname{dom} T \to L^2(\Sigma), \qquad u \mapsto h,\\ \Gamma_1 &: \operatorname{dom} T \to L^2(\Sigma), \qquad u \mapsto \operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u. \end{aligned} \tag{3.18}$$

The model space $\mathcal{K}$ is hence given by $\mathcal{K} := L^2(\mathbb{R}^d)\times L^2(\Sigma)\times L^2(\Sigma)$ and equipped with the inner product

$$\left[\!\!\left[\begin{bmatrix} u\\ f\\ f'\end{bmatrix}, \begin{bmatrix} v\\ g\\ g'\end{bmatrix} \right]\!\!\right]_{\mathcal{K}} := \langle u,v\rangle_{L^2(\mathbb{R}^d)} + \langle f,g'\rangle_{L^2(\Sigma)} + \langle f',g\rangle_{L^2(\Sigma)}.$$

The linear relation $\tilde{T}$ in $\mathcal{K}$ is given by

$$\begin{aligned} \tilde{T} &= \left\{ \left\{ \begin{bmatrix} u\\ \Gamma_1 u\\ g\end{bmatrix}, \begin{bmatrix} Tu\\ f'\\ \Gamma_0 u\end{bmatrix} \right\} : \begin{array}{l} u\in\operatorname{dom}T,\\ f',g\in L^2(\Sigma)\end{array} \right\}\\ &= \left\{ \left\{ \begin{bmatrix} u\\ \operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u\\ g\end{bmatrix}, \begin{bmatrix} (-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma\\ \varphi\\ h\end{bmatrix} \right\} : \right.\\ &\qquad\qquad \left. \begin{array}{l} u\in L^2(\mathbb{R}^d) : \exists h\in L^2(\Sigma) \text{ with}\\ (-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma \in L^2(\mathbb{R}^d),\\ \varphi,g\in L^2(\Sigma)\end{array} \right\} \end{aligned}$$

and the boundary maps $\tilde{\Gamma} := \binom{\tilde{\Gamma}_0}{\tilde{\Gamma}_1}$ are given by

$$\tilde{\Gamma}\left\{ \begin{bmatrix} u\\ \operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u\\ g\end{bmatrix}, \begin{bmatrix} (-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma\\ \varphi\\ h\end{bmatrix} \right\} = \begin{bmatrix} g\\ \varphi\end{bmatrix}.$$

According to Theorem 3.11 the linear relation

$$\begin{aligned} H_0 = \ker\tilde{\Gamma}_0 &= \left\{ \left\{ \begin{bmatrix} u\\ \operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u\\ 0\end{bmatrix}, \begin{bmatrix} (-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma\\ \varphi\\ h\end{bmatrix} \right\} : \right.\\ &\qquad\qquad \left. \begin{array}{l} u\in L^2(\mathbb{R}^d) : \exists h\in L^2(\Sigma) \text{ with}\\ (-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma \in L^2(\mathbb{R}^d),\\ \varphi\in L^2(\Sigma)\end{array} \right\} \end{aligned}$$

is selfadjoint and its spectrum is given by $\sigma(H_0)=\sigma_c(H_0)=[1,\infty[$. Moreover, with the γ-field γ and the Weyl function M of the generalized boundary triple $(L^2(\Sigma),\Gamma_0,\Gamma_1)$ in (3.18) the resolvent of H_0 can be written as

$$(H_0-\lambda)^{-1} = \left[\begin{array}{c|c|c} (A-\lambda)^{-1} & 0 & \gamma(\lambda)\\ \gamma(\bar{\lambda})^* & 0 & M(\lambda)\\ 0 & 0 & 0 \end{array}\right], \qquad \lambda\in\mathbb{C}\setminus[1,\infty[,$$

The Weyl function $\tilde{M}$ of $(L^2(\Sigma),\tilde{\Gamma}_0,\tilde{\Gamma}_1)$ satisfies $\tilde{M}(\lambda)=\lambda^2 M(\lambda)$ for all $\lambda\in\mathbb{C}\setminus[1,\infty[$ and the γ-field $\tilde{\gamma}$ is given by

$$\tilde{\gamma}(\lambda)g=\begin{bmatrix}\lambda\gamma(\lambda)g\\ \lambda M(\lambda)g\\ g\end{bmatrix},\qquad g\in L^2(\Sigma),\quad \lambda\in\mathbb{C}\setminus[1,\infty[.$$

Its adjoint is given by

$$\tilde{\gamma}(\lambda)^+(v;h;k)=\overline{\lambda}\gamma(\lambda)^*v+h+\overline{\lambda}M(\lambda)^*k$$

for $(v;h;k)\in\mathcal{K}=L^2(\mathbb{R}^d)\times L^2(\Sigma)\times L^2(\Sigma)$ and $\lambda\in\mathbb{C}\setminus[1,\infty[$.

Analogously as in Corollary 3.14 we can also obtain a Schatten-von Neumann estimate: Let Θ be a closed linear relation in $L^2(\Sigma)$ and $\lambda\in\mathbb{C}\setminus[1,\infty[$ with $0\in\rho[\Theta-\tilde{M}(\lambda)]$. With Lemma 2.6 we obtain $\gamma(\overline{\lambda})^*\in\mathcal{L}(L^2(\mathbb{R}^d),L^2(\Sigma))$ and

$$\gamma(\overline{\lambda})^*=\Gamma_1(A-\overline{\lambda})^{-1}=\operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}(-\Delta+1-\overline{\lambda})^{-1}.$$

In particular $\operatorname{ran}\gamma(\overline{\lambda})^*\subseteq\operatorname{ran}\operatorname{tr}_\Sigma^3=H^1(\Sigma)$, cf. Lemma 2.20. As we have assumed that Σ is a compact C^∞-manifold of codimension 4 it follows $\gamma(\overline{\lambda})^*\in\mathfrak{S}_q(L^2(\mathbb{R}^d),L^2(\Sigma))$ for $q>d-4$, cf. Lemma 2.23. Hence it follows with Corollary 2.2 in Chapter II of [37] and with equation (3.14) from Theorem 3.13

$$P_{\mathcal{H}^0}(H_\Theta-\lambda)^{-1}E_{\mathcal{H}^0}-(A-\lambda)^{-1}=\lambda^{2j}\gamma(\lambda)\big[\Theta-\lambda^{2j}M(\lambda)\big]^{-1}\gamma(\overline{\lambda})^*\in\mathfrak{S}_p\big(L^2(\mathbb{R}^d)\big)$$

for $p:=\frac{q}{2}>\frac{d-4}{2}$. Here the linear relation H_Θ is given by

$$H_\Theta=\left\{\left\{\begin{bmatrix}u\\ \operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u\\ g\end{bmatrix},\begin{bmatrix}(-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma\\ \varphi\\ h\end{bmatrix}\right\}:\begin{array}{l}u\in L^2(\mathbb{R}^d):\exists h\in L^2(\Sigma)\text{ with}\\ (-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma\in L^2(\mathbb{R}^d),\\ \{g,\varphi\}\in\Theta\end{array}\right\}.$$

For the special case that Θ is an operator also H_Θ is an operator and given by

$$\begin{aligned}\operatorname{dom}H_\Theta=\{&(u;\operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u;g):g\in\operatorname{dom}\Theta,u\in L^2(\mathbb{R}^d)\text{ s.t.}\\ &\exists h\in L^2(\Sigma)\text{ with }(-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma\in L^2(\mathbb{R}^d)\},\\ H_\Theta(u;\operatorname{tr}_\Sigma^3(-\Delta+1)^{-1}u;g)&=((-\Delta+1)u-(-\Delta+1)^{-1}h\delta_\Sigma;\Theta g;h).\end{aligned}$$

4 SCHRÖDINGER OPERATORS WITH δ-INTERACTIONS ON MANIFOLDS OF CODIMENSION 2

The aim of this chapter is to apply the approaches from Chapter 3 to describe and investigate Schrödinger operators with δ-interactions supported on compact C^2-manifolds of codimension 2 without boundary. Therefore we construct in the first section a generalized boundary triple which is a special case of the one from Theorem 3.5. Moreover we show some properties of the corresponding γ-field γ and Weyl function M. In Section 4.2 we investigate the operators A_Θ which are parametrized with the generalized boundary triple from Section 4.1 by linear relations in $L^2(\Sigma)$. Moreover we show that these operators can also be understood as Schrödinger operators with δ-interactions of singular strength on a manifold of codimension 1.

A natural question which appears here is how the parameter Θ has to be chosen such that the operator A_Θ coincides with a Schrödinger operator with δ-interaction of a given strength. For this we have to introduce the concept of the generalized trace which allows us to define $\operatorname{tr}_\Sigma u$ also for functions $u \in L^2(\mathbb{R}^d)$ which are not smooth enough to define their trace in the classical sense. This is done in Section 4.3. Moreover we define in this section the Schrödinger operator $-\Delta_{\Sigma,\alpha}$ with δ-interaction of strength $\frac{1}{\alpha}$ supported on Σ and provide a Schatten–von Neumann property for the resolvent difference with the free Laplacian. In Section 4.4 we consider the special case of a closed curve in $\mathbb{R}^3$. A deeper analysis of the objects from Section 4.3 for this case allows us to improve the Schatten–von Neumann property. Moreover we provide estimates on the number of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$ and an isoperimetric inequality for the principal eigenvalues.

Throughout the whole chapter Σ is a compact C^2-manifolds of codimension 2 without boundary (in particular $H^{-s}(\Sigma)$ is the dual space of $H^s(\Sigma)$). If necessary, further restrictions on Σ are made before the corresponding statements or sections. Recall that the trace operator $\operatorname{tr}_\Sigma^2 : H^2(\mathbb{R}^d) \to H^1(\Sigma)$ is continuous and bijective, cf. Lemma 2.20.

4.1 The generalized boundary triple

In this section we construct a generalized boundary triple which is a special case of the one in Theorem 3.5. For this we have to chose at first suitable candidates for the objects $\mathcal{H}^0$, A, $\mathcal{G}$ and $\mathcal{G}$ appearing in Section 3.1. As in Example 3.8 we set $\mathcal{H}^0 := L^2(\mathbb{R}^d)$ and consider the selfadjoint operator A in $L^2(\mathbb{R}^d)$ given by by

$$Au := (-\Delta + 1)u, \qquad \operatorname{dom} A := H^2(\mathbb{R}^d).$$

Obviously $A \geq 1$ and the chain of Hilbert spaces induced by A coincides with the Sobolev spaces $H^s(\mathbb{R}^d)$, $s \in \mathbb{Z}$, cf. Example 3.3. Moreover, if we interpret Δ as distributional derivatives, we have

$$\langle(-\Delta+1)u, v\rangle_{H^{-s-2}(\mathbb{R}^d),H^{s+2}(\mathbb{R}^d)} = \langle u, (-\Delta+1)v\rangle_{H^{-s}(\mathbb{R}^d),H^s(\mathbb{R}^d)}$$

for all $s \in \mathbb{Z}$, $u \in H^{-s}(\mathbb{R}^d)$ and $v \in H^{s+2}(\mathbb{R}^d)$. Hence the operators A_{-s} for $s \in \mathbb{N}_0$ are given by

$$A_{-s} : H^{-s}(\mathbb{R}^d) \to H^{-s-2}(\mathbb{R}^d), \qquad u \mapsto (-\Delta+1)u.$$

Furthermore, we set $\mathcal{G} := L^2(\Sigma)$ and define for $h \in H^{-1}(\Sigma)$ the distribution $h\delta_\Sigma$ via

$$(h\delta_\Sigma, \varphi)_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)} := (h, \operatorname{tr}^2_\Sigma \varphi)_{H^{-1}(\Sigma),H^1(\Sigma)}, \qquad \varphi \in H^2(\mathbb{R}^d).$$

In particular for $h \in L^2(\Sigma)$ we get

$$(h\delta_\Sigma, \varphi)_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)} = (h, \operatorname{tr}^2_\Sigma \varphi)_{H^{-1}(\Sigma),H^1(\Sigma)} = (h, \operatorname{tr}^2_\Sigma \varphi)_{L^2(\Sigma)}, \qquad \varphi \in H^2(\mathbb{R}^d),$$

Lemma 4.1. *The operator*

$$G : L^2(\Sigma) \to H^{-2}(\mathbb{R}^d), \quad h \mapsto h\delta_\Sigma,$$

is a bounded, injective operator and satisfies $\operatorname{ran} G \cap H^{-1}(\mathbb{R}^d) = \{0\}$*. The adjoint operator* $G^* : H^2(\mathbb{R}^d) \to L^2(\Sigma)$ *is given by* $G^*u = \operatorname{tr}^2_\Sigma u$ *and* $\operatorname{ran} G^* = H^1(\Sigma)$*. If we denote by* $G^\circledast$ *the operator*

$$G^\circledast : H^2(\mathbb{R}^d) \to H^1(\Sigma), \qquad u \mapsto G^*u,$$

then the adjoint of $G^\circledast$ *is given by*

$$(G^\circledast)^* : H^{-1}(\Sigma) \to H^{-2}(\mathbb{R}^d), \quad h \mapsto h\delta_\Sigma,$$

and satisfies $\operatorname{ran}(G^\circledast)^* \cap H^{-1}(\mathbb{R}^d) = \{0\}$*. In particular* $Gh = (G^\circledast)^*h$ *for all* $h \in L^2(\Sigma)$*.*

Proof. The fact that G is a bounded, injective operator from $L^2(\Sigma)$ to $H^{-2}(\mathbb{R}^d)$ with $\operatorname{ran} G \cap H^{-1}(\mathbb{R}^d) = \{0\}$ follows from Lemma 2.21 with $\varepsilon = 1$. Furthermore, we get for arbitrary $h \in L^2(\Sigma)$ and $u \in H^2(\mathbb{R}^d)$

$$\langle h, G^*u\rangle_{L^2(\Sigma)} = \langle Gh, u\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)} = \langle h, \operatorname{tr}^2_\Sigma u\rangle_{L^2(\Sigma)},$$

from which we conclude $G^*u = \operatorname{tr}^2_\Sigma u$. Hence we observe for the space $\mathcal{G}^+$ defined in the proof of Theorem 3.5

$$\mathcal{G}^+ := \operatorname{ran}(G^* \upharpoonright H^2(\mathbb{R}^d)) = \operatorname{tr}^2_\Sigma H^2(\mathbb{R}^d) = H^1(\Sigma).$$

Moreover we have for all $h \in H^{-1}(\Sigma)$ and all $u \in H^2(\mathbb{R}^d)$

$$\begin{aligned}\langle (G^{\circledast})^* h, u\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)} &= \langle h, G^{\circledast} u\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} \\ &= \langle h, G^* u\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} \\ &= \langle h, \mathrm{tr}_\Sigma^2 u\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} = \langle h\delta_\Sigma, u\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)}\end{aligned}$$

and hence $(G^{\circledast})^* h = h\delta_\Sigma$. To prove $\operatorname{ran}(G^{\circledast})^* \cap H^{-1}(\mathbb{R}^d) = \{0\}$ let $h \in H^{-1}(\Sigma)$ with $(G^{\circledast})^* h \in H^{-1}(\mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$. Let $(\varphi_n)_n \subset \mathscr{S}(\mathbb{R}^d)$ be again a sequence with $\|\varphi_n\|_{H^1(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0$ and $\varphi_n = 1$ on Σ, cf. the proof of Lemma 2.21. Hence $\|\psi\varphi_n\|_{H^1(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0$ and

$$\begin{aligned}\langle (G^{\circledast})^* h, \psi\rangle_{-1,1} = \langle (G^{\circledast})^* h, \psi\rangle_{-2,2} = \langle h, G^{\circledast}\psi\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} = \langle h, G^*\psi\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} \\ = \langle h, \mathrm{tr}_\Sigma^2 \psi\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} = \langle h, \mathrm{tr}_\Sigma^2(\psi\varphi_n)\rangle_{H^{-1}(\Sigma),H^1(\Sigma)} = \langle (G^{\circledast})^* h, \psi\varphi_n\rangle_{-1,1} \xrightarrow{n\to\infty} 0.\end{aligned}$$

As $\psi \in C_0^\infty(\mathbb{R}^d)$ was arbitrary we get $(G^{\circledast})^* h = 0$. Hence $\operatorname{ran}(G^{\circledast})^* \cap H^{-1}(\mathbb{R}^d) = 0$. □

Due to Lemma 4.1 the operator G satisfies all conditions in (3.1) on page 29 for $k = 2$ and $j = 0$. Analogously as in Lemma 3.4 we define in $L^2(\mathbb{R}^d)$ the operator

$$\begin{aligned}Tu &= (-\Delta + 1)u - h\delta_\Sigma, \\ \operatorname{dom} T &= \{u \in L^2(\mathbb{R}^d) : \exists h \in L^2(\Sigma) \text{ with } (-\Delta+1)u - h\delta_\Sigma \in L^2(\mathbb{R}^d)\}.\end{aligned}$$

According to Lemma 2.1 we have $\operatorname{dom} T = \operatorname{dom} A \dot{+} \ker T = H^2(\mathbb{R}^d) \dot{+} \ker T$. Hence every $u \in \operatorname{dom} T$ can be written as $u = u_c + u_s$ with $u_c \in H^2(\mathbb{R}^d)$ and $u_s \in \ker T$. Using this decomposition we define the mappings

$$\begin{aligned}\Gamma_0 &: \operatorname{dom} T \to L^2(\Sigma), \qquad u \mapsto h, \\ \Gamma_1 &: \operatorname{dom} T \to L^2(\Sigma), \qquad u \mapsto \mathrm{tr}_\Sigma^2 u_c,\end{aligned}$$

cf. Theorem 3.5. Note that the space $\mathcal{G}^+ := \operatorname{ran}(G^* \restriction H^2(\mathbb{R}^d)) = H^1(\Sigma)$ is dense in $L^2(\Sigma)$ and $\mathcal{G}^- := (\mathcal{G}^+)^* = H^{-1}(\Sigma)$. Hence a direct consequence of Lemma 4.1, Lemma 3.4, Theorem 3.5 and Theorem 3.6 is the following corollary.

Corollary 4.2. *The triple* $(L^2(\Sigma), \Gamma_0, \Gamma_1)$ *is a generalized boundary triple for* $\overline{T} = S^*$ *with*

$$Su = (-\Delta+1)u, \qquad \operatorname{dom} S = \{u \in H^2(\mathbb{R}^d) : \mathrm{tr}_\Sigma^2 u = 0\} = \ker\Gamma_0 \cap \ker\Gamma_1.$$

The operator S^* *is given by*

$$\begin{aligned}S^* u &= (-\Delta+1)u - h\delta_\Sigma, \\ \operatorname{dom} S^* &= \{u \in L^2(\mathbb{R}^d) : \exists h \in H^{-1}(\Sigma) \text{ with } (-\Delta+1)u - h\delta_\Sigma \in L^2(\mathbb{R}^d)\}.\end{aligned}$$

For $\lambda < 0$ denote by G_λ the integral kernel of the resolvent of the free Laplacian, i.e. $(-\Delta - \lambda)^{-1}u = G_\lambda * u$ for all $u \in L^2(\mathbb{R}^d)$. According to [64, Chapter 7.4] we have

$$G_\lambda(x) = \frac{1}{(2\pi)^{d/2}} \left(\frac{\sqrt{-\lambda}}{|x|} \right)^{d/2-1} K_{\frac{d}{2}-1}\big(\sqrt{-\lambda}|x|\big), \qquad x \in \mathbb{R}^d \setminus \{0\},$$

where K_ν denotes the ν-th modified Bessel function of the second kind. Using (7.44) in [64, Chapter 7.4] we get

$$\begin{aligned} G_\lambda(x) &= \frac{1}{(2\pi)^{d/2}} \left(\frac{\sqrt{-\lambda}}{|x|} \right)^{d/2-1} \left(\frac{\Gamma(\frac{d}{2}-1)}{2} \left(\frac{\sqrt{-\lambda}|x|}{2} \right)^{-d/2+1} + O\Big((\sqrt{-\lambda}|x|)^{-d/2+3} \Big) \right) \\ &= \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}|x|^{d-2}} + \frac{1}{(2\pi)^{d/2}} O\left(\frac{-\lambda}{|x|^{d-4}} \right) \end{aligned}$$

for $x \in \mathbb{R}^d \setminus \{0\}$, $d \geq 3$, and $\lambda \to 0$. Hence, for $d \geq 3$ the function G_0 defined by

$$G_0(x) := \lim_{\lambda \to 0} G_\lambda(x) = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}|x|^{d-2}}, \qquad x \in \mathbb{R}^d \setminus \{0\},$$

is the fundamental solution of the Laplace operator. Moreover we can define (by analytic continuation, cf. [64, Chapter 7.4]) G_λ for all $\lambda \in \mathbb{C} \setminus [0, \infty[$ such that $(-\Delta - \lambda)^{-1}u = G_\lambda * u$ remains true for all $u \in L^2(\mathbb{R}^d)$ and $\overline{G_\lambda} = G_{\overline{\lambda}}$ holds.

The next lemma gives an explicit representation of the γ-field γ and an estimate for its norms.

Lemma 4.3. *Let $\lambda \in \mathbb{C} \setminus [1, \infty[$ and $h \in L^2(\Sigma)$. Then*

$$\big(\gamma(\lambda)h\big)(x) = \int_\Sigma h(y) G_{\lambda-1}(x-y)\, d\sigma(y) \tag{4.1}$$

holds for almost all $x \in \mathbb{R}^d$. Moreover we have for $\lambda < 1$ and $\varepsilon \in]0,1]$ the estimate

$$\|\gamma(\lambda)\| \leq \frac{\min\{|\lambda-1|, 1\}^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}} \| \operatorname{tr}_\Sigma^{1+\varepsilon} \|.$$

In particular $\lim_{\lambda \to -\infty} \|\gamma(\lambda)\| = 0$. If we assume additionally that Σ is a compact C^∞-manifold then $\gamma(\lambda) \in \mathfrak{S}_p(L^2(\Sigma), L^2(\mathbb{R}^d))$ for all $p > d-2$ and $\lambda \in \mathbb{C} \setminus [1, \infty[$.

Proof. Using $\operatorname{ran}(A - \overline{\lambda})^{-1} = H^2(\mathbb{R}^d)$, $A - \overline{\lambda} = -\Delta - (\overline{\lambda} - 1)$ and $\gamma(\lambda)^* = \Gamma_1(A - \overline{\lambda})^{-1}$, cf. Lemma 2.6, we get for all $h \in L^2(\Sigma)$ and $u \in L^2(\mathbb{R}^d)$

$$\langle \gamma(\lambda)h, u \rangle_{L^2(\mathbb{R}^d)} = \langle h, \gamma(\lambda)^* u \rangle_{L^2(\Sigma)} = \langle h, \Gamma_1(A - \overline{\lambda})^{-1} u \rangle_{L^2(\Sigma)} = \langle h, \operatorname{tr}_\Sigma^2 (G_{\overline{\lambda}-1} * u) \rangle_{L^2(\Sigma)}.$$

With Fubini's theorem it follows

$$\begin{aligned}\langle \gamma(\lambda)h, u\rangle_{L^2(\mathbb{R}^d)} &= \int_\Sigma h(s)\overline{(G_{\overline{\lambda}-1} * u)(y)}\, d\sigma(y) \\ &= \int_\Sigma h(s)\overline{\left(\int_{\mathbb{R}^d} G_{\overline{\lambda}-1}(x-y)u(x)\, dx\right)}\, d\sigma(y) \\ &= \int_{\mathbb{R}^d}\left(\int_\Sigma h(y)G_{\lambda-1}(x-y)\, d\sigma(y)\right)\overline{u(x)}\, dx.\end{aligned}$$

In particular

$$\langle \gamma(\lambda)|h|, \mathbb{1}_K\rangle_{L^2(\mathbb{R}^d)} = \int_K\int_\Sigma |h(y)|G_{\lambda-1}(x-y)\, d\sigma(y)\, dx \geq \int_K\left|\int_\Sigma h(y)G_{\lambda-1}(x-y)\, d\sigma(y)\right| dx$$

for every compact set $K \subseteq \mathbb{R}^d$. Hence $x \mapsto \int_\Sigma h(y)G_{\lambda-1}(x-y)\, d\sigma(y)$ is a function in $L^1_{loc}(\mathbb{R}^d)$ which coincides with $\gamma(\lambda)h$ in the distributional sense. Hence they coincide also in $L^2(\mathbb{R}^d)$ and equation (4.1) follows.

For $\lambda < 1$ and $\varepsilon \in]0,1]$ we have

$$\|\gamma(\lambda)\|_{\mathcal{L}(L^2(\Sigma),L^2(\mathbb{R}^d))} = \|\gamma(\lambda)^*\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^2(\Sigma))} \leq \|\gamma(\lambda)^*\|_{\mathcal{L}(L^2(\mathbb{R}^d),H^\varepsilon(\Sigma))}.$$

Using again $\gamma(\lambda)^* = \operatorname{tr}^2_\Sigma(A-\overline{\lambda})^{-1} = \operatorname{tr}^2_\Sigma(A-\lambda)^{-1}$ we get with Lemma 2.18

$$\begin{aligned}\|\gamma(\lambda)\|_{\mathcal{L}(L^2(\Sigma),L^2(\mathbb{R}^d))} &\leq \|\operatorname{tr}^2_\Sigma(A-\lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d),H^\varepsilon(\Sigma))} \\ &\leq \|\operatorname{tr}^{\varepsilon+1}_\Sigma\|_{\mathcal{L}(H^{\varepsilon+1}(\mathbb{R}^d),H^\varepsilon(\Sigma))}\|(-\Delta-(\lambda-1))^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d),H^{\varepsilon+1}(\mathbb{R}^d))} \\ &\leq \|\operatorname{tr}^{\varepsilon+1}_\Sigma\|_{\mathcal{L}(H^{\varepsilon+1}(\mathbb{R}^d),H^\varepsilon(\Sigma))}\frac{\min\{|\lambda-1|,1\}^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}}.\end{aligned}$$

Choosing for example $\varepsilon = \frac{1}{2}$ we get

$$\|\gamma(\lambda)\|_{\mathcal{B}(L^2(\Sigma),L^2(\mathbb{R}^d))} \leq \frac{\|\operatorname{tr}^{3/2}_\Sigma\|_{\mathcal{B}(H^{3/2}(\mathbb{R}^d),H^{1/2}(\Sigma))}}{|\lambda-1|^{1/4}} \xrightarrow{\lambda\to-\infty} 0.$$

According to Lemma 2.6 the operator $\gamma(\lambda)^*$ belongs to $\mathcal{L}(L^2(\mathbb{R}^d),L^2(\Sigma))$ for all $\lambda \in \rho(A) = \mathbb{C}\setminus[1,\infty[$ and $\operatorname{ran}\gamma(\lambda)^* \subseteq H^1(\Sigma)$. If we assume additionally that Σ is a compact C^∞-manifold then Lemma 2.23 implies $\gamma(\overline{\lambda+1})^* \in \mathfrak{S}_p(L^2(\mathbb{R}^d),L^2(\Sigma))$ for all $p > d-2$. As the singular values of $\gamma(\lambda)^*$ and $\gamma(\lambda)$ coincide also the last statement is proven. □

The next lemma provides some properties of the Weyl function M.

Lemma 4.4. *Let $\lambda \in \rho(A) = \mathbb{C} \setminus [1,\infty[$. Then the operator $M(\lambda)$ can be written as*

$$M(\lambda) = \lambda \gamma(0)^* \gamma(\lambda) = \operatorname{tr}_\Sigma^2 \big(\gamma(\lambda) - \gamma(0)\big)$$

and satisfies

$$\big(M(\lambda)h\big)(x) = \int_\Sigma h(y) \Big(G_{\lambda-1}(|x-y|) - G_{-1}(|x-y|) \Big)\, d\sigma(y) \tag{4.2}$$

for all $h \in L^2(\Sigma)$ and almost all $x \in \Sigma$. In particular $M(0) = 0$. If $\lambda \neq 0$ then $M(\lambda)^{-1}$ is an unbounded operator in $L^2(\Sigma)$. Furthermore we have for all $\lambda < 1$ and $\varepsilon \in]0,1]$ the estimate

$$\|M(\lambda)\| \leq |\lambda| \cdot \frac{\big(\min\{|\lambda-1|,1\}\big)^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}} \cdot \|\operatorname{tr}_\Sigma^{1+\varepsilon}\|^2_{\mathcal{L}(H^{1+\varepsilon}(\mathbb{R}^d), H^\varepsilon(\Sigma))}.$$

If we assume additionally that Σ is a compact C^∞-manifold then $M(\lambda) \in \mathfrak{S}_p(L^2(\Sigma))$ for all $p > \frac{d}{2} - 1$ and $\lambda \in \rho(A)$.

Proof. Let $h \in L^2(\Sigma)$ be arbitrary. As Γ_0 is surjective there exists $u \in \operatorname{dom} T$ such that $h = \Gamma_0 u = \Gamma_0 u_s$, where we have used the decomposition $u = u_c + u_s \in \operatorname{dom} A \dotplus \ker T$ and $\operatorname{dom} A = \ker \Gamma_0$. Hence we get with the definition of $\gamma(\lambda)$ in Lemma 2.6

$$M(0)h = \Gamma_1 \gamma(0) \Gamma_0 u_s = \Gamma_1 (\Gamma_0 \restriction \ker T)^{-1} \Gamma_0 u_s = \Gamma_1 u_s = \operatorname{tr}_\Sigma^2 (u_s)_c = \operatorname{tr}_\Sigma^2 0 = 0.$$

As $h \in L^2(\Sigma)$ was arbitrary it follows $M(0) = 0$. Using Lemma 2.7 we obtain now

$$M(\lambda) = M(\lambda) - M(0)^* = (\lambda - \overline{0}) \gamma(0)^* \gamma(\lambda) = \lambda \gamma(0)^* \gamma(\lambda)$$

for all $\lambda \in \rho(A) = \mathbb{C} \setminus [1,\infty[$. Furthermore we get with Lemma 2.6 (by interchanging λ and μ)

$$\gamma(0) - \gamma(\lambda) = (0 - \lambda)(A - 0)^{-1} \gamma(\lambda) = -\lambda A^{-1} \gamma(\lambda).$$

Using $\gamma(0)^* = \operatorname{tr}_\Sigma^2 (A-0)^{-1}$ we get hence

$$M(\lambda) = \lambda \gamma(0)^* \gamma(\lambda) = \operatorname{tr}_\Sigma^2 \lambda A^{-1} \gamma(\lambda) = \operatorname{tr}_\Sigma^2 \big(\gamma(\lambda) - \gamma(0)\big).$$

Together with equation (4.1) from Lemma 4.3 we get now (4.2). Furthermore we get with Lemma 4.3 for $\lambda \in \rho(A) \cap \mathbb{R}$ an estimate for the norm of $M(\lambda)$:

$$\begin{aligned}
\|M(\lambda)\| &= \|\lambda \gamma(0)^* \gamma(\lambda)\| \leq |\lambda| \cdot \|\gamma(0)^*\| \cdot \|\gamma(\lambda)\| = |\lambda| \cdot \|\gamma(0)\| \cdot \|\gamma(\lambda)\| \\
&\leq |\lambda| \cdot \|\operatorname{tr}_\Sigma^{1+\varepsilon}\| \frac{\min\{|0-1|,1\}^{-\frac{1+\varepsilon}{2}}}{|0-1|^{\frac{1-\varepsilon}{2}}} \cdot \|\operatorname{tr}_\Sigma^{1+\varepsilon}\| \frac{\min\{|\lambda-1|,1\}^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}} \\
&= |\lambda| \cdot \|\operatorname{tr}_\Sigma^{1+\varepsilon}\|^2 \frac{\min\{|\lambda-1|,1\}^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}}.
\end{aligned}$$

Next we show that $M(\lambda)^{-1}$ is an unbounded operator if $\lambda \in \rho(A) \setminus \{0\}$. For this let $f \in L^2(\Sigma) \setminus \{0\}$. At first we consider the case $\lambda \in \mathbb{R}$. Using the continuity of the γ-field on $\rho(A)$, of. Lemma 2.6, we get for $\mu \in \rho(A) \cap \mathbb{R} =]-\infty, -1[$

$$\begin{aligned} \frac{\langle M(\lambda)f,f\rangle_{L^2(\Sigma)} - \langle M(\mu)f,f\rangle_{L^2(\Sigma)}}{\lambda-\mu} &= \frac{\langle [M(\lambda)-M(\mu)]f,f\rangle_{L^2(\Sigma)}}{\lambda-\mu} \\ &= \frac{\langle (\lambda-\overline{\mu})[\gamma(\mu)^*\gamma(\lambda)]f,f\rangle_{L^2(\Sigma)}}{\lambda-\mu} \\ &= \langle \gamma(\lambda)f,\gamma(\mu)f\rangle_{L^2(\Sigma)} \xrightarrow{\mu\to\lambda} \|\gamma(\lambda)f\|^2_{L^2(\Sigma)}. \end{aligned}$$

Hence $\frac{d}{d\lambda}\langle M(\lambda)f,f\rangle_{L^2(\Sigma)} = \|\gamma(\lambda)f\|^2_{L^2(\Sigma)} > 0$ because $\gamma(\lambda)$ is injective, i.e. the function $]-\infty,1[\ni \lambda \mapsto \langle M(\lambda)f,f\rangle_{L^2(\Sigma)}$ is strictly increasing. As $M(0)=0$ this means $\langle M(\lambda)f,f\rangle_{L^2(\Sigma)} \neq 0$ if $\lambda \neq 0$. In particular $M(\lambda)f \neq 0$ for all $f \neq 0$ and therefore $\ker M(\lambda) = \{0\}$. If $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we get with Lemma 2.7 for $f \neq 0$

$$\begin{aligned} \operatorname{Im}\Big(\langle M(\lambda)f,f\rangle_{L^2(\Sigma)}\Big) &= \frac{1}{2i}\Big(\langle M(\lambda)f,f\rangle_{L^2(\Sigma)} - \langle f,M(\lambda)f\rangle_{L^2(\Sigma)}\Big) \\ &= \frac{1}{2i}\big\langle \big(M(\lambda)-M(\lambda)^*\big)f,f\big\rangle_{L^2(\Sigma)} \\ &= \frac{1}{2i}\Big\langle (\lambda-\overline{\lambda})\gamma(\lambda)^*\gamma(\lambda)f,f\Big\rangle_{L^2(\Sigma)} = \operatorname{Im}(\lambda)\|\gamma(\lambda)f\|^2_{L^2(\Sigma)} \neq 0. \end{aligned}$$

Hence also in this case $M(\lambda)f \neq 0$ and therefore $\ker M(\lambda) = \{0\}$. This means that $M(\lambda)^{-1}$ is an operator. Furthermore we have

$$\big(\operatorname{dom} M(\lambda)^{-1}\big)^{\perp} = \big(\operatorname{ran} M(\lambda)\big)^{\perp} = \ker M(\lambda)^* = \ker M(\overline{\lambda}) = \{0\},$$

i.e. $\operatorname{dom} M(\lambda)^{-1}$ is dense in $L^2(\Sigma)$. On the other hand we have

$$\operatorname{dom} M(\lambda)^{-1} = \operatorname{ran} M(\lambda) \subseteq \operatorname{ran}\Gamma_1 \subseteq H^1(\Sigma) \neq L^2(\Sigma).$$

Hence $\operatorname{dom} M(\lambda)^{-1}$ is not closed. As $M(\lambda)^{-1}$ is closed (because $M(\lambda) = M(\overline{\lambda})^*$ is closed) it follows that $M(\lambda)^{-1}$ is not continuous.

It remains to show that $M(\lambda)$ belongs to $\mathfrak{S}_p(L^2(\Sigma))$ for all $p > \frac{d}{2}-1$ if Σ is a compact C^∞-manifold. For this recall that $\gamma(\lambda) \in \mathfrak{S}_q(L^2(\Sigma), L^2(\mathbb{R}^d))$ and $\gamma(\lambda)^* \in \mathfrak{S}_q(L^2(\mathbb{R}^d), L^2(\Sigma))$ for all $q > d-2$, cf. Lemma 4.3. Hence we get with Corollary 2.2 in Chapter II of [37]

$$M(\lambda) = \lambda\gamma(0)\gamma(\lambda)^* \in \mathfrak{S}_{\frac{q}{2}}(L^2(\Sigma)).$$

With $p := \frac{q}{2} > \frac{d}{2} - 1$ the desired result follows. □

The following Lemma shows again, that the codimension of Σ plays an important role. Whereas in the case that Σ is a closed manifold of codimension 1 which separates $\mathbb{R}^d$ into an interior domain Ω_i and an exterior domain Ω_e the Friedrichs extension of S is given by the orthogonal sums of the Dirichlet operators on Ω_i and on Ω_e the situation is different if the codimension is 2.

Theorem 4.5. *The operator A is the Friedrichs extension of S.*

Proof. Define the form $\mathfrak{s}$ by

$$\mathfrak{s}(u,v) := \langle Su, v\rangle, \qquad \operatorname{dom}\mathfrak{s} = \operatorname{dom} S.$$

Then the domain of $\overline{\mathfrak{s}}$ is given by

$$\begin{aligned}\operatorname{dom}\overline{\mathfrak{s}} &= \{u \in L^2(\mathbb{R}^d) : \exists (u_k)_k \subseteq \operatorname{dom} S \text{ with } \|u_k - u\|_{L^2} \xrightarrow{k\to\infty} 0 \text{ and } s[u_k - u_l] \xrightarrow{k,l\to\infty} 0\} \\ &= \{u \in L^2(\mathbb{R}^d) : \exists (u_k)_k \subseteq \operatorname{dom} S \text{ with } \|u_k - u\|_{H^1} \xrightarrow{k\to\infty} 0\} = \overline{\operatorname{dom} S}^{H^1(\mathbb{R}^d)}.\end{aligned}$$

Let now $u \in H^1(\mathbb{R}^d)$ be arbitrary. Hence there exists a sequence $(\varphi_n)_n \subseteq C_0^\infty(\mathbb{R}^d)$ with $\|u - \varphi_n\|_{H^1(\mathbb{R}^d)} \leq (2n)^{-1}$. Note that Σ has codimension 2. Therefore the H^1-capacity of Σ is 0, cf. Corollary 3.3.4 and Corollary 5.1.15 in [2]. This means in particular that we can find for every $n \in \mathbb{N}$ a function $\psi_n \in \mathscr{S}(\mathbb{R}^d)$ which is equal to 1 on an open neighborhood of Σ and satisfies $\|\psi_n\|_{H^1(\mathbb{R}^d)} \leq (2n\|\varphi_n\|_{C^1(\mathbb{R}^d)})^{-1}$. Defining $u_n := (1-\psi_n)\varphi_n \in \operatorname{dom} S$ we get

$$\begin{aligned}\|u - u_n\|_{H^1(\mathbb{R}^d)} &\leq \|u - \varphi_n\|_{H^1(\mathbb{R}^d)} + \|\psi_n\varphi_n\|_{H^1(\mathbb{R}^d)} \\ &\leq \frac{1}{2n} + \|\psi_n\|_{C^1(\mathbb{R}^d)}\|\varphi_n\|_{H^1(\mathbb{R}^d)} \\ &\leq \frac{1}{2n} + \|\psi_n\|_{C^1(\mathbb{R}^d)}\frac{1}{2n\|\varphi_n\|_{C^1(\mathbb{R}^d)}} = \frac{1}{n}.\end{aligned}$$

Hence $H^1(\mathbb{R}^d) \subseteq \overline{\operatorname{dom} S}^{H^1(\mathbb{R}^d)}$. As the converse inclusion is obvious we get $\operatorname{dom}\overline{\mathfrak{s}} = H^1(\mathbb{R}^d)$. Therefore we have

$$\overline{\mathfrak{s}}(u,v) = \lim_{k\to\infty}\mathfrak{s}(u_k,v_k) = \lim_{k\to\infty}\langle Su_k, v_k\rangle = \lim_{k\to\infty}\sum_{j=1}^n\langle\partial_j u_k, \partial_j v_k\rangle = \sum_{j=1}^n\langle\partial_j u, \partial_j v\rangle$$

for all $u,v \in H^1(\mathbb{R}^d)$. Hence we get for all $u \in H^2(\mathbb{R}^d) = \operatorname{dom} A$ and all $v \in H^1(\mathbb{R}^d) = \operatorname{dom}\overline{\mathfrak{s}}$

$$\langle Au, v\rangle = \langle -\Delta u, v\rangle = \sum_{j=1}^n\langle\partial_j u, \partial_j v\rangle = \overline{\mathfrak{s}}(u,v).$$

According to Corollary 2.4 in [41, Ch.VI] this means that A is contained in the representing operator of $\overline{\mathfrak{s}}$, i.e. $A \subseteq S_F$. As both operators are selfadjoint they coincide. □

4.2 The operators A_Θ

In this section we investigate the operators A_Θ generated by the generalized boundary triple constructed in the previous section. We give criteria for selfadjointness of A_Θ and estimates for the spectrum $\sigma(A_\Theta)$ in dependence on the parameter Θ (note that the parameter Θ might be an operator or even a linear relation in $L^2(\Sigma)$). Moreover we show that (under certain conditions) the operators A_Θ can also be parametrized within the setting of a δ-interaction on a manifold of codimension 1.

We start with a first criterion for selfadjointness.

Theorem 4.6. *Let Θ be a closed symmetric linear relation in $L^2(\Sigma)$ with $H^1(\Sigma) \subseteq \operatorname{ran}\Theta$ and $0 \notin \sigma_p(\Theta)$. Then the operator*

$$A_\Theta u = Tu, \qquad \operatorname{dom} A_\Theta = \{u \in \operatorname{dom} T : \Gamma u \in \Theta\} = \{u \in \operatorname{dom} T : \Theta^{-1}\Gamma_1 u = \Gamma_0 u\}$$

is a selfadjoint operator in $L^2(\mathbb{R}^d)$. If we assume additionally that $0 \notin \sigma(\Theta)$ and that Σ is a compact C^∞-manifold then

$$(A_\Theta - \lambda)^{-1} - (A - \lambda)^{-1} \in \mathfrak{S}_p(L^2(\mathbb{R}^d))$$

holds for all $\lambda \in \rho(A_\Theta) \cap \rho(A)$ and $p > \frac{d}{2} - 1$.

Proof. Keeping in mind $\operatorname{ran}\gamma(\overline{\lambda})^* \subseteq \operatorname{ran}\operatorname{tr}_\Sigma^2 = H^1(\Sigma)$ the selfadjointness of A_Θ follows directly from Theorem 2.8 for $\lambda = 0$. Moreover we get $0 \in \rho(A_\Theta)$ and

$$A_\Theta^{-1} - A^{-1} = \gamma(0)\Theta^{-1}\gamma(0)^*.$$

If $0 \notin \sigma(\Theta)$ then $\Theta^{-1} \in \mathcal{L}(L^2(\Sigma))$. Moreover, if Σ is a compact C^∞-manifold then $\gamma(0) \in \mathfrak{S}_q(L^2(\Sigma), L^2(\mathbb{R}^d))$ and $\gamma(0)^* \in \mathfrak{S}_q(L^2(\mathbb{R}^d), L^2(\Sigma))$ for all $q > d - 2$, cf. Lemma 4.3. Hence we get with Corollary 2.2 in Chapter II of [37] and $p := \frac{q}{2} > \frac{d}{2} - 1$

$$A_\Theta^{-1} - A^{-1} = \gamma(0)\Theta^{-1}\gamma(0)^* \in \mathfrak{S}^{\frac{q}{2}}(L^2(\mathbb{R}^d)) = \mathfrak{S}^p(L^2(\mathbb{R}^d)).$$

For arbitrary $\lambda \in \rho(A_\Theta) \cap \rho(A)$ note that $I + \lambda(A_\Theta - \lambda)^{-1}$ and $I + \lambda(A - \lambda)^{-1}$ belong both to

$\mathcal{L}(L^2(\mathbb{R}^d))$ and hence

$$\begin{aligned}
\mathfrak{S}^p(L^2(\mathbb{R}^d)) \ni & \Big(I+\lambda(A_\Theta-\lambda)^{-1}\Big)\Big(A_\Theta^{-1}-A^{-1}\Big)\Big(I+\lambda(A-\lambda)^{-1}\Big) \\
&= \Big(A_\Theta^{-1}-A^{-1}+(A_\Theta-\lambda)^{-1}-A_\Theta^{-1}-\lambda(A_\Theta-\lambda)^{-1}A^{-1}\Big)\Big(I+\lambda(A-\lambda)^{-1}\Big) \\
&= \Big((A_\Theta-\lambda)^{-1}-A^{-1}-\lambda(A_\Theta-\lambda)^{-1}A^{-1}\Big)\Big(I+\lambda(A-\lambda)^{-1}\Big) \\
&= \Big((A_\Theta-\lambda)^{-1}-A^{-1}-\lambda(A_\Theta-\lambda)^{-1}A^{-1}\Big)+\lambda(A_\Theta-\lambda)^{-1}(A-\lambda)^{-1} \\
&+\Big(I+\lambda(A_\Theta-\lambda)^{-1}\Big)(-\lambda)A^{-1}(A-\lambda)^{-1} \\
&= \Big((A_\Theta-\lambda)^{-1}-A^{-1}-\lambda(A_\Theta-\lambda)^{-1}A^{-1}\Big)+\lambda(A_\Theta-\lambda)^{-1}(A-\lambda)^{-1} \\
&+\Big(I+\lambda(A_\Theta-\lambda)^{-1}\Big)\Big(A^{-1}-(A-\lambda)^{-1}\Big) \\
&= \Big((A_\Theta-\lambda)^{-1}-A^{-1}-\lambda(A_\Theta-\lambda)^{-1}A^{-1}\Big)+\lambda(A_\Theta-\lambda)^{-1}(A-\lambda)^{-1} \\
&+A^{-1}-(A-\lambda)^{-1}+\lambda(A_\Theta-\lambda)^{-1}A^{-1}-\lambda(A_\Theta-\lambda)^{-1}(A-\lambda)^{-1} \\
&= (A_\Theta-\lambda)^{-1}-(A-\lambda)^{-1},
\end{aligned}$$

where we have used the well-known resolvent identity $(\lambda-\mu)(A_\Theta-\lambda)^{-1}(A_\Theta-\mu)^{-1} = (A_\Theta-\lambda)^{-1}-(A_\Theta-\mu)^{-1}$ and an analog identity for A. □

In the next theorem we show that A_Θ is semibounded from below if the parameter Θ is uniformly positive.

Theorem 4.7. *Let Θ be a selfadjoint operator in $L^2(\Sigma)$ with $\Theta \geq \theta$ for some $\theta > 0$. Then A_Θ is selfadjoint and $\sigma(A_\Theta) \subseteq [\frac{\theta}{\theta+c^2},\infty[$ with $c := \|\operatorname{tr}_\Sigma^{1+\varepsilon}\|$ for $\varepsilon \in]0,1]$.*

Proof. As seen in the proof of Lemma 4.4 the function $]-\infty,1[\ni\lambda\mapsto\langle M(\lambda)f,f\rangle_{L^2(\Sigma)}$ is strictly increasing for all $f\neq 0$. With $M(0)=0$, cf. Lemma 4.4, we have therefore $\langle M(\lambda)f,f\rangle_{L^2(\Sigma)}<0$ for all $\lambda<0$. Hence

$$\langle[\Theta-M(\lambda)]f,f\rangle_{L^2(\Sigma)} = \langle\Theta f,f\rangle_{L^2(\Sigma)} - \underbrace{\langle M(\lambda)f,f\rangle_{L^2(\Sigma)}}_{\leq 0} \geq \theta\|f\|^2.$$

Hence $[\Theta-M(\lambda)]\geq\theta$ and $[\Theta-M(\lambda)]^{-1}\in\mathcal{L}(L^2(\Sigma))$. Therefore A_Θ is selfadjoint and $\lambda\in\rho(A_\Theta)$ for all $\lambda<0$, cf. Theorem 2.8.

Let now $\lambda\in[0,1[$. If $\|M(\lambda)\|<\theta$ we have

$$\begin{aligned}
\langle[\Theta-M(\lambda)]f,f\rangle_{L^2(\Sigma)} &= \langle\Theta f,f\rangle_{L^2(\Sigma)} - \langle M(\lambda)f,f\rangle_{L^2(\Sigma)} \\
&\geq \theta\|f\|^2-\|M(\lambda)\|\cdot\|f\|^2 = (\theta-\|M(\lambda)\|)\cdot\|f\|^2.
\end{aligned}$$

Hence $[\Theta - M(\lambda)]^{-1} \in \mathcal{L}(L^2(\Sigma))$ and $\lambda \in \rho(A_\Theta)$. It remains to verify that $\|M(\lambda)\| < \theta$ holds for $0 \le \lambda < \frac{\theta}{\theta+c^2}$. For $\lambda = 0$ this is obvious. Note that $0 < \lambda < \frac{\theta}{\theta+c^2}$ implies

$$\frac{|\lambda-1|}{|\lambda|} = \frac{1-\lambda}{\lambda} = \frac{1}{\lambda} - 1 > \frac{\theta+c^2}{\theta} - 1 = \frac{c^2}{\theta}.$$

Hence we get with Lemma 4.4

$$\|M(\lambda)\| \le |\lambda| \cdot \frac{\big(\min\{|\lambda-1|,1\}\big)^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}} \cdot \|\operatorname{tr}_\Sigma^{1+\varepsilon}\|^2 \le \frac{|\lambda|}{|\lambda-1|} \cdot c^2 < \frac{\theta}{c^2} \cdot c^2 = \theta.$$

Consequently $]-\infty, \frac{\theta}{\theta+c^2}[\, \subseteq \rho(A_\Theta)$ and therefore $\sigma(A_\Theta) \subseteq [\frac{\theta}{\theta+c^2}, \infty[$. □

Next we give an analog of the previous theorem for uniformly negative parameter.

Theorem 4.8. *Let Θ be a selfadjoint operator in $L^2(\Sigma)$ with $\Theta \le \theta$ for some $\theta < 0$. Then A_Θ is selfadjoint and $\rho(A_\Theta) \supseteq\,]\frac{\theta}{c^2}, 1[$ with $c := \|\operatorname{tr}_\Sigma^{1+\varepsilon}\|$ for some $\varepsilon \in]0,1]$. If $\Theta\big(H^1(\Sigma)\big) \subseteq H^1(\Sigma)$ then A_Θ is unbounded from below.*

Proof. As in the proof of the previous theorem we observe for all $\lambda \in]0,1[$

$$\langle \big(\Theta - M(\lambda)\big) f, f\rangle_{L^2(\Sigma)} = \langle \Theta f, f\rangle_{L^2(\Sigma)} - \underbrace{\langle M(\lambda) f, f\rangle_{L^2(\Sigma)}}_{\ge 0} \le \theta \|f\|^2 + 0.$$

Hence $[\Theta - M(\lambda)] \le \theta$ and $[\Theta - M(\lambda)]^{-1} \in \mathcal{L}(L^2(\Sigma))$. Therefore A_Θ is selfadjoint and $\lambda \in \rho(A_\Theta)$ for all $\lambda \in]0,1[$, cf. Theorem 2.8.

Let now $\frac{\theta}{c^2} < \lambda \le 0$. Then we get with Lemma 4.4

$$\|M(\lambda)\| \le |\lambda| \cdot \frac{\big(\min\{|\lambda-1|,1\}\big)^{-\frac{1+\varepsilon}{2}}}{|\lambda-1|^{\frac{1-\varepsilon}{2}}} \cdot \|\operatorname{tr}_\Sigma^{1+\varepsilon}\|^2 = |\lambda| c^2 < -\theta.$$

Hence $\theta + \|M(\lambda)\| < 0$. Using $\langle M(\lambda) f, f\rangle_{L^2(\Sigma)} \ge -\|M(\lambda)\| \cdot \|f\|^2$ we get

$$\langle [\Theta - M(\lambda)] f, f\rangle_{L^2(\Sigma)} = \langle \Theta f, f\rangle_{L^2(\Sigma)} - \langle M(\lambda) f, f\rangle_{L^2(\Sigma)} \le \big(\theta + \|M(\lambda)\|\big)\|f\|^2.$$

Hence $\Theta - M(\lambda) \le \theta + \|M(\lambda) < 0$. Therefore $[\Theta - M(\lambda)]^{-1} \in \mathcal{L}(L^2(\Sigma))$ and $\lambda \in \rho(A_\Theta)$ for all $\lambda \in]\frac{\theta}{c^2}, 0]$.

It remains to show that A_Θ is unbounded from below under the additional condition $\Theta\big(H^1(\Sigma)\big) \subseteq H^1(\Sigma)$. For this assume the converse, i.e. that A_Θ is bounded from below. As A is the Friedrichs extension of S (see Theorem 4.5) we know $A \ge A_\Theta$ (cf. Problem 2.22 in [41, Ch.VI]). Hence

$$(A - \lambda)^{-1} \le (A_\Theta - \lambda)^{-1}$$

for all $\lambda < \min\{1, \inf\sigma(A_\Theta)\}$. Using Krein's resolvent formula we get

$$0 \le (A_\Theta - \lambda)^{-1} - (A-\lambda)^{-1} = \gamma(\lambda)\big[\Theta - M(\lambda)\big]^{-1}\gamma(\lambda)^*.$$

Hence we have for all $u \in L^2(\mathbb{R}^d)$

$$0 \le \langle \gamma(\lambda)\big[\Theta - M(\lambda)\big]^{-1}\gamma(\lambda)^* u, u\rangle_{L^2(\mathbb{R}^d)} = \langle \big[\Theta - M(\lambda)\big]^{-1}\gamma(\lambda)^* u, \gamma(\lambda)^* u\rangle_{L^2(\Sigma)}.$$

As $\operatorname{ran}\gamma(\lambda)^* = H^1(\Sigma)$ the above estimate can be written as

$$0 \le \langle \big[\Theta - M(\lambda)\big]^{-1} g, g\rangle_{L^2(\Sigma)}, \qquad \forall\, g \in H^1(\Sigma).$$

If $f \in H^1(\Sigma)$ then we have $g := [\Theta - M(\lambda)]f \in H^1(\Sigma)$, because $\operatorname{ran} M(\lambda) \subseteq \operatorname{ran}\Gamma_1 = H^1(\Sigma)$ and $\Theta\big(H^1(\Sigma)\big) \subseteq H^1(\Sigma)$. Hence we get

$$0 \le \langle f, [\Theta - M(\lambda)]f\rangle_{L^2(\Sigma)}, \qquad \forall\, f \in H^1(\Sigma),$$

and therefore

$$\langle M(\lambda)f, f\rangle_{L^2(\Sigma)} \le \langle \Theta f, f\rangle_{L^2(\Sigma)}, \qquad \forall\, f \in H^1(\Sigma). \tag{4.3}$$

Choose a sequence $(f_n)_n \subseteq H^1(\Sigma)$ with $\|f_n\|_{L^2(\Sigma)} = 1$ and $\|f_n\|_{H^{-1}(\Sigma)} \xrightarrow{n\to\infty} 0$ (such a sequence exists because otherwise the norms $\|\cdot\|_{L^2(\Sigma)}$ and $\|\cdot\|_{H^{-1}(\Sigma)}$ would be equivalent). Note that $M(\lambda)$ can be considered as an continuous operator from $L^2(\Sigma)$ to $H^1(\Sigma)$. Hence we get

$$|\langle M(\lambda)f_n, f_n\rangle_{L^2(\Sigma)}| = |\langle M(\lambda)f_n, f_n\rangle_{H^1(\Sigma),H^{-1}(\Sigma)}| \le \|M(\lambda)\|\cdot\|f_n\|_{L^2(\Sigma)}\|f_n\|_{H^{-1}(\Sigma)} \xrightarrow{n\to\infty} 0.$$

Therefore (4.3) implies

$$0 > \theta = \theta\|f_n\|^2_{L^2(\Sigma)} \ge \langle \Theta f_n, f_n\rangle_{L^2(\Sigma)} \ge \langle M(\lambda)f_n, f_n\rangle_{L^2(\Sigma)} \xrightarrow{n\to\infty} 0,$$

which is a contradiction. Hence A_Θ is unbounded from below. □

Finally we will discuss in this section how a δ-interaction supported on Σ can be understood as a δ-interaction supported on a manifold of codimension 1 with singular strength. For this we assume that Σ is contained in a compact C^2-manifold $\mathcal{T}$ of codimension 1 without boundary. By $\operatorname{tr}_{\mathcal{T}\upharpoonright\Sigma}$ we denote the trace from $H^{3/2}(\mathcal{T})$ to $H^1(\Sigma)$. For $h \in L^2(\Sigma)$ we define the distribution $hd_\Sigma \in H^{-3/2}(\mathcal{T})$ via

$$\langle hd_\Sigma, \varphi\rangle_{H^{-3/2}(\mathcal{T}),H^{3/2}(\mathcal{T})} = \langle h, \operatorname{tr}_{\mathcal{T}\upharpoonright\Sigma}\varphi\rangle_{L^2(\Sigma)}, \qquad \varphi \in H^{3/2}(\mathcal{T}).$$

Analogously as for $h\delta_\Sigma$ one checks that hd_Σ belongs in fact to $H^{-3/2}(\mathcal{T})$:

$$\begin{aligned}
|\langle hd_\Sigma, \varphi\rangle_{H^{-3/2}(\mathcal{T}),H^{3/2}(\mathcal{T})}| &= |\langle h, \operatorname{tr}_{\mathcal{T}\upharpoonright\Sigma}\varphi\rangle_{L^2(\Sigma)}| \\
&\le \|h\|_{L^2(\Sigma)}\|\operatorname{tr}_{\mathcal{T}\upharpoonright\Sigma}\varphi\|_{L^2(\Sigma)} \\
&\le \|h\|_{L^2(\Sigma)}\|\operatorname{tr}_{\mathcal{T}\upharpoonright\Sigma}\varphi\|_{H^1(\Sigma)} \\
&\le \|h\|_{L^2(\Sigma)}\|\operatorname{tr}_{\mathcal{T}\upharpoonright\Sigma}\|\cdot\|\varphi\|_{H^{3/2}(\mathcal{T})}.
\end{aligned}$$

The calculation above shows in particular that the operator

$$\tilde{G} : L^2(\Sigma) \to H^{-3/2}(\mathcal{T}), \quad h \mapsto h d_\Sigma,$$

is continuous. Its adjoint is given by

$$\tilde{G}^* : H^{3/2}(\mathcal{T}) \to L^2(\Sigma), \quad \varphi \mapsto \varphi|_\Sigma.$$

Recall the definitions of the operators S and T

$$\begin{aligned} Su &= (-\Delta+1)u, & \operatorname{dom} S &= \{u \in H^2(\mathbb{R}^d) : \operatorname{tr}^2_\Sigma u = 0\} \\ Tu &= (-\Delta+1)u - h\delta_\Sigma, & \operatorname{dom} T &= \{u \in L^2(\mathbb{R}^d) : \exists h \in L^2(\Sigma) \\ &&& \quad \text{with } (-\Delta+1)u - h\delta_\Sigma \in L^2(\mathbb{R}^d)\} \end{aligned}$$

and the definitions of the boundary maps Γ_0 and Γ_1

$$\begin{aligned} &\Gamma_0 : \operatorname{dom} T \to L^2(\Sigma), && u \mapsto h, \\ &\Gamma_1 : \operatorname{dom} T \to L^2(\Sigma), && u \mapsto \operatorname{tr}^2_\Sigma u_c, \qquad u = u_c + u_s \in H^2(\mathbb{R}^d) \dotplus \ker T = \operatorname{dom} T. \end{aligned}$$

The generalized boundary triple for S^* used in this chapter is $(L^2(\Sigma), \Gamma_0, \Gamma_1)$, cf. Corollary 4.2.

As $\mathcal{T}$ is a manifold of codimension 1 we know by Example 3.8 that

$$G_\mathcal{T} : L^2(\mathcal{T}) \to H^{-1}(\mathbb{R}^d), \qquad f \mapsto f\delta_\mathcal{T}$$

is a continuous operator and that the operator $S_\mathcal{T} := A \restriction \ker(G^\mathcal{T})^*$ is given by

$$S_\mathcal{T} u = (-\Delta+1)u, \qquad \operatorname{dom} S_\mathcal{T} = \{u \in H^2(\mathbb{R}^d) : \operatorname{tr}^2_\mathcal{T} u = 0\}.$$

Its adjoint contains the operator $T_\mathcal{T}$ defined by

$$\begin{aligned} T_\mathcal{T} u &= (-\Delta+1)u - f\delta_\mathcal{T}, \\ \operatorname{dom} T_\mathcal{T} &= \{u \in L^2(\mathbb{R}^d) : \exists f \in L^2(\Sigma) \text{ with } (-\Delta+1)u - f\delta_\mathcal{T} \in L^2(\mathbb{R}^d)\}, \end{aligned}$$

and $(L^2(\Sigma), \Gamma_0^\mathcal{T}, \Gamma_1^\mathcal{T})$ with the boundary maps

$$\begin{aligned} &\Gamma_0^\mathcal{T} : \operatorname{dom} T_\mathcal{T} \to L^2(\mathcal{T}), && u \mapsto f, \\ &\Gamma_1^\mathcal{T} : \operatorname{dom} T_\mathcal{T} \to L^2(\mathcal{T}), && u \mapsto \operatorname{tr}^1_\mathcal{T} u. \end{aligned}$$

as in (3.6) is a generalized boundary triple for $S^*_\mathcal{T}$. Due to $\operatorname{ran}(G^\circledast_\mathcal{T})^* \cap L^2(\mathbb{R}^d) = \emptyset$ we can even construct an ordinary boundary for $S^*_\mathcal{T}$ as in Theorem 3.6: Let $\iota_- : H^{-3/2}(\mathcal{T}) \to L^2(\mathcal{T})$ and $\iota_+ : H^{3/2}(\mathcal{T}) \to L^2(\mathcal{T})$ be isomorphisms as in (3.4), i.e.

$$\langle u, v \rangle_{H^{3/2}(\mathcal{T}), H^{-3/2}(\mathcal{T})} = \langle \iota_+ u, \iota_- v \rangle_{L^2(\mathcal{T})} \qquad \forall\, u \in H^{3/2}(\mathcal{T}), v \in H^{-3/2}(\mathcal{T}).$$

Then the triple $(L^2(\mathcal{T}), \hat{\Gamma}_0^{\mathcal{T}}, \hat{\Gamma}_1^{\mathcal{T}})$ with

$$\begin{aligned}
&\hat{\Gamma}_0^{\mathcal{T}} : \operatorname{dom} S_{\mathcal{T}}^* \to L^2(\mathcal{T}), \quad u \mapsto \iota_- f,\\
&\hat{\Gamma}_1^{\mathcal{T}} : \operatorname{dom} S_{\mathcal{T}}^* \to L^2(\mathcal{T}), \quad u \mapsto \iota_+ \operatorname{tr}_{\mathcal{T}}^2 u_c, \quad u = u_c + u_s \in H^2(\mathbb{R}^d) \dotplus \ker S_{\mathcal{T}}^* = \operatorname{dom} S_{\mathcal{T}}^*
\end{aligned}$$

is an ordinary boundary triple for the operator $S_{\mathcal{T}}^*$, which is given by

$$\begin{aligned}
S_{\mathcal{T}}^* u &= (-\Delta+1)u - f\delta_{\mathcal{T}},\\
\operatorname{dom} S_{\mathcal{T}}^* &= \{u \in L^2(\mathbb{R}^d) : \exists f \in H^{-3/2}(\mathcal{T}) \text{ with } (-\Delta+1)u - f\delta_{\mathcal{T}} \in L^2(\mathbb{R}^d)\}.
\end{aligned}$$

Note that $S_{\mathcal{T}} \subseteq S$ and hence $T \subseteq S^* \subseteq S_{\mathcal{T}}^*$. Therefore $\ker T \subseteq \ker S_{\mathcal{T}}^*$. This means in particular that for $u \in \operatorname{dom} T$ the decomposition $u = u_c + u_s \in H^2(\mathbb{R}^d) \dotplus \ker S_{\mathcal{T}}^*$ is also a decomposition with respect to $H^2(\mathbb{R}^d) \dotplus \ker T$.

As mentioned in Example 3.8 the operators which are known in the literature as Schrödinger operators with δ-interactions supported on $\mathcal{T}$ are restrictions of $T_{\mathcal{T}}$ and can be parameterized with the generalized boundary triple $(L^2(\Sigma), \Gamma_0^{\mathcal{T}}, \Gamma_1^{\mathcal{T}})$. Note that the representation of $S_{\mathcal{T}}^*$ only differs from the representation of $T_{\mathcal{T}}$ by the fact that the functions f can be in $H^{-3/2}(\mathcal{T})$ and not only in $L^2(\mathcal{T})$. Hence, it is reasonable in a certain way to call the operators parameterize by the ordinary boundary triple $(L^2(\mathcal{T}), \hat{\Gamma}_0^{\mathcal{T}}, \hat{\Gamma}_1^{\mathcal{T}})$ *Schrödinger operators with δ-interactions of singular strengths*. We will discuss in the next section how the parameter ϑ must be chosen such that A_ϑ becomes a Schrödinger operators with δ-interactions supported on Σ. The next theorem shows how both concepts are connected. Roughly speaking a Schrödinger operator with δ-interaction supported on the manifold Σ of codimension 2 is a Schrödinger operator with δ-interaction with singular strength supported on the manifold $\mathcal{T}$ of codimension 1. The singular strength is again a δ-interaction.

Theorem 4.9. *Let ϑ be a symmetric linear relation in $L^2(\Sigma)$ such that $A_\vartheta \subseteq T$ with $\operatorname{dom} A_\vartheta = \left\{u \in \operatorname{dom} T : \begin{bmatrix}\Gamma_0 u\\ \Gamma_1 u\end{bmatrix} \in \vartheta\right\}$ is selfadjoint. Define the symmetric linear relation*

$$\Theta := \iota_+ \big(\tilde{G}\vartheta^{-1}\tilde{G}^*\big)^{-1}\iota_-^{-1} \subseteq L^2(\mathcal{T}) \times L^2(\mathcal{T}).$$

Then the operators A_ϑ and $A_\Theta \subseteq S_{\mathcal{T}}^*$ with $\operatorname{dom} A_\Theta = \left\{u \in \operatorname{dom} S_{\mathcal{T}}^* : \begin{bmatrix}\hat{\Gamma}_0^{\mathcal{T}} u\\ \hat{\Gamma}_1^{\mathcal{T}} u\end{bmatrix} \in \Theta\right\}$ coincide.

Proof. Due to $\iota_+ = (\iota_-^{-1})^*$ the linear relation Θ is symmetric. Hence A_Θ is symmetric too. Let now $u \in \operatorname{dom} A_\vartheta$ with $u = u_c + u_s \in H^2(\mathbb{R}^d) \dotplus \ker T \subseteq H^2(\mathbb{R}^d) \dotplus \ker S_{\mathcal{T}}^*$. Hence

$$\begin{aligned}
\begin{bmatrix} h\\ \operatorname{tr}_\Sigma^2 u_c\end{bmatrix} = \begin{bmatrix}\Gamma_0 u\\ \Gamma_1 u\end{bmatrix} \in \vartheta \quad &\Longrightarrow \quad \begin{bmatrix}\operatorname{tr}_\Sigma^2 u_c\\ h\end{bmatrix} \in \vartheta^{-1} \quad \Longrightarrow \quad \begin{bmatrix}\operatorname{tr}_{\mathcal{T}}^2 u_c\\ h\end{bmatrix} \in \vartheta^{-1}\tilde{G}^*\\
&\Longrightarrow \quad \begin{bmatrix}\operatorname{tr}_{\mathcal{T}}^2 u_c\\ \tilde{G}h\end{bmatrix} \in \tilde{G}\vartheta^{-1}\tilde{G}^* \quad \Longrightarrow \quad \begin{bmatrix}\tilde{G}h\\ \operatorname{tr}_{\mathcal{T}}^2 u_c\end{bmatrix} \in \big(\tilde{G}\vartheta^{-1}\tilde{G}^*\big)^{-1}\\
&\Longrightarrow \quad \begin{bmatrix}\iota_-\tilde{G}h\\ \iota_+\operatorname{tr}_{\mathcal{T}}^2 u_c\end{bmatrix} \in \iota_+\big(\tilde{G}\vartheta^{-1}\tilde{G}^*\big)^{-1}\iota_-^{-1}.
\end{aligned}$$

Note that

$$\begin{aligned}\langle \tilde{G}h\delta_{\mathcal{T}}, \varphi\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)} &= \langle \tilde{G}h, \mathrm{tr}^2_{\mathcal{T}}\,\varphi\rangle_{H^{-3/2}(\mathcal{T}),H^{3/2}(\mathcal{T})} = \langle h d_\Sigma, \mathrm{tr}^2_{\mathcal{T}}\,\varphi\rangle_{H^{-3/2}(\mathcal{T}),H^{3/2}(\mathcal{T})}\\ &= \langle h, \mathrm{tr}_{\mathcal{T}\restriction\Sigma}(\mathrm{tr}^2_{\mathcal{T}}\,\varphi)\rangle_{L^2(\Sigma)} = \langle h, \mathrm{tr}^2_\Sigma\,\varphi\rangle_{L^2(\Sigma)} = \langle h\delta_\Sigma, \varphi\rangle_{H^{-2}(\mathbb{R}^d),H^2(\mathbb{R}^d)}\end{aligned}$$

holds for all $\varphi \in H^2(\mathbb{R}^d)$. This implies

$$(-\Delta+1)u - \tilde{G}h\delta_{\mathcal{T}} = (-\Delta+1)u - h\delta_\Sigma \in L^2(\mathbb{R}^d) \tag{4.4}$$

and therefore $\hat{\Gamma}_0^{\mathcal{T}} u = \iota_- \tilde{G}h$. Hence we get

$$\begin{bmatrix}\Gamma_0 u\\ \Gamma_1 u\end{bmatrix} \in \vartheta \quad\Longrightarrow\quad \begin{bmatrix}\iota_-\tilde{G}h\\ \iota_+\,\mathrm{tr}^2_{\mathcal{T}}\,u_c\end{bmatrix} \in \iota_+\big(\tilde{G}\vartheta^{-1}\tilde{G}^*\big)^{-1}\iota_-^{-1} \quad\Longrightarrow\quad \begin{bmatrix}\hat{\Gamma}_0^{\mathcal{T}} u\\ \hat{\Gamma}_1^{\mathcal{T}} u\end{bmatrix} \in \Theta,$$

i.e. $u \in \mathrm{dom} A_\Theta$. Moreover (4.4) implies $A_\vartheta u = A_\Theta u$. Consequently $A_\vartheta \subseteq A_\Theta$. As A_ϑ is selfadjoint and A_Θ is symmetric both operators coincide. □

A natural question appearing now is the following: Which parameters Θ lead to Schrödinger operators with δ-interaction? To answer this question we have to introduce the concept of the generalized trace, which is done in the following section.

4.3 The generalized trace and δ-interactions on Σ

According to Lemma 2.1 every element $u \in \mathrm{dom}\,T$ can be written uniquely as $u = u_c + u_s$ with $u_c \in \mathrm{dom} A = H^2(\mathbb{R}^d)$ and $u_s \in \ker T$. Setting $h := \Gamma_0 u$ we have $u_s = \gamma(0)h$. Consequently, the trace of u to Σ should be "$u|_\Sigma = u_c|_\Sigma + (\gamma(0)h)|_\Sigma$", but a look at (4.1) shows that there is a problem: Due to the singularity of G_{-1} it is in general not possible to evaluate

$$\big(\gamma(0)h\big)(x) = \int_\Sigma h(y)\overline{G_{-1}(x-y)}\,d\sigma(y)$$

at $x \in \Sigma$. A possible solution is to "cut out" the singularity, see Definition 4.12 below.

For this we require that Σ is a compact, regular C^2-manifold without selfintersections and without boundary. Furthermore the corresponding parametrizations should satisfy the following conditions.

(C1) There exist bounded open sets $\Omega_i \subseteq \mathbb{R}^{d-2}$, relatively open sets $\Sigma_i \subseteq \Sigma$ and homeomorphism $\sigma_i : \Omega_i \to \Sigma_i$ for $i \in \{1,\dots,m\}$, such that each σ_i is $C^2(\overline{\Omega_i})$, σ_i^{-1} is Lipschitz continuous and $\bigcup_{i=1}^m \Sigma_i = \Sigma$.

(C2) For each $i \in \{1,\dots,m\}$ and each $\xi \in \overline{\Omega}_i$ the Jacobian matrix $D\sigma_i(\xi) \in \mathbb{R}^{d,d-2}$ has full rank.

(C3) For each $i \in \{1,\dots,m\}$ exists a continuous function $F_i : \Omega_i \times \Omega_i \to \mathbb{R}^d$ and a constant $C_i > 0$ such that we have for all $s,t \in \Omega_i$

$$\sigma_i(s) = \sigma_i(t) + [D\sigma_i(t)](s-t) + F_i(s,t) \quad \text{and} \quad |F_i(s,t)| \le C_i|s-t|^2.$$

As $D\sigma_i(\xi) \in \mathbb{R}^{d,d-2}$ has full rank there exists $P_i(\xi) \in \mathbb{R}^{d-2,d-2}$ with full rank such that $D\sigma_i(\xi) \cdot P_i(\xi)$ is an isometric matrix, e.g.

$$D\sigma_i(\xi) \cdot P_i(\xi) = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}.$$

The construction of the matrix $P_i(\xi)$ can be done with the help of the singular value decomposition. We require from the matrices $P_i(\xi)$ the following additional condition.

(C4) For each $i \in \{1,\dots,m\}$ the matrix valued function $P_i : \Omega_i \to \mathbb{R}^{d-2,d-2}$, $\xi \mapsto P_i(\xi)$, is in $C^1(\overline{\Omega_i})$, i.e. each component is in $C^1(\overline{\Omega_i})$.

Remark 4.10. We make same definitions and remarks concerning the conditions above.

(i) Condition (C1) implies in particular that also all σ_i are Lipschitz continuous. We will denote by $L > 1$ a common Lipschitz constant of all σ_i and all σ_i^{-1}.

(ii) There exists $M > 1$ such that for all $i \in \{1,\dots,m\}$ and all $s,t \in \Omega_i$ the estimates

$$\frac{1}{M}|t-s| \le \big|[D\sigma_i(t)](s-t)\big| \le M|t-s|$$

hold.

(iii) For each $i \in \{1,\dots,m\}$ the function $s \mapsto \sqrt{\det\left([D\sigma_i(s)]^\top [D\sigma_i(s)]\right)}$ belongs to $C^1(\overline{\Omega_i})$ and is hence Lipschitz continuous. Denote by $K > 0$ a common Lipschitz constant.

(iv) Let $Q > 1$ be such that $\|P_i(\xi)\| \le Q$ and $\|P_i(\xi)^{-1}\| \le Q$ hold for all $i \in \{1,\dots,m\}$ and $\xi \in \Omega_i$. Furthermore set $C := \max\{1, C_1, \dots, C_m\}$.

(v) There exists $\varepsilon > 0$ such that for each $x \in \Sigma$ exists $i \in \{1,\dots,m\}$ with $B_\varepsilon(x) \cap \Sigma \subseteq \Sigma_i$.

For $\delta < \varepsilon$ and $x \in \Sigma$ we define

$$\Sigma_\delta(x) := \Sigma \setminus B_\delta(x) = \Sigma \setminus \{\sigma_i(s) : s \in \Omega_i \ \wedge \ |\sigma_i(s) - x| < \delta\},$$

where $i \in \{1,\dots,m\}$ was chosen as in item (v) of Remark 4.10.

Lemma 4.11. *Let $\lambda \leq 1$. The function k_λ defined by*

$$k_\lambda(x) := \lim_{\delta \to 0} \left[\int_{\Sigma_\delta(x)} G_{\lambda-1}(x-y)\, d\sigma(y) + \frac{\ln \delta}{2\pi} \right]$$

is bounded and satisfies $\sup_{x \in \Sigma} k_\lambda(x) \xrightarrow{\lambda \to -\infty} -\infty$.

Proof. Let i be as in item (v) of Remark 4.10 and $\xi := \sigma_i^{-1}(x) \in \Omega_i$. Define $\tilde{\sigma}_i : \tilde{\Omega}_i \to \Sigma_i$ by $\tilde{\sigma}_i := \sigma_i \circ P_i(\xi)$, where $\tilde{\Omega}_i := [P_i(\xi)]^{-1}\Omega_i$, and set $t := \tilde{\sigma}_i^{-1}(x) \in \tilde{\Omega}_i$. Hence $\xi = P_i(\xi)t$ because $\sigma_i(\xi) = x = \tilde{\sigma}_i(t) = \sigma_i(P_i(\xi)t)$. The parametrization $\tilde{\sigma}_i$ has the important property that for all $s \in \mathbb{R}^{d-2}$ the identity

$$|D\tilde{\sigma}_i(t)s| = |D\sigma_i(\xi) \cdot P_i(\xi) \cdot s| = |s|.$$

holds. Moreover we have with L and Q from Remark 4.10 for all $s \in \tilde{\Omega}_i$ the estimate

$$\begin{aligned} |\tilde{\sigma}_i(t) - \tilde{\sigma}_i(s)| &= |\sigma_i(P_i(\xi)t) - \sigma_i(P_i(\xi)s)| \leq L|P_i(\xi)t - P_i(\xi)s| \leq LQ|t-s|, \\ |\tilde{\sigma}_i(t) - \tilde{\sigma}_i(s)| &= |\sigma_i(P_i(\xi)t) - \sigma_i(P_i(\xi)s)| \geq \frac{1}{L}|P_i(\xi)t - P_i(\xi)s| \geq \frac{1}{LQ}|t-s|. \end{aligned} \tag{4.5}$$

Note that (C3) implies for all $s \in \tilde{\Omega}_i$

$$\begin{aligned} \tilde{\sigma}_i(s) = \sigma_i(P_i(\xi)s) &= \sigma_i(P_i(\xi)t) + [D\sigma_i(t)]\big(P_i(\xi)s - P_i(\xi)t\big) + F_i\big(P_i(\xi)s, P_i(\xi)t\big) \\ &= \tilde{\sigma}_i(t) + [D\tilde{\sigma}_i(t)](s-t) + \tilde{F}_i(s,t) \end{aligned} \tag{4.6}$$

with $\tilde{F}_i(s,t) := F_i\big(P_i(\xi)s, P_i(\xi)t\big)$. Moreover we get with $\tilde{C} := CQ^2 > 1$

$$|\tilde{F}_i(s,t)| = |F_i\big(P_i(\xi)s, P_i(\xi)t\big)| \leq C|P_i(\xi)s - P_i(\xi)t|^2 \leq CQ^2|s-t|^2 = \tilde{C}|s-t|^2.$$

As $\tilde{\Omega}_i$ is open we can assume without loss of generality that ε from item (v) of Remark 4.10 is so small such that

$$\varepsilon < \frac{1}{2\tilde{C}QL} \qquad \text{and} \qquad \{s \in \mathbb{R}^{d-2} : |s-t| \leq \varepsilon\} \subseteq \tilde{\Omega}_i \tag{4.7}$$

hold.

We split the integral in the definition of k_λ into several parts:

$$\int_{\Sigma_\delta(x)} G_{\lambda-1}(x-y)\,d\sigma(y) + \frac{\ln\delta}{2\pi}$$

$$= \int_{\Sigma_\delta(x)} G_{\lambda-1}(x-y) - G_0(x-y)\,d\sigma(y) \qquad + \int_{\Sigma_\delta(x)\setminus B_\varepsilon(x)} G_0(x-y)\,d\sigma(y) \tag{4.8}$$

$$-\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int_{\{s\in\tilde{\Omega}_i:\varepsilon\geq|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta\}} \frac{1}{|t-s|^{d-2}}\,ds \qquad + \int_{\Sigma_\delta(x)\cap B_\varepsilon(x)} G_0(x-y)\,d\sigma(y) \tag{4.9}$$

$$-\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\,\wedge\,(s\in\tilde{\Omega}_i^c\,\vee\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|>\varepsilon)\}} \frac{1}{|t-s|^{d-2}}\,ds \tag{4.10}$$

$$+\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\,\wedge\,(s\in\tilde{\Omega}_i^c\,\vee\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta)\}} \frac{1}{|t-s|^{d-2}}\,ds \qquad + \frac{\ln\delta}{2\pi}. \tag{4.11}$$

We show that the limits of (4.8), (4.9), (4.10), and (4.11) for $\delta\to 0$ exist, are finite and can be estimated by constants independent of x. We start with the first integral in (4.8). Its integrand can be written as

$$G_{\lambda-1}(x) - G_0(x) = \frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}|x|^{d-2}} \int_0^\infty \frac{\cos(\sqrt{1-\lambda}|x|t)-1}{(t^2+1)^{\frac{d-1}{2}}}\,dt \leq 0,$$

where we have used

$$G_0(x) = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}|x|^{d-2}} = \frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}|x|^{d-2}} \cdot \frac{\sqrt{\pi}\Gamma(\frac{d}{2}-1)}{2\Gamma(\frac{d-1}{2})} = \frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}|x|^{d-2}} \int_0^\infty \frac{1}{(t^2+1)^{\frac{d-1}{2}}}\,dt$$

and

$$\begin{aligned} G_{\lambda-1}(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\frac{\sqrt{1-\lambda}}{|x|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(\sqrt{-\lambda}|x|) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{1-\lambda}}{|x|}\right)^{\frac{d}{2}-1} \frac{\Gamma(\frac{d-1}{2})2^{\frac{d}{2}-1}}{(\sqrt{1-\lambda}|x|)^{\frac{d}{2}-1}\Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos(\sqrt{1-\lambda}|x|t)}{(t^2+1)^{\frac{d-1}{2}}}\,dt \\ &= \frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}|x|^{d-2}} \int_0^\infty \frac{\cos(\sqrt{1-\lambda}|x|t)}{(t^2+1)^{\frac{d-1}{2}}}\,dt, \end{aligned}$$

cf. the integral representation given in [38, 8.432 5.]. Hence

$$\begin{aligned}
0 &\geq \int\limits_{\Sigma_\delta(x)} G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) \\
&= \int\limits_{\Sigma_\delta(x)} \frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}|x-y|^{d-2}} \int\limits_0^\infty \frac{\cos(\sqrt{1-\lambda}|x-y|q)-1}{(q^2+1)^{\frac{d-1}{2}}}\, dq\, d\sigma(y) \\
&= -\frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}} \int\limits_{\Sigma_\delta(x)} \frac{1}{|x-y|^{d-2}} \int\limits_0^\infty \frac{\sin^2\left(\frac{\sqrt{1-\lambda}|x-y|}{2}q\right)}{(q^2+1)^{\frac{d-1}{2}}}\, dq\, d\sigma(y),
\end{aligned} \tag{4.12}$$

where we have used $1-\cos a = \cos 0 - \cos a = 2\sin\frac{a+0}{2}\sin\frac{a-0}{2} = 2\sin^2\frac{a}{2}$. At first we consider the case $d=3$. For this consider the function $f:]0,\infty[\to \mathbb{R}$ defined by $f(s) := \frac{e^{-s}\sinh(s)}{s} > 0$. This function satisfies

$$\begin{aligned}
f'(s) &= \frac{\left[-e^{-s}\sinh(s) + e^{-s}\cosh(s)\right]s - e^{-s}\sinh(s)}{s^2} \\
&= -\frac{e^{-s}}{s}\left[\sinh(s) - \cosh(s) + \frac{\sinh(s)}{s}\right] = -\frac{e^{-s}}{s}\left[\frac{e^s - e^{-s}}{2s} - e^{-s}\right] \leq 0
\end{aligned}$$

and

$$\lim_{s\searrow 0} f(s) = \lim_{s\searrow 0} \frac{e^{-s}\sinh(s)}{s} = \lim_{s\searrow 0} \frac{-e^{-s}\sinh(s) + e^{-s}\cosh(s)}{1} = 1.$$

Hence $f(s) \leq 1$ for all $s > 0$. Using this and $\int_0^\infty \frac{\sin^2(aq)}{(q^2+1)^{\frac{3-1}{2}}}\, dq = \frac{\pi}{2}e^{-a}\sinh(a)$ we observe from (4.12)

$$\begin{aligned}
0 &\geq \int\limits_{\Sigma_\delta(x)} G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) \\
&= -\frac{1}{2} \int\limits_{\Sigma_\delta(x)} \frac{e^{-\frac{\sqrt{1-\lambda}|x-y|}{2}} \sinh\left(\frac{\sqrt{1-\lambda}|x-y|}{2}\right)}{|x-y|}\, d\sigma(y) \\
&\geq -\frac{1}{2} \int\limits_{\Sigma_\delta(x)} \frac{\sqrt{1-\lambda}}{2}\, d\sigma(y) \geq -\frac{\sqrt{1-\lambda}}{4}|\Sigma|.
\end{aligned}$$

As the integrand is nonpositive the above estimates shows that the first integral in (4.8) converges for $\delta \to 0$. Note that the bounds given above are independent of x.

For the case $d > 3$ we make use of $2\sin^2 \frac{a}{2} \leq a$ for $a \geq 0$. Hence (4.12) implies

$$\begin{aligned} 0 &\geq \int_{\Sigma_\delta(x)} G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) \\ &\geq \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \int_0^\infty \frac{1}{(q^2+1)^{\frac{d-1}{2}}} \int_{\Sigma_\delta(x)} \frac{-\sqrt{1-\lambda}|x-y|q}{|x-y|^{d-2}}\, d\sigma(y)\, dq \\ &= -\sqrt{1-\lambda}\frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \left(\int_0^\infty \frac{q}{(q^2+1)^{\frac{d-1}{2}}}\, dq \right) \left(\int_{\Sigma_\delta(x)} \frac{1}{|x-y|^{d-3}}\, d\sigma(y) \right). \end{aligned} \tag{4.13}$$

The first integral in (4.13) converges for $d > 3$ and equals $\frac{1}{d-3}$. We split the remaining integral again into two parts and use $\Sigma_\delta(x) \setminus B_\varepsilon(x) = \Sigma \setminus B_\varepsilon(x)$:

$$\begin{aligned} \int_{\Sigma_\delta(x)} \frac{1}{|x-y|^{d-3}}\, d\sigma(y) &= \int_{\Sigma \setminus B_\varepsilon(x)} \frac{1}{|x-y|^{d-3}}\, d\sigma(y) + \int_{\Sigma_\delta(x) \cap B_\varepsilon(x)} \frac{1}{|x-y|^{d-3}}\, d\sigma(y) \\ &\leq \frac{|\Sigma|}{\varepsilon^{d-3}} + \int_{\{y \in \Sigma_i : \varepsilon \geq |x-y| \geq \delta\}} \frac{1}{|x-y|^{d-3}}\, d\sigma(y). \end{aligned} \tag{4.14}$$

For $s := \sigma^{-1}(y)$ we get $|s-t| = |\sigma_i^{-1}(x) - \sigma_i^{-1}(y)| \leq L|x-y| = L|\sigma_i(s) - \sigma_i(t)|$ with the Lipschitz constant L from item (i) of Remark 4.10. Hence $\frac{1}{|x-y|^{d-3}} \leq \frac{L^{d-3}}{|s-t|^{d-3}}$ and

$$\{s \in \Omega_i : \varepsilon \geq |\sigma_i(t) - \sigma_i(s)| \geq \delta\} \subseteq \{s \in \mathbb{R}^{d-2} : L\varepsilon \geq |t-s|\}.$$

With $\det\left([D\sigma_i(s)]^\top [D\sigma_i(s)]\right) \leq \|D\sigma_i(s)\|^{2(d-2)} \leq M^{2(d-2)}$ and polar coordinates we get

$$\begin{aligned} \int_{\{y \in \Sigma_i : \varepsilon \geq |x-y| \geq \delta\}} \frac{1}{|x-y|^{d-3}}\, d\sigma(y) &= \int_{\{s \in \Omega_i : \varepsilon \geq |\sigma_i(t) - \sigma_i(s)| \geq \delta\}} \frac{\sqrt{\det\left([D\sigma_i(s)]^\top [D\sigma_i(s)]\right)}}{|\sigma_i(t) - \sigma_i(s)|^{d-3}}\, ds \\ &\leq \int_{\{s \in \mathbb{R}^{d-2} : L\varepsilon \geq |t-s|\}} \frac{L^{d-3}M^{d-2}}{|t-s|^{d-3}}\, ds \\ &= \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}-1)} \int_0^{L\varepsilon} \frac{L^{d-3}M^{d-2}}{r^{d-3}} \cdot r^{d-3}\, dr = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}-1)} (LM)^{d-2}\varepsilon. \end{aligned} \tag{4.15}$$

Hence we observe from (4.14), (4.13) and (4.15)

$$0 \geq \int_{\Sigma_\delta(x)} G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) \geq \frac{-\sqrt{1-\lambda}}{d-3} \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \left[\frac{|\Sigma|}{\varepsilon^{d-2}} + \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}-1)} (LM)^{d-2}\varepsilon \right].$$

As the integrand is nonpositive the above estimates shows that the first integral in (4.8) converges for $\delta \to 0$. Note that the bounds given above are independent of x.

The estimate for the second integral in (4.8) is easier. As $\delta < \varepsilon$ this integral is in fact independent of δ. Moreover we get the (x-independent) estimates

$$0 \leq \int_{\Sigma_\delta(x)\setminus B_\varepsilon(x)} G_0(x-y)\,d\sigma(y) = \int_{\Sigma\setminus B_\varepsilon(x)} \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}|x-y|^{d-2}}\,d\sigma(y) \leq \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}\varepsilon^{d-2}}|\Sigma|.$$

Next we consider (4.9). As $\{s \in \tilde{\Omega}_i : \varepsilon = |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\}$ is a set of measure zero we get

$$\begin{aligned}(4.9) &= -\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int_{\{s\in\tilde{\Omega}_i:\varepsilon>|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta\}} \frac{1}{|t-s|^{d-2}}\,ds + \int_{\Sigma_\delta\cap B_\varepsilon(x)} G_0(x-y)\,d\sigma(y) \\ &= \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int_{\{s\in\tilde{\Omega}_i:\varepsilon>|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta\}} \frac{\sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top[D\tilde{\sigma}_i(s)]\right)}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|^{d-2}} - \frac{1}{|t-s|^{d-2}}\,ds. \end{aligned} \tag{4.16}$$

Note that (4.5) and (4.7) imply for all $s \in \tilde{\Omega}_i$ with $\varepsilon > |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|$

$$|t-s| \leq LQ|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \leq LQ\varepsilon < \frac{1}{2\tilde{C}} < \frac{1}{2}. \tag{4.17}$$

It follows from (4.6) that $|t-s| - \tilde{C}|t-s|^2 \leq |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|$. Hence we get for the integrand of (4.16) the estimate

$$\begin{aligned} \frac{\sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top[D\tilde{\sigma}_i(s)]\right)}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|^{d-2}} - \frac{1}{|t-s|^{d-2}} &\leq \frac{\sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top[D\tilde{\sigma}_i(s)]\right)}}{\left(|t-s|-\tilde{C}|t-s|^2\right)^{d-2}} - \frac{1}{|t-s|^{d-2}} \\ &= \left[\frac{\sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top[D\tilde{\sigma}_i(s)]\right)}}{\left(1-\tilde{C}|t-s|\right)^{d-2}} - 1\right]\frac{1}{|t-s|^{d-2}} \\ &= \left[\frac{\sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top[D\tilde{\sigma}_i(s)]\right)}-1}{\left(1-\tilde{C}|t-s|\right)^{d-2}} + \frac{1-\left(1-\tilde{C}|t-s|\right)^{d-2}}{\left(1-\tilde{C}|t-s|\right)^{d-2}}\right]\frac{1}{|t-s|^{d-2}}. \end{aligned} \tag{4.18}$$

Note that $1-\tilde{C}|t-s| > \frac{1}{2}$, cf. (4.17). Therefore all denominators above are positive. Moreover we have

$$\begin{aligned} \sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top[D\tilde{\sigma}_i(s)]\right)} &= \sqrt{\det\left([D\sigma_i(P_i(\xi)s)\cdot P_i(\xi)]^\top[D\sigma_i(P_i(\xi)s)]P_i(\xi)\right)} \\ &= \sqrt{\det P_i(\xi)^\top}\sqrt{\det\left([D\sigma_i(P_i(\xi)s)]^\top[D\sigma_i(P_i(\xi)s)]\right)}\sqrt{\det P_i(\xi)} \\ &= |\det P_i(\xi)|\sqrt{\det\left([D\sigma_i(P_i(\xi)s)]^\top[D\sigma_i(P_i(\xi)s)]\right)} \end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
1 &= \sqrt{\det\big([D\sigma_i(\xi)\cdot P_i(\xi)]^\top [D\sigma_i(\xi)] P_i(\xi)\big)} \\
&= \sqrt{\det P_i(\xi)^\top}\sqrt{\det\big([D\sigma_i(\xi)]^\top [D\sigma_i(\xi)]\big)}\sqrt{\det P_i(\xi)} \\
&= |\det P_i(\xi)|\sqrt{\det\big([D\sigma_i(\xi)]^\top [D\sigma_i(\xi)]\big)}.
\end{aligned}
$$

Hence we get with $\sigma_i(\xi) = x = \tilde{\sigma}_i(t) = \sigma_i(P_i(\xi)t)$, the constants K and Q from item (iii) and (iv) of Remark 4.10 and with $|\det P_i(\xi)| \le \|P_i(\xi)\|^{d-s} \le Q^{d-2}$ the estimate

$$
\begin{aligned}
&\sqrt{\det\big([D\tilde{\sigma}_i(s)]^\top [D\tilde{\sigma}_i(s)]\big)} - 1 \\
&\quad = |\det P_i(\xi)| \left[\sqrt{\det\big([D\sigma_i(P_i(\xi)s)]^\top [D\sigma_i(P_i(\xi)s)]\big)} - \sqrt{\det\big([D\sigma_i(\xi)]^\top [D\sigma_i(\xi)]\big)}\right] \\
&\quad \le Q^{d-2}\cdot K|P_i(\xi)s - \xi| = Q^{d-2}\cdot K|P_i(\xi)s - P_i(\xi)t| \le KQ^{d-1}\cdot|s-t|. \qquad (4.20)
\end{aligned}
$$

Moreover we have for $|t-s| \le 1$

$$
\begin{aligned}
1 - \big(1 - \tilde{C}|t-s|\big)^{d-2} &= 1 - \sum_{k=0}^{d-2}\binom{d-2}{k} 1^{d-2-k}\big(-\tilde{C}|t-s|\big)^k \\
&= -\sum_{k=1}^{d-2}\binom{d-2}{k}(-1)^k \tilde{C}^k |t-s|^k = |t-s|\sum_{k=1}^{d-2}\binom{d-2}{k}(-1)^{k-1}\tilde{C}^k|t-s|^{k-1} \\
&\le |t-s|\sum_{k=1}^{d-2}\binom{d-2}{k}\tilde{C}^k = |t-s|\Big((1+\tilde{C})^{d-2} - 1\Big). \qquad (4.21)
\end{aligned}
$$

Moreover we have due to $\tilde{C}|t-s| < \frac{1}{2}$

$$
\big(1 - \tilde{C}|t-s|\big)^{d-2} > \big(1 - \frac{1}{2}\big)^{d-2} = 2^{d-2} \qquad (4.22)
$$

Hence we get with (4.18), (4.20), (4.21), (4.22) and $R := \frac{KQ^{d-1} + (1+\tilde{C})^{d-2} - 1}{2^{d-2}}$

$$
\begin{aligned}
&\frac{\sqrt{\det\big([D\tilde{\sigma}_i(s)]^\top [D\tilde{\sigma}_i(s)]\big)}}{|\tilde{\sigma}_i(t) - \tilde{\sigma}_i(s)|^{d-2}} - \frac{1}{|t-s|^{d-2}} \\
&\quad \le \left[\frac{\sqrt{\det\big([D\tilde{\sigma}_i(s)]^\top [D\tilde{\sigma}_i(s)]\big)} - 1}{\big(1 - \tilde{C}|t-s|\big)^{d-2}} + \frac{1 - \big(1 - \tilde{C}|t-s|\big)^{d-2}}{\big(1 - \tilde{C}|t-s|\big)^{d-2}}\right]\frac{1}{|t-s|^{d-2}} \\
&\quad \le \left[\frac{KQ^{d-1}\cdot|s-t|}{2^{d-2}} + \frac{|t-s|\big((1+\tilde{C})^{d-2} - 1\big)}{2^{d-2}}\right]\frac{1}{|t-s|^{d-2}} \\
&\quad = \frac{KQ^{d-1} + (1+\tilde{C})^{d-2} - 1}{2^{d-2}}\cdot\frac{1}{|t-s|^{d-3}} = \frac{R}{|t-s|^{d-3}}.
\end{aligned}
$$

Analogously we get

$$\frac{\sqrt{\det\left([D\tilde{\sigma}_i(s)]^\top [D\tilde{\sigma}_i(s)]\right)}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|^{d-2}} - \frac{1}{|t-s|^{d-2}} \geq -\frac{R}{|t-s|^{d-3}}.$$

Note that the constant is independent of x. Hence the absolute value of the integrand in (4.16) can be estimated by the function $s \mapsto \frac{R}{|t-s|^{d-3}}$. This function is integrable because (4.17) implies $\{s \in \tilde{\Omega}_i : \varepsilon > |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\} \subseteq \{s \in \mathbb{R}^{d-2} : \frac{1}{2} > |t-s|\}$ and hence

$$\int\limits_{\{s\in\tilde{\Omega}_i:\varepsilon>|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\}} \frac{R}{|t-s|^{d-3}}\,ds \leq \int\limits_{\{s\in\mathbb{R}^{d-2}:\frac{1}{2}>|t-s|\}} \frac{R}{|t-s|^{d-3}}\,ds$$
$$= \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \int_0^{\frac{1}{2}} \frac{R}{r^{d-3}} \cdot r^{d-3}\,ds = \frac{\pi^{\frac{d-2}{2}} R}{\Gamma\left(\frac{d}{2}-1\right)}.$$

By Lebesgue's dominated convergence theorem the limit of (4.9) for $\delta \to 0$ exists and can be estimated by $\pm\frac{\pi^{\frac{d-2}{2}} R}{\Gamma(\frac{d}{2}-1)}$. Also these bounds are independent of x.

The term in (4.10) is independent of δ. Hence it suffices to show that the integral in (4.10) converges and can be bounded by a constant independent of t. Note that

$$\{s \in \tilde{\Omega}_i : 1 \geq |s-t| \,\wedge\, |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| > \varepsilon\} \subseteq \left\{s \in \tilde{\Omega}_i : 1 \geq |s-t| > \frac{\varepsilon}{LQ}\right\},$$

cf. (4.5). Hence we get with (4.7)

$$\begin{aligned}
\{s \in \mathbb{R}^{d-2} &: 1 \geq |s-t| \,\wedge\, (s \in \tilde{\Omega}_i^c \,\vee\, |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| > \varepsilon)\} \\
&= \{s \in \tilde{\Omega}_i^c : 1 \geq |s-t|\} \cup \{s \in \tilde{\Omega}_i : 1 \geq |s-t| \,\wedge\, |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| > \varepsilon\} \\
&\subseteq \left\{s \in \tilde{\Omega}_i^c : 1 \geq |s-t| > \frac{\varepsilon}{LQ}\right\} \cup \left\{s \in \tilde{\Omega}_i : 1 \geq |s-t| > \frac{\varepsilon}{LQ}\right\} \\
&= \left\{s \in \mathbb{R}^{d-2} : 1 \geq |s-t| > \frac{\varepsilon}{LQ}\right\}
\end{aligned}$$

and therefore

$$0 \leq \int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\,\wedge\,(s\in\tilde{\Omega}_i^c\,\vee\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|>\varepsilon)\}} \frac{1}{|t-s|^{d-2}}\,ds$$
$$\leq \int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|>\frac{\varepsilon}{LQ}\}} \frac{1}{|t-s|^{d-2}}\,ds = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}-1)} \int_{\frac{\varepsilon}{LQ}}^{1} \frac{1}{r^{d-2}} \cdot r^{d-3}\,ds = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}-1)} \ln\frac{LQ}{\varepsilon}.$$

Note that this estimate is independent of x. Next we consider (4.11). Note at first that

$$\frac{\Gamma(\frac{d}{2}-1)}{2\pi^{\frac{d-2}{2}}}\int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|t-s|\geq\delta\}}\frac{1}{|t-s|^{d-2}}\,ds=\int\limits_{\delta}^{1}\frac{1}{r^{d-2}}\cdot r^{d-3}\,ds=-\ln\delta. \tag{4.23}$$

Recall that $|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\leq|t-s|\big(1+\tilde{C}|t-s|\big)$ holds for all $s\in\tilde{\Omega}_i$, cf. (4.6). Moreover we can assume that δ is so small that $\{s\in\mathbb{R}^{d-2}:|s-t|<\delta\}\subseteq\tilde{\Omega}_i$. Hence

$$\begin{aligned}\{s\in&\mathbb{R}^{d-2}:1\geq|s-t|\,\wedge\,(s\in\tilde{\Omega}_i^c\,\vee\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta)\}\\&=\{s\in\tilde{\Omega}_i^c:1\geq|s-t|\geq\delta\}\cup\{s\in\tilde{\Omega}_i:1\geq|s-t|\,\wedge\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta\}\\&\subseteq\left\{s\in\tilde{\Omega}_i^c:1\geq|s-t|\geq\frac{\delta}{1+\tilde{C}|t-s|}\right\}\cup\left\{s\in\tilde{\Omega}_i:1\geq|s-t|\geq\frac{\delta}{1+\tilde{C}|t-s|}\right\}\\&=\left\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\geq\frac{\delta}{1+\tilde{C}|t-s|}\right\}.\end{aligned} \tag{4.24}$$

With (4.23) and (4.24) we get now

$$\begin{aligned}&\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\,\wedge\,(s\in\tilde{\Omega}_i^c\,\vee\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta)\}}\frac{1}{|t-s|^{d-2}}\,ds+\frac{\ln\delta}{2\pi}\\&\leq\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\left[\int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\geq\frac{\delta}{1+\tilde{C}|t-s|}\}}\frac{1}{|t-s|^{d-2}}\,ds-\int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|t-s|\geq\delta\}}\frac{1}{|t-s|^{d-2}}\,ds\right]\\&=\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\int\limits_{\{s\in\mathbb{R}^{d-2}:\delta>|s-t|\geq\frac{\delta}{1+\tilde{C}|t-s|}\}}\frac{1}{|t-s|^{d-2}}\,ds\leq\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\int\limits_{\{s\in\mathbb{R}^{d-2}:\delta>|s-t|\geq\frac{\delta}{1+\tilde{C}\delta}\}}\frac{1}{|t-s|^{d-2}}\,ds\\&=\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\cdot\frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})}\int\limits_{\frac{\delta}{1+\tilde{C}\delta}}^{\delta}\frac{1}{r}\,dr=\frac{1}{2\pi}\left(\ln\delta-\ln\frac{\delta}{1+\tilde{C}\delta}\right)=\frac{\ln(1+\tilde{C}\delta)}{2\pi}.\end{aligned} \tag{4.25}$$

We have $|s-t|\leq LQ|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|$ for all $s\in\tilde{\Omega}_i$, cf. equation (4.5). Hence $|s-t|>\delta LQ$ implies $|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|>\delta$ for all $s\in\tilde{\Omega}_i$. Hence

$$\begin{aligned}\{s\in\mathbb{R}^{d-2}:1\geq|s-t|>\delta LQ\,\wedge\,(s\in\tilde{\Omega}_i^c\,\vee\,|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta)\}\\\supseteq\{s\in\mathbb{R}^{d-2}:1\geq|s-t|>\delta LQ\}.\end{aligned} \tag{4.26}$$

As we are just interested in the limit $\delta\to0$ we can assume in the following that δ is so small that $\{s\in\mathbb{R}^{d-2}:|s-t|\leq\delta LM\}\subseteq\tilde{\Omega}_i$ holds, which is possible because $\tilde{\Omega}_i$ is open. Moreover recall that $|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \geq |t-s|-\tilde{C}|t-s|^2$ holds for all $s \in \tilde{\Omega}_i$. Hence

$$\begin{aligned}
\{s \in \mathbb{R}^{d-2} : \delta LQ \geq |s-t| \wedge (s \in \tilde{\Omega}_i^c \vee |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \geq \delta)\} \\
= \{s \in \mathbb{R}^{d-2} : \delta LQ \geq |s-t| \wedge |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \geq \delta\} \\
\supseteq \{s \in \mathbb{R}^{d-2} : \delta LQ \geq |s-t| \wedge |t-s|-\tilde{C}|t-s|^2 \geq \delta\} \\
= \{s \in \mathbb{R}^{d-2} : \delta LQ \geq |s-t| \geq \delta+\tilde{C}|t-s|^2\} \\
\supseteq \{s \in \mathbb{R}^{d-2} : \delta LQ \geq |s-t| \geq \delta+\tilde{C}(\delta LQ)^2\}.
\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) we get for $\delta LQ < 1$

$$\begin{aligned}
\{s \in & \mathbb{R}^{d-2} : 1 \geq |s-t| \wedge (s \in \tilde{\Omega}_i^c \vee |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \geq \delta)\} \\
&= \{s \in \mathbb{R}^{d-2} : 1 \geq |s-t| > \delta LQ \wedge (s \in \tilde{\Omega}_i^c \vee |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \geq \delta)\} \\
&\quad \cup \{s \in \mathbb{R}^{d-2} : \delta LQ \geq |s-t| \wedge (s \in \tilde{\Omega}_i^c \vee |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)| \geq \delta)\} \\
&\supseteq \{s \in \mathbb{R}^{d-2} : 1 \geq |s-t| \geq \delta+\tilde{C}(\delta LQ)^2\}.
\end{aligned} \tag{4.28}$$

With (4.23) and (4.28) we get now

$$\begin{aligned}
&\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t| \wedge (s\in\tilde{\Omega}_i^c \vee |\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|\geq\delta)\}} \frac{1}{|t-s|^{d-2}}\, ds + \frac{\ln\delta}{2\pi} \\
&\geq \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \left[\int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|s-t|\geq\delta+\tilde{C}(\delta LQ)^2\}} \frac{1}{|t-s|^{d-2}}\, ds - \int\limits_{\{s\in\mathbb{R}^{d-2}:1\geq|t-s|\geq\delta\}} \frac{1}{|t-s|^{d-2}}\, ds \right] \\
&= \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \int\limits_{\{s\in\mathbb{R}^{d-2}:\delta+\tilde{C}(\delta LQ)^2>|s-t|\geq\delta\}} \frac{-1}{|t-s|^{d-2}}\, ds \\
&= \frac{1}{2\pi} \int\limits_{\delta}^{\delta+\tilde{C}(\delta LQ)^2} \frac{-1}{r}\, dr = \frac{-\left(\ln\left(\delta+\tilde{C}(\delta LQ)^2\right)-\ln\delta\right)}{2\pi} = -\frac{\ln\left(1+\tilde{C}\delta(LQ)^2\right)}{2\pi}.
\end{aligned} \tag{4.29}$$

From (4.25) and (4.29) we conclude that (4.11) tends to 0 if $\delta \to 0$. Note that also this convergence is independent of x.

It remains to show $\sup_{x\in\Sigma} k_\lambda(x) \xrightarrow{\lambda\to-\infty} -\infty$. As the first integral in (4.8) is the only term depending on λ it suffices to consider just this term. For this we use the following representation resulting from (4.12):

$$\begin{aligned}
&\int_\Sigma G_{\lambda-1}(x-y)-G_0(x-y)\,d\sigma(y)\\
&\qquad=-\frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}}\int_\Sigma\frac{1}{|x-y|^{d-2}}\int_0^\infty\frac{\sin^2\big(\frac{\sqrt{1-\lambda}|x-y|}{2}q\big)}{(q^2+1)^{\frac{d-1}{2}}}\,dq\,d\sigma(y)\\
&\qquad=-\frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}}\int_0^\infty\frac{1}{(q^2+1)^{\frac{d-1}{2}}}\int_\Sigma\frac{\sin^2\big(\frac{\sqrt{1-\lambda}|x-y|}{2}q\big)}{|x-y|^{d-2}}\,d\sigma(y)\,dq\\
&\qquad\le-\frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}}\int_0^\infty\frac{1}{(q^2+1)^{\frac{d-1}{2}}}\int_{\Sigma\cap B_\varepsilon(x)}\frac{\sin^2\big(\frac{\sqrt{1-\lambda}|x-y|}{2}q\big)}{|x-y|^{d-2}}\,d\sigma(y)\,dq. \qquad (4.30)
\end{aligned}$$

Due to (4.7) and (4.5) we have

$$\Big\{s\in\mathbb{R}^{d-2}:|s-t|<\frac{\varepsilon}{LQ}\Big\}=\Big\{s\in\tilde\Omega_i:|s-t|<\frac{\varepsilon}{LQ}\Big\}\subseteq\{s\in\tilde\Omega_i:|\tilde\sigma_i(t)-\tilde\sigma_i(s)|<\varepsilon\}$$

and therefore

$$\Sigma\cap B_\varepsilon(x)=\tilde\sigma_i\big(\{s\in\tilde\Omega_i:|\tilde\sigma_i(t)-\tilde\sigma_i(s)|\le\varepsilon\}\big)\supseteq\tilde\sigma_i\Big(\Big\{s\in\mathbb{R}^{d-2}:|s-t|\le\frac{\varepsilon}{LQ}\Big\}\Big).$$

Hence we get for the inner integral in (4.30) the estimate

$$\begin{aligned}
&\int_{\Sigma\cap B_\varepsilon(x)}\frac{\sin^2\big(\frac{\sqrt{1-\lambda}|x-y|}{2}q\big)}{|x-y|^{d-2}}\,d\sigma(y)\\
&\qquad\ge\int_{\{s\in\mathbb{R}^{d-2}:|s-t|<\frac{\varepsilon}{LQ}\}}\frac{\sin^2\frac{\sqrt{1-\lambda}|\tilde\sigma_i(t)-\tilde\sigma_i(s)|q}{2}}{|\tilde\sigma_i(t)-\tilde\sigma_i(s)|^{d-2}}\sqrt{\det\big([D\tilde\sigma_i(s)]^\top[D\tilde\sigma_i(s)]\big)}\,ds. \qquad (4.31)
\end{aligned}$$

According to item (iii) in Remark 4.10 there exists a constant $m_i\in\mathbb{R}$ with

$$\sqrt{\det\big([D\sigma_i(P_i(\xi)s)]^\top[D\sigma_i(P_i(\xi)s)]\big)}\ge m_i$$

for all $s\in\tilde\Omega_i$. From condition (C2) it follows $m_i>0$. Set $m_0:=\min\{m_1,\dots,m_m\}$. Due to $|\det P_i(\xi)|=|\det[P_i(\xi)]^{-1}|^{-1}$ and $|\det[P_i(\xi)]^{-1}|\le Q^{d-2}$ with the constant Q from item (iv) in Remark 4.10 we observe hence from equation (4.19)

$$\sqrt{\det\big([D\tilde\sigma_i(s)]^\top[D\tilde\sigma_i(s)]\big)}=|\det P_i(\xi)|\sqrt{\det\big([D\sigma_i(P_i(\xi)s)]^\top[D\sigma_i(P_i(\xi)s)]\big)}\ge\frac{m_0}{Q^{d-2}}.$$

Therefore we can conclude from (4.31)

$$\int_{\Sigma\cap B_\varepsilon(x)} \frac{\sin^2\left(\frac{\sqrt{1-\lambda}|x-y|}{2}q\right)}{|x-y|^{d-2}}\, d\sigma(y) \geq \frac{m_0}{Q^{d-2}} \int_{\{s\in\mathbb{R}^{d-2}:|s-t|<\frac{\varepsilon}{LQ}\}} \frac{\sin^2 \frac{\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|q}{2}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|^{d-2}}\, ds. \tag{4.32}$$

Next we define the function $\Phi_\lambda : \mathbb{R}^{d-2} \to \mathbb{R}^{d-2}$ via

$$\Phi_\lambda(s) = t + \frac{s-t}{\sqrt{1-\lambda}}.$$

Note that $[D\Phi_\lambda](s) = \frac{1}{\sqrt{1-\lambda}} I_{d-2}$ and $\det[D\Phi_\lambda] = \frac{1}{\sqrt{1-\lambda}^{d-2}}$. Moreover

$$\Phi_\lambda\left(\left\{s\in\mathbb{R}^{d-2}: |s-t| < \frac{\varepsilon\sqrt{1-\lambda}}{LQ}\right\}\right) = \left\{s\in\mathbb{R}^{d-2}: |s-t| < \frac{\varepsilon}{LQ}\right\}.$$

Hence we get for every $R>1$ and every sufficiently large λ the estimate

$$\begin{aligned}
\int_{\{s\in\mathbb{R}^{d-2}:|s-t|<\frac{\varepsilon}{LQ}\}} \frac{\sin^2 \frac{\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|q}{2}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|^{d-2}}\, ds &= \int_{\Phi_\lambda\left(\left\{s\in\mathbb{R}^{d-2}:|s-t|<\frac{\varepsilon\sqrt{1-\lambda}}{LQ}\right\}\right)} \frac{\sin^2 \frac{\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|q}{2}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i(s)|^{d-2}}\, ds \\
&= \int_{\left\{s\in\mathbb{R}^{d-2}:|s-t|<\frac{\varepsilon\sqrt{1-\lambda}}{LQ}\right\}} \frac{\sin^2 \frac{\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)|q}{2}}{|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)|^{d-2}} \cdot \frac{1}{\sqrt{1-\lambda}^{d-2}}\, ds \\
&\geq \int_{\left\{s\in\mathbb{R}^{d-2}:\frac{1}{R}\leq|s-t|\leq R\right\}} \frac{\sin^2 \frac{\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)|q}{2}}{\left(\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)|\right)^{d-2}}\, ds.
\end{aligned} \tag{4.33}$$

According to (4.6) we get

$$\begin{aligned}
\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)| &= \sqrt{1-\lambda}\left|[D\tilde{\sigma}_i(t)]\big(\Phi_\lambda(s)-t\big) + \tilde{F}_i\big(\Phi_\lambda(s),t\big)\right| \\
&\leq \sqrt{1-\lambda}\left|[D\tilde{\sigma}_i(t)]\frac{s-t}{\sqrt{1-\lambda}}\right| + \sqrt{1-\lambda}\tilde{C}\left|\frac{s-t}{\sqrt{1-\lambda}}\right|^2 \\
&= |s-t| + \frac{\tilde{C}|s-t|^2}{\sqrt{1-\lambda}}
\end{aligned}$$

and analogously

$$\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)| \geq |s-t| - \frac{\tilde{C}|s-t|^2}{\sqrt{1-\lambda}}.$$

Hence we get for $t \in \{s \in \mathbb{R}^{d-2} : \frac{1}{R} \le |s-t| \le R\}$ the estimate

$$\left| \sqrt{1-\lambda}|\tilde{\sigma}_i(t) - \tilde{\sigma}_i \circ \Phi_\lambda(s)| - |s-t| \right| \le \frac{\tilde{C}|s-t|^2}{\sqrt{1-\lambda}} \le \frac{\tilde{C}R^2}{\sqrt{1-\lambda}}.$$

Note that this estimate is independent of x. Hence the integrand in (4.33) converges for $\lambda \to -\infty$ uniformly against

$$\frac{\sin^2 \frac{|s-t|q}{2}}{|s-t|^{d-2}}.$$

Note that also this convergence can be estimated independently of x.

As the set $\{s \in \mathbb{R}^{d-2} : \frac{1}{R} \le |s-t| \le R\}$ is compact also the integral converges against

$$\begin{aligned}\int\limits_{\left\{s\in\mathbb{R}^{d-2}:\frac{1}{R}\le|s-t|\le R\right\}} \frac{\sin^2 \frac{|s-t|q}{2}}{|s-t|^{d-2}}\, ds &= \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \int\limits_{\frac{1}{R}}^{R} \frac{\sin^2 \frac{rq}{2}}{r^{d-2}} \cdot r^{d-3}\, dr \\ &= \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \int\limits_{\frac{q}{2R}}^{\frac{Rq}{2}} \frac{\sin^2 p}{\frac{2}{q}p} \cdot \frac{2}{q}\, dp = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \int\limits_{\frac{q}{2R}}^{\frac{Rq}{2}} \frac{\sin^2 p}{p}\, dp\end{aligned}$$

where we have used the substitution $p := \frac{rq}{2}$. Again this convergence can be estimated independently of x. Together with (4.30), (4.32) and (4.33) we get

$$\begin{aligned}&\int\limits_\Sigma G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) \le \\ &-\frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}} \int\limits_0^\infty \frac{1}{(q^2+1)^{\frac{d-1}{2}}} \frac{m_0}{Q^{d-2}} \int\limits_{\left\{s\in\mathbb{R}^{d-2}:\frac{1}{R}\le|s-t|\le R\right\}} \frac{\sin^2 \frac{\sqrt{1-\lambda}|\tilde{\sigma}_i(t)-\tilde{\sigma}_i\circ\Phi_\lambda(s)|q}{2}}{\left(\sqrt{1-\lambda}|\tilde{\sigma}_i(t) - \tilde{\sigma}_i \circ \Phi_\lambda(s)|\right)^{d-2}}\, ds\, dq \\ &\xrightarrow{\lambda\to-\infty} \quad -\frac{m_0}{Q^{d-2}} \frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d+1}{2}}} \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}-1\right)} \int\limits_0^\infty \frac{1}{(q^2+1)^{\frac{d-1}{2}}} \int\limits_{\frac{q}{2R}}^{\frac{Rq}{2}} \frac{\sin^2 p}{p}\, dp\, dq.\end{aligned}$$

Also this convergence can be estimated independently of x. Hence

$$\lim_{\lambda\to-\infty} \sup_{x\in\Sigma} \int\limits_\Sigma G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) \le -\hat{C} \int\limits_0^\infty \frac{1}{(q^2+1)^{\frac{d-1}{2}}} \int\limits_{\frac{q}{2R}}^{\frac{Rq}{2}} \frac{\sin^2 p}{p}\, dp\, dq.$$

with the corresponding constant $\hat{C} > 0$. As the integral $\int_0^\infty \frac{\sin^2 p}{p}\, dp$ does not converge and $R > 1$ was chosen arbitrary we conclude

$$\lim_{\lambda\to-\infty} \sup_{x\in\Sigma} \int\limits_\Sigma G_{\lambda-1}(x-y) - G_0(x-y)\, d\sigma(y) = -\infty.$$

□

Definition 4.12. For $x = \sigma(s_0) \in \Sigma$ let $B_\delta^\Sigma(x) := \{\sigma(s) : |s - s_0| < \delta\}$ be the open ball in Σ with center x and radius δ. For $\lambda \leq 1$, $x \in \Sigma$ and $h \in C^{0,1}(\Sigma)$ define

$$(B_\lambda h)(x) := \lim_{\delta \searrow 0} \left[\int_{\Sigma \setminus B_\delta^\Sigma(x)} h(y) G_{\lambda-1}(x-y)\, d\sigma(y) + h(x) \frac{\ln \delta}{2\pi} \right].$$

Lemma 4.13. *The definition above gives rise to a well defined operator B_λ in $L^2(\Sigma)$ with domain* $\operatorname{dom} B_\lambda = C^{0,1}(\Sigma) \subseteq L^2(\Sigma)$. *The operator B_λ is symmetric and bounded from above by the finite number* $\sup_{x\in\Sigma} k_\lambda(x)$.

Proof. Let $h \in C^{0,1}(\Sigma)$. Note that we can write $\big(B_\lambda h\big)(x)$ as

$$\begin{aligned}(B_\lambda h)(x) &= \lim_{\delta \searrow 0} \Bigg[\int_{\Sigma \setminus B_\delta^\Sigma(x)} \big[h(y) - h(x)\big] G_{\lambda-1}(x-y)\, d\sigma(y) \\ &\qquad\qquad + \int_{\Sigma \setminus B_\delta^\Sigma(x)} h(x) G_{\lambda-1}(x-y)\, d\sigma(y) + h(x) \frac{\ln \delta}{2\pi} \Bigg] \\ &= \lim_{\delta \searrow 0} \left[\int_{\Sigma \setminus B_\delta^\Sigma(x)} \big[h(y) - h(x)\big] G_{\lambda-1}(x-y)\, d\sigma(y) \right] + h(x) k_\lambda(x) \qquad (4.34)\end{aligned}$$

with $k_\lambda(x)$ defined as in Lemma 4.11. Denoting by L a Lipschitz constant of h we get for the integral in (4.34) the estimate

$$\begin{aligned}&\left| \int_{\Sigma \setminus B_\delta^\Sigma(x)} \big[h(y) - h(x)\big] G_{\lambda-1}(x-y)\, d\sigma(y) \right| \\ &\quad \leq \int_\Sigma |h(y) - h(x)| \cdot \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{\sqrt{-(\lambda-1)}}{|x-y|} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\big(\sqrt{-(\lambda-1)}|x-y|\big)\, d\sigma(y) \\ &\quad \leq L \frac{(1-\lambda)^{\frac{d-2}{4}}}{(2\pi)^{\frac{d}{2}}} \int_\Sigma \frac{K_{\frac{d}{2}-1}\big(\sqrt{(1-\lambda)}|x-y|\big)}{|x-y|^{\frac{d}{2}-2}}\, d\sigma(y).\end{aligned}$$

The singularity of the integrand at x is in $O(|x-y|^{-d+3})$, cf. (7.44) in [64, Chapter 7.4]. As Σ is a compact $(d-2)$-dimensional manifold the integral converges. Hence the limit in (4.34) exists and we get

$$(B_\lambda h)(x) = \int_\Sigma \big[h(y) - h(x)\big] G_{\lambda-1}(x-y)\, d\sigma(y) + h(x) k_\lambda(x) \qquad (4.35)$$

and the estimate

$$|(B_\lambda h)(x)| \leq L\frac{(1-\lambda)^{\frac{d-2}{4}}}{(2\pi)^{\frac{d}{2}}}\int_\Sigma \frac{K_{\frac{d}{2}-1}\big(\sqrt{(1-\lambda)}|x-y|\big)}{|x-y|^{\frac{d}{2}-2}}\,d\sigma(y) + |h(x)|\cdot\|k_\lambda\|_\infty.$$

Hence we get the estimate

$$\|B_\lambda h\|_{L^2(\Sigma)} \leq L\frac{(1-\lambda)^{\frac{d-2}{4}}}{(2\pi)^{\frac{d}{2}}}\sqrt{|\Sigma|}\int_\Sigma \frac{K_{\frac{d}{2}-1}\big(\sqrt{(1-\lambda)}|x-y|\big)}{|x-y|^{\frac{d}{2}-2}}\,d\sigma(y) + \|h\|_{L^2(\Sigma)}\cdot\|k_\lambda\|_\infty.$$

Thus B_λ is a well defined operator in $L^2(\Sigma)$.
Let now $h, g \in C^{0,1}(\Sigma)$. Then

$$\begin{aligned}
\langle B_\lambda h, g\rangle_{L^2(\Sigma)} - \langle h, B_\lambda g\rangle_{L^2(\Sigma)} &= \langle [B_\lambda - k_\lambda]h, g\rangle_{L^2(\Sigma)} - \langle h, [B_\lambda - k_\lambda]g\rangle_{L^2(\Sigma)} \\
&= \int_\Sigma \left(\int_\Sigma \big[h(y)-h(x)\big]G_{\lambda-1}(x-y)\,d\sigma(y)\right)\overline{g(x)}\,d\sigma(x) \\
&\quad - \int_\Sigma h(y)\overline{\left(\int_\Sigma \big[g(x)-g(y)\big]G_{\lambda-1}(x-y)\,d\sigma(x)\right)}\,d\sigma(y) \\
&= \int_\Sigma\int_\Sigma \big[h(y)\overline{g(y)} - h(x)\overline{g(x)}\big]G_{\lambda-1}(x-y)\,d\sigma(y)\,d\sigma(x) = 0,
\end{aligned}$$

where we have used for the last equality that the integrand is skew-symmetric with respect to s and x. Hence B_λ is symmetric.
Due to the symmetry we observe now

$$\begin{aligned}
2\langle [B_\lambda - k_\lambda]h, h\rangle_{L^2(\Sigma)} &= \langle [B_\lambda - k_\lambda]h, h\rangle_{L^2(\Sigma)} + \langle h, [B_\lambda - k_\lambda]h\rangle_{L^2(\Sigma)} \\
&= \int_\Sigma \left(\int_\Sigma \big[h(y)-h(x)\big]G_{\lambda-1}(x-y)\,d\sigma(y)\right)\overline{h(x)}\,d\sigma(x) \\
&\quad + \int_\Sigma h(y)\overline{\left(\int_\Sigma \big[h(x)-h(y)\big]G_{\lambda-1}(x-y)\,d\sigma(x)\right)}\,d\sigma(y) \\
&= \int_\Sigma\int_\Sigma G_{\lambda-1}(x-y)\left(2h(y)\overline{h(x)} - |h(x)|^2 - |h(y)|^2\right)\,d\sigma(y)\,d\sigma(x).
\end{aligned}$$

Setting $u := \mathrm{Re}(h)$ and $v := \mathrm{Im}(h)$ we observe

$$\begin{aligned}
&2h(y)\overline{h(x)} - |h(x)|^2 - |h(y)|^2 \\
&\quad = 2\big[u(y)u(x) + iv(y)u(x) - iu(y)v(x) + v(y)v(x)\big] - u(x)^2 - v(x)^2 - u(y)^2 - v(y)^2 \\
&\quad = -\big[u(y)-u(x)\big]^2 - \big[v(y)-v(x)\big]^2 + 2i\big[v(y)u(x) - u(y)v(x)\big].
\end{aligned}$$

Hence we get

$$2\langle [B_\lambda - k_\lambda]h,h\rangle_{L^2(\Sigma)} = -\int_\Sigma\int_\Sigma G_{\lambda-1}(x-y)\Big(\big[u(y)-u(x)\big]^2 + \big[v(y)-v(x)\big]^2\Big)\,d\sigma(y)\,d\sigma(x)$$
$$+\int_\Sigma\int_\Sigma 2iG_{\lambda-1}(x-y)\Big(v(y)u(x)-u(y)v(x)\Big)\,d\sigma(y)\,d\sigma(x).$$

Note that $2\langle [B_\lambda - k_\lambda]h,h\rangle_{L^2(\Sigma)}$ is real whereas the second integral above is purely imaginary and thus zero. The integrand of the first integral above is nonnegative which implies $2\langle [B_\lambda - k_\lambda]h,h\rangle_{L^2(\Sigma)} \leq 0$ or, equivalently, $\langle B_\lambda h,h\rangle_{L^2(\Sigma)} \leq \langle k_\lambda h,h\rangle_{L^2(\Sigma)}$. From this we observe $B_\lambda \leq \sup_{x\in\Sigma} k_\lambda(x)$. Recall that $\sup_{x\in\Sigma} k_\lambda(x)$ is finite as k_λ is bounded, cf. Lemma 4.11. □

Define now the operator $\widetilde{B_\lambda} := -F(-B_\lambda)$, where $F(-B_\lambda)$ is the Friedrichs extension of $-B_\lambda$. Note that also $\widetilde{B_\lambda}$ is bounded from above by $\sup_{x\in\Sigma} k_\lambda(x)$.

Lemma 4.14. *Let $\lambda < 1$. Then $\widetilde{B_\lambda} = \widetilde{B_0} + M(\lambda)$. In particular* $\operatorname{dom}\widetilde{B_\lambda}$ *is λ-independent.*

Proof. For all $h \in C^{0,1}(\Sigma) \subseteq L^2(\Sigma) = \operatorname{dom} M(\lambda)$ we have

$$\begin{aligned}\big(B_\lambda h\big)(x) - \big(M(\lambda)h\big)(x) &= \lim_{\delta\searrow 0}\Bigg[\int_{\Sigma\setminus B^\Sigma_\delta(x)} h(y)G_{\lambda-1}(x-y)\,d\sigma(y) + h(x)\frac{\ln\delta}{2\pi}\Bigg]\\
&\quad - \lim_{\delta\searrow 0}\int_{\Sigma\setminus B^\Sigma_\delta(x)} h(y)\big[G_{\lambda-1}(x-y) - G_{-1}(x-y)\big]\,d\sigma(y)\\
&= \lim_{\delta\searrow 0}\Bigg[\int_{\Sigma\setminus B^\Sigma_\delta(x)} h(y)G_{-1}(x-y)\,d\sigma(y) + h(x)\frac{\ln\delta}{2\pi}\Bigg]\\
&= \big(B_0 h\big)(x).\end{aligned}$$

Hence $B_\lambda = B_0 + M(\lambda)$. Recall that $M(\lambda)$ is bounded and selfadjoint. Hence we can apply Lemma 2.16 and get

$$\widetilde{B_\lambda} = -F(-B_\lambda) = -F(-B_0 - M(\lambda)) = -F(-B_0) + M(\lambda) = \widetilde{B_0} + M(\lambda).$$

In particular $\operatorname{dom}\widetilde{B_\lambda} = \operatorname{dom}\widetilde{B_0} \cap \operatorname{dom} M(\lambda) = \operatorname{dom}\widetilde{B_0} \cap L^2(\Sigma) = \operatorname{dom}\widetilde{B_0}$. □

Now we are ready to define the generalized trace for a large class of elements in the domain of T. For this recall that for $\lambda < 1$ every element $u \in \operatorname{dom} T$ can be written uniquely as $u = u_c^\lambda + u_s^\lambda$ with $u_c^\lambda \in H^2(\mathbb{R}^d)$ and $u_s^\lambda \in \ker(T-\lambda)$, cf. Lemma 2.1. Moreover $u_s^\lambda = \gamma(\lambda)h$ for some $h \in L^2(\Sigma)$, cf. the definition of $\gamma(\lambda)$ in Lemma 2.6.

Definition 4.15. Let $\lambda < 1$. For $h \in \operatorname{dom} \widetilde{B_\lambda}$ we define the generalized trace of $\gamma(\lambda)h$ via

$$\operatorname{tr}_\Sigma (\gamma(\lambda)h) := \widetilde{B_\lambda} h.$$

Hence for an element $u = u_c^\lambda + \gamma(\lambda)h \in \operatorname{dom} T$ with $h \in \operatorname{dom} \widetilde{B_\lambda}$ we define

$$\operatorname{tr}_\Sigma u := \operatorname{tr}_\Sigma^2 u_c^\lambda + \widetilde{B_\lambda} h.$$

The definition above has the disadvantage that it seems to depend on $\lambda < 1$. However, this is not the case as we will show in the following lemma.

Lemma 4.16. *The definition of* $\operatorname{tr}_\Sigma u$ *is independent of the particular choice of* $\lambda < 1$.

Proof. Let $\lambda, \mu < 1$ and $u = u_c + \gamma(\lambda)h \in \operatorname{dom} T$ with $u_c \in H^2(\mathbb{R})$ and $h \in \operatorname{dom} \widetilde{B_\lambda} = \operatorname{dom} \widetilde{B_\mu}$, cf. Lemma 4.14. Hence we can write u as $u = \tilde{u}_c + \gamma(\mu)h$ with

$$\tilde{u}_c := u_c + \gamma(\lambda)h - \gamma(\mu)h = u_c + (\lambda - \mu)(A - \lambda)^{-1}\gamma(\mu)h \ \in H^2(\mathbb{R}^3),$$

where we have used the formula $\gamma(\lambda) = \gamma(\mu) + (\lambda - \mu)(A - \lambda)^{-1}\gamma(\mu)$, cf. Lemma 4.14. Hence $u = \tilde{u}_c + \gamma(\mu)h$ is also a decomposition of u as $u = u_c + \gamma(\lambda)h$. Moreover we get with Lemma 4.4 and Lemma 4.14

$$\begin{aligned} \operatorname{tr}_\Sigma \big(\gamma(\lambda)h - \gamma(\mu)h\big) &= \operatorname{tr}_\Sigma \big(\gamma(\lambda)h - \gamma(0)h\big) - \operatorname{tr}_\Sigma \big(\gamma(\mu)h - \gamma(0)h\big) \\ &= M(\lambda)h - M(\mu)h \\ &= \big[\widetilde{B_0} + M(\lambda)\big]h - \big[\widetilde{B_0} + M(\mu)\big]h = \widetilde{B_\lambda}h - \widetilde{B_\mu}h. \end{aligned}$$

Hence we get

$$\operatorname{tr}_\Sigma u_c + \widetilde{B_\lambda}h = \operatorname{tr}_\Sigma \big(u - \gamma(\lambda)\big) + \widetilde{B_\lambda}h = \operatorname{tr}_\Sigma \big(\tilde{u}_c + \gamma(\mu)h - \gamma(\lambda)\big) + \widetilde{B_\lambda}h = \operatorname{tr}_\Sigma \tilde{u}_c + \widetilde{B_\mu}h.$$

From this we observe that $\operatorname{tr}_\Sigma u$ is independent of the particular choice of $\lambda < 1$. □

Next we have to specify an operator Θ in $L^2(\Sigma)$ such that the operator A_Θ as defined in Section 4.2 coincides with a Schrödinger operator with δ-interaction of strength $\frac{1}{\alpha}$ on Σ, i.e. with an operator which acts formally like

$$\left(-\Delta - \frac{1}{\alpha}\delta_\Sigma\right) u = -\Delta u - \frac{1}{\alpha} u|_\Sigma \cdot \delta_\Sigma.$$

On the other hand, the action of the operator $T - 1$ is given by $-\Delta u - h\delta_\Sigma$. Equating both expressions we get

$$h = \frac{1}{\alpha} u|_\Sigma = \frac{1}{\alpha} \operatorname{tr}_\Sigma \big(u_c + \gamma(0)h\big) = \frac{1}{\alpha}\big(u_c|_\Sigma + \widetilde{B_0}h\big).$$

Hence $\alpha h - \widetilde{B_0}h = u_c|_\Sigma$. Using the generalized boundary triple from Corollary 4.2 this equation is equivalent to $(\alpha - \widetilde{B_0})\Gamma_0 u = \Gamma_1 u$. This heuristic explanation motivates the following definition.

Definition 4.17. For $\alpha \in \mathbb{R} \setminus \{0\}$ define the Schrödinger operator $-\Delta_{\Sigma,\alpha}$ in $L^2(\mathbb{R}^d)$ with δ-interaction of strength $\frac{1}{\alpha}$ supported on Σ as $-\Delta_{\Sigma,\alpha} := A_\Theta - 1$ with $\Theta := \alpha - \widetilde{B_0}$. This means

$$\operatorname{dom} -\Delta_{\Sigma,\alpha} = \operatorname{dom} A_\Theta = \{u \in \operatorname{dom} T : (\alpha - \widetilde{B_0})\Gamma_0 u = \Gamma_1 u\} = \{u \in \operatorname{dom} T : h = \frac{1}{\alpha} \operatorname{tr}_\Sigma u\},$$
$$-\Delta_{\Sigma,\alpha} u = (A_\Theta - 1)u = -\Delta u - h\delta_\Sigma = -\Delta u - \frac{1}{\alpha} \operatorname{tr}_\Sigma u \cdot \delta_\Sigma.$$

If $\alpha > \sup_{x\in\Sigma} k_0(x)$ then $\Theta \geq \alpha - \sup_{x\in\Sigma} k_0(x) > 0$, cf. Lemma 4.13. Hence we know from Theorem 4.6 and Theorem 4.7 that the operator A_Θ is selfadjoint and bounded from below. Moreover, if Σ is a compact C^∞-manifold, the resolvent difference with A belongs to $\mathfrak{S}_p(L^2(\mathbb{R}^d))$ for $p > \frac{d}{2} - 1$. Obviously $-\Delta_{\Sigma,\alpha}$ has the same properties. The following theorem shows that the assumption $\alpha > \sup_{x\in\Sigma} k_0(x)$ is not necessary for this.

Theorem 4.18. *Let $\lambda_0 < 1$ be such that $\sup_{x\in\Sigma} k_\lambda(x) < \alpha$ holds for all $\lambda < \lambda_0$, cf. Lemma 4.11. Then the Schrödinger operator $-\Delta_{\Sigma,\alpha}$ is selfadjoint in $L^2(\mathbb{R}^d)$ and bounded from below by $\lambda_0 - 1$. If we assume additionally that Σ is a compact C^∞-manifold then*

$$(-\Delta_{\Sigma,\alpha} - \lambda)^{-1} - (-\Delta_{free} - \lambda)^{-1} \in \mathfrak{S}_p(L^2(\mathbb{R}^d))$$

holds for all $\lambda \in \rho(-\Delta_{\Sigma,\alpha}) \cap \rho(-\Delta_{free})$ and $p > \frac{d}{2} - 1$. In particular this resolvent difference is compact and $\sigma_{ess}(-\Delta_{\Sigma,\alpha}) = [0, \infty[$.

Proof. As in Definition 4.17 set $\Theta := \alpha - \widetilde{B_0}$. As $M(\lambda)$ is bounded and selfadjoint for $\lambda \in \mathbb{R}$, cf. Lemma 2.7, $\Theta - M(\lambda)$ is selfadjoint too. Moreover we have for all $\lambda < \lambda_0$

$$\Theta - M(\lambda) = \alpha - \big(\widetilde{B_0 + M(\lambda)}\big) = \alpha - \widetilde{B_\lambda} \geq \alpha - \sup_{x\in\Sigma} k_\lambda(x) > 0,$$

i.e. $0 \in \rho(\Theta - M(\lambda))$. By Theorem 2.8 the operator A_Θ is selfadjoint in $L^2(\mathbb{R}^d)$ and $\lambda \in \rho(A_\Theta)$. As this is true for all $\lambda < \lambda_0$ we get $A_\Theta \geq \lambda_0$. Hence also $-\Delta_{\Sigma,\alpha} = A_\Theta - 1$ is selfadjoint and bounded from below by $\lambda_0 - 1$.

Note that also by Theorem 2.8 Krein's resolvent formula (2.1) holds for all $\lambda < \lambda_0$. Hence

$$(-\Delta_{\Sigma,\alpha} - \lambda)^{-1} - (-\Delta_{\text{free}} - \lambda)^{-1} = \gamma(\lambda + 1)[\Theta - M(\lambda + 1)]^{-1}\gamma(\overline{\lambda + 1})^*$$

holds for all $\lambda < \lambda_0 - 1$. If we assume that Σ is a compact C^∞-manifold then Lemma 2.23 implies $\gamma(\overline{\lambda + 1}) \in \mathfrak{S}_q(L^2(\Sigma), L^2(\mathbb{R}^d))$ and $\gamma(\overline{\lambda + 1})^* \in \mathfrak{S}_q(L^2(\mathbb{R}^d), L^2(\Sigma))$ for all $q > d - 2$. As $[\Theta - M(\lambda + 1)]^{-1} \in \mathcal{L}(L^2(\Sigma))$ we get with Lemma 2.3 in [13] (see also III.§7.2.2 in [37])

$$(-\Delta_{\Sigma,\alpha} - \lambda)^{-1} - (-\Delta_{\text{free}} - \lambda)^{-1} = \gamma(\lambda + 1)[\Theta - M(\lambda + 1)]^{-1}\gamma(\overline{\lambda + 1})^* \in \mathfrak{S}_p(L^2(\mathbb{R}^d))$$

for $p := \frac{q}{2} > \frac{d-2}{2}$ and all $\lambda < \lambda_0 - 1$. Analogously as in the proof of Theorem 4.6 we get for an arbitrary $\mu \in \rho(-\Delta_{\Sigma,\alpha}) \cap \rho(-\Delta_{\text{free}})$

$$(-\Delta_{\Sigma,\alpha} - \mu)^{-1} - (-\Delta_{\text{free}} - \mu)^{-1} = U_1\big((-\Delta_{\Sigma,\alpha} - \lambda)^{-1} - (-\Delta_{\text{free}} - \lambda)^{-1}\big)U_2 \in \mathfrak{S}_p(L^2(\mathbb{R}^d))$$

with the two bounded operators

$$U_1 := \big(I + (\mu - \lambda)(-\Delta_{\Sigma,\alpha} - \mu)^{-1}\big) \quad \text{and} \quad U_2 := \big(I + (\mu - \lambda)(-\Delta_{\text{free}} - \mu)^{-1}\big).$$

As the resolvent difference belongs to $\mathfrak{S}_p(L^2(\mathbb{R}^d))$ it is in particular compact. Hence we get with Theorem 6.19 from [64] $\sigma_{\text{ess}}(-\Delta_{\Sigma,\alpha}) = \sigma_{\text{ess}}(-\Delta_{\text{free}}) = [0,\infty[$. □

For a better analysis of $-\Delta_{\Sigma,\alpha}$ a deeper understanding of the operator $\widetilde{B_\lambda}$ is needed. In particular a better knowledge of the eigenvalues of $\widetilde{B_\lambda}$ is helpful to describe the eigenvalues of $-\Delta_{\Sigma,\alpha}$ more accurate. As the eigenvalues of $\widetilde{B_\lambda}$ are dependent on the dimension we restrict ourselves to the case that Σ is a closed curve in $\mathbb{R}^3$, which is done in the next section.

4.4 Application to δ-interactions on closed curves in $\mathbb{R}^3$

Throughout this section Σ is a compact, closed, regular C^2-curve in $\mathbb{R}^3$ of length $L > 0$ without self-intersections. Of course it is possible to find a set of parametrizations σ_i satisfying the conditions (C1) to (C4) from Section 4.3, but for our purpose it is more convenient to use a C^2-parametrization $\sigma : [0,L] \to \mathbb{R}^3$ of Σ with $|\dot{\sigma}(s)| = 1$ for all $s \in [0,L]$. Moreover, we define for $x = \sigma(t) \in \Sigma$ and $\delta > 0$ the open interval in Σ

$$I_\delta^\Sigma(x) := \{\sigma(s) : s \in\,]t-\delta, t+\delta[\}.$$

(If $t = 0$ or $t = L$ we have to replace σ by its L-periodic extension or by a shifted parametrization. However, this case is not important as it just concerns a set of measure 0.) As σ is Lipschitz-continuous with Lipschitz constant 1 we observe $I_\delta^\Sigma(x) \subseteq \Sigma \cap B_\delta(x)$ and hence $\Sigma \setminus I_\delta^\Sigma(x) \supseteq \Sigma \setminus B_\delta(x)$. In general, these sets do not coincide, but when δ tends to 0 they become similar. This allows us to give an alternative representation of the function k_λ in the following Lemma.

Lemma 4.19. *Let $\lambda \leq 1$ and let k_λ be the function defined in Lemma 4.11. Then*

$$k_\lambda(x) = \lim_{\delta \to 0} \left[\int_{\Sigma \setminus I_\delta^\Sigma(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) - \frac{\ln \delta}{2\pi} \right]$$

holds for all $x \in \Sigma$.

Proof. Let $x \in \Sigma$ and $t \in [0,L]$ such that $\sigma(t) = x$. As mentioned above we can assume $t \neq 0$ and $t \neq L$. Furthermore we assume in the following that δ is sufficiently small such that $\delta <$

$\min\{t, L_t\}$. Due to $\Sigma \setminus I_\delta^\Sigma(x) = [\Sigma \setminus B_\delta(x)] \,\dot\cup\, [(\Sigma \cap B_\delta(x)) \setminus I_\delta^\Sigma(x)]$ we have

$$
\begin{aligned}
&\int_{\Sigma \setminus I_\delta^\Sigma(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) \\
&\quad = \int_{\Sigma \setminus B_\delta(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) + \int_{(\Sigma \cap B_\delta(x)) \setminus I_\delta^\Sigma(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) \\
&\quad = \int_{\Sigma \setminus B_\delta(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) + \int_{\{s\in[0,L]: |\sigma(t)-\sigma(s)|<\delta\le|s-t|\}} \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\, ds.
\end{aligned}
$$

Hence we get with the definition of k_λ in Lemma 4.11

$$
\begin{aligned}
&\lim_{\delta\to 0}\left[\int_{I_\delta^\Sigma(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) - \frac{\ln\delta}{2\pi}\right] \\
&\quad = \lim_{\delta\to 0}\left[\int_{(\Sigma \cap B_\delta(x)) \setminus I_\delta^\Sigma(x)} \frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y) - \frac{\ln\delta}{2\pi}\right. \\
&\qquad\qquad \left. + \int_{\{s\in[0,L]: |\sigma(t)-\sigma(s)|<\delta\le|s-t|\}} \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\, ds\right] \\
&\quad = k_\lambda(x) + \lim_{\delta\to 0} \int_{\{s\in[0,L]: |\sigma(t)-\sigma(s)|<\delta\le|s-t|\}} \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\, ds.
\end{aligned}
\tag{4.36}
$$

As Σ is a C^2-curve we can apply Taylor's theorem to each component of σ and get for some suitable ζ_1, ζ_2 and ζ_3

$$
\sigma(t) = \begin{bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{bmatrix} = \sigma(s) + \sigma'(s)(t-s) + \begin{bmatrix} \sigma_1''(\zeta_1) \\ \sigma_2''(\zeta_2) \\ \sigma_3''(\zeta_3) \end{bmatrix} \frac{(t-s)^2}{2}.
$$

With the constant $C_\sigma := \sqrt{\|\sigma_1''\|_\infty^2 + \|\sigma_2''\|_\infty^2 + \|\sigma_3''\|_\infty^2}$ and a local Lipschitz constant L of σ^{-1} we get now

$$
\begin{aligned}
|\sigma(t)-\sigma(s)| &\ge |\sigma'(s)|\cdot|t-s| - \left|\begin{bmatrix} \sigma_1''(\xi_1) \\ \sigma_2''(\xi_2) \\ \sigma_3''(\xi_3) \end{bmatrix}\right| \frac{(t-s)^2}{2} \ge |t-s| - \frac{C_\sigma}{2}|t-s|^2 \\
&= |t-s|\Big(1 - \frac{C_\sigma}{2}|t-s|\Big) \ge |t-s|\Big(1 - \frac{C_\sigma}{2}L|\sigma(t)-\sigma(s)|\Big) > |t-s|\Big(1 - \frac{C_\sigma}{2}L\delta\Big)
\end{aligned}
$$

for all $s \in [0,L]$ with $|\sigma(t)-\sigma(s)| < \delta \leq |t-s|$, if we assume that δ is sufficiently small such that $1-\frac{C_\sigma}{2}L\delta > 0$. Hence

$$\{s\in[0,L]:|\sigma(s)-\sigma(t)|<\delta\leq|s-t|\} \subseteq \left\{s\in[0,L]:|t-s|\left(1-\frac{C_\sigma}{2}L\delta\right)<\delta\leq|s-t|\right\} = \left\{s\in[0,L]:\delta\leq|t-s|<\frac{\delta}{1-\frac{C_\sigma}{2}L\delta}\right\}$$

and therefore

$$\int\limits_{\{s\in[0,L]:|\sigma(t)-\sigma(s)|<\delta\leq|s-t|\}} \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds$$
$$\leq \int\limits_{\left\{s\in[0,L]:\delta\leq|t-s|<\frac{\delta}{1-\frac{C_\sigma}{2}L\delta}\right\}} \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds \leq \int\limits_{\left\{s\in[0,L]:\delta\leq|t-s|<\frac{\delta}{1-\frac{C_\sigma}{2}L\delta}\right\}} \frac{L}{4\pi|t-s|}\,ds$$
$$= \frac{L}{2\pi}\int_\delta^{\frac{\delta}{1-\frac{C_\sigma}{2}L\delta}} \frac{1}{s}\,ds = \frac{L}{2\pi}\left(\ln\frac{\delta}{1-\frac{C_\sigma}{2}L\delta}-\ln\delta\right) = -\frac{L}{2\pi}\ln\left(1-\frac{C_\sigma}{2}L\delta\right) \xrightarrow{\delta\to0} 0.$$

The assertion follows now with (4.36). □

Note that with Lemma 4.19 and with equation (4.35) in the proof of Lemma 4.13 we also get the following alternative representation

$$\big(B_\lambda h\big)(x) = \lim_{\delta\to0}\left[\int\limits_{\Sigma\setminus I_\delta^\Sigma(x)} h(y)\frac{e^{-\sqrt{-(\lambda-1)}|x-y|}}{4\pi|x-y|}\,d\sigma(y) - h(x)\frac{\ln\delta}{2\pi}\right] \tag{4.37}$$

of the operator B_λ. We will use in the following this representation of B_λ because it is easier to handle than the one given in Definition 4.12.

Next we consider at first the case that the closed curve is a circle of radius $R>0$ in $\mathbb{R}^3$. In order to distinguish it from a more general closed curve Σ the circle is denoted by $\mathcal{T}$. Without loss of generality we assume that $\mathcal{T}$ is parametrized by the function

$$\tau:[0,2\pi R]\to\mathbb{R}^3,\quad t\mapsto R\big(\cos(t/R),\sin(t/R),0\big).$$

Furthermore we will use the formula

$$|\tau(t)-\tau(s)| = 2R\sin\left(\frac{|s-t|}{2R}\right). \tag{4.38}$$

At first we will show the following Lemma.

Lemma 4.20. *The function k_λ defined as in Lemma 4.11 is independent of x. In particular $k_1(x) = \frac{\ln(4R)}{2\pi}$ for all $x \in \mathcal{T}$.*

Proof. Due to the symmetry of the circle $\mathcal{T}$ we observe that k_1 in fact is independent of x. Hence we can choose in the following w.l.o.g. $x = \tau(0)$. Moreover we get with formula (4.38) and the substitution $s := \frac{t}{2R}$

$$\begin{aligned}
\int_{\mathcal{T}_\delta(x)} G_0(x-y)\, d\sigma(y) &= \int_{\mathcal{T}_\delta(x)} \frac{1}{4\pi|x-y|}\, d\sigma(y) = \int_{\delta}^{2\pi R-\delta} \frac{1}{4\pi|\tau(t)-\tau(0)|}\, dt \\
&= \int_{\delta}^{2\pi R-\delta} \frac{1}{4\pi \cdot 2R\sin\left(\frac{t}{2R}\right)}\, dt = \int_{\frac{\delta}{2R}}^{\pi-\frac{\delta}{2R}} \frac{1}{8\pi R \sin s} \cdot 2R\, ds \\
&= \int_{\frac{\delta}{2R}}^{\pi-\frac{\delta}{2R}} \frac{1}{4\pi \sin s}\, ds = \int_{\frac{\delta}{2R}}^{\frac{\pi}{2}} \frac{1}{2\pi \sin s}\, ds = \frac{1}{2\pi}\Big[-\ln(\cos(t/2)) + \ln(\sin(t/2))\Big]_{\frac{\delta}{2R}}^{\frac{\pi}{2}} \\
&= \frac{1}{2\pi}\Big[-\ln(\cos(\pi/4)) + \ln(\sin(\pi/4)) + \ln(\cos(\delta/4R)) - \ln(\sin(\delta/4R))\Big] \\
&= \frac{1}{2\pi}\Big[\ln(\cos(\delta/4R)) - \ln(\sin(\delta/4R))\Big] \\
&= \frac{1}{2\pi}\left[\ln(\cos(\delta/4R)) + \ln\left(\frac{\delta}{\sin(\delta/4R)}\right) - \ln\delta\right].
\end{aligned}$$

Recalling the alternative representation of k_λ in Lemma 4.19 we get hence

$$\begin{aligned}
k_1(x) &= \lim_{\delta\to 0}\left[\int_{\mathcal{T}_\delta(x)} G_0(x-y)\, d\sigma(y) + \frac{\ln\delta}{2\pi}\right] \\
&= \frac{1}{2\pi}\lim_{\delta\to 0}\left[\ln(\cos(\delta/4R)) + \ln\left(\frac{\delta}{\sin(\delta/4R)}\right)\right] \\
&= \frac{1}{2\pi}\lim_{\delta\to 0}\ln\left(\frac{4R\delta}{\sin\delta}\right) = \frac{\ln(4R)}{2\pi} + \frac{1}{2\pi}\lim_{\delta\to 0}\ln\left(\frac{\delta}{\sin\delta}\right) = \frac{\ln(4R)}{2\pi}.
\end{aligned}$$

□

Next we consider the operator B_1 defined by Definition 4.12 or equation (4.37) for the case of a circle. In order to distinguish it from the case of a general closed curve it is denoted by $B_1^{\mathcal{T}}$.

Lemma 4.21. *The operator $B_1^{\mathcal{T}}$ defined by*

$$(B_1^{\mathcal{T}} h)(x) = \lim_{\delta\to 0}\left[\int_{\mathcal{T}_\delta(x)} \frac{h(y)}{4\pi|x-y|}\, d\tau(y) + h(x)\frac{\ln\delta}{2\pi}\right] = \int_{\mathcal{T}} \frac{h(y)-h(x)}{4\pi|x-y|}\, d\tau(y) + h(x)k_1(x)$$

is essentially selfadjoint in $L^2(\mathcal{T})$. Its closure $\overline{B_1^{\mathcal{T}}}$ is semibounded from above, has a compact resolvent, and its eigenvalues (ordered nonincreasingly and counted with multiplicity) are given by

$$\nu_1(1) = \frac{\ln(4R)}{2\pi}, \qquad \nu_{2k}(1) = \nu_{2k+1}(1) = \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{k} \frac{1}{2j-1}, \quad k \in \mathbb{N}.$$

Proof. At first we calculate the eigenvalues of $B_1^{\mathcal{T}}$. If h is a constant function we get obviously $[B_1^{\mathcal{T}} - k_0]h = 0$. Consider next the function h_k defined by $h_k(x) := \sin(kt/R)$ with $t := \tau^{-1}(x)$ and $k \in \mathbb{N}$. With the identity (4.38) and $\sin(ks/R) - \sin(kt/R) = 2\sin\left(\frac{ks-kt}{2R}\right)\cos\left(\frac{ks+kt}{2R}\right)$ we obtain

$$\begin{aligned}
\big([B_1^{\mathcal{T}} - k_0]h_k\big)(x) &= \int_{\mathcal{T}} \frac{h(y)-h(x)}{4\pi|x-y|}\, d\tau(y) = \int_0^{2\pi R} \frac{\sin(ks/R)-\sin(kt/R)}{4\pi\cdot 2R\sin\left(\frac{|s-t|}{2R}\right)}\, ds \\
&= \int_0^{2\pi R} \frac{\sin\left(\frac{k(s-t)}{2R}\right)\cos\left(\frac{k(s+t)}{2R}\right)}{4\pi R\sin\left(\frac{|s-t|}{2R}\right)}\, ds \\
&= \int_0^{t} \frac{\sin\left(\frac{k(s-t)}{2R}\right)\cos\left(\frac{k(s+t)}{2R}\right)}{4\pi R\sin\left(\frac{|s-t|}{2R}\right)}\, ds + \int_t^{2\pi R} \frac{\sin\left(\frac{k(s-t)}{2R}\right)\cos\left(\frac{k(s+t)}{2R}\right)}{4\pi R\sin\left(\frac{|s-t|}{2R}\right)}\, ds. \qquad (4.39)
\end{aligned}$$

With the fact, that sin is an odd function, the substitution $z := s - t + 2\pi R$ and the formulas $\sin(\alpha+\pi) = -\sin(\alpha)$ and $\cos(\alpha+\pi) = -\cos(\alpha)$ the first integral in 4.39 becomes

$$\begin{aligned}
\int_0^{t} \frac{\sin\left(\frac{k(s-t)}{2R}\right)\cos\left(\frac{k(s+t)}{2R}\right)}{4\pi R\sin\left(\frac{|s-t|}{2R}\right)}\, ds &= \int_0^{t} \frac{\sin\left(\frac{k(s-t)}{2R}\right)\cos\left(\frac{k(s+t)}{2R}\right)}{-4\pi R\sin\left(\frac{s-t}{2R}\right)}\, ds \\
= \int_{2\pi R-t}^{2\pi R} \frac{\sin\left(\frac{kz}{2R}-k\pi\right)\cos\left(\frac{kt}{R}+\frac{kz}{2R}-k\pi\right)}{-4\pi R\sin\left(\frac{z}{2R}-\pi\right)}\, dz &= \int_{2\pi R-t}^{2\pi R} \frac{\sin\left(\frac{kz}{2R}\right)\cos\left(\frac{kt}{R}+\frac{kz}{2R}\right)}{4\pi R\sin\left(\frac{z}{2R}\right)}\, dz.
\end{aligned}$$

Analogously we get with the substitution $z := s - t$ for the second integral in 4.39

$$\int_t^{2\pi R} \frac{\sin\left(\frac{k(s-t)}{2R}\right)\cos\left(\frac{k(s+t)}{2R}\right)}{4\pi R\sin\left(\frac{|s-t|}{2R}\right)}\, ds = \int_0^{2\pi R-t} \frac{\sin\left(\frac{kz}{2R}\right)\cos\left(\frac{kt}{R}+\frac{kz}{2R}\right)}{4\pi R\sin\left(\frac{z}{2R}\right)}\, dz.$$

Combining these two results we obtain from (4.39) with the substitution $s := \frac{z}{2R}$ and the formula

$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$

$$\begin{aligned}
\big([B_1^{\mathcal{T}}-k_0]h_k\big)(x) &= \int_0^{2\pi R} \frac{\sin\left(\frac{kz}{2R}\right)\cos\left(\frac{kt}{R}+\frac{kz}{2R}\right)}{4\pi R\sin\left(\frac{z}{2R}\right)}\,dz = \int_0^{\pi} \frac{\sin(ks)\cos\left(\frac{kt}{R}+ks\right)}{2\pi\sin(s)}\,ds \\
&= \int_0^{\pi} \frac{\sin(ks)}{2\pi\sin(s)}\left[\cos\left(\frac{kt}{R}\right)\cdot\cos(ks)-\sin\left(\frac{kt}{R}\right)\cdot\sin(ks)\right]ds \\
&= \cos\left(\frac{kt}{R}\right)\int_0^{\pi}\frac{\sin(ks)\cos(ks)}{2\pi\sin(s)}\,ds-\sin\left(\frac{kt}{R}\right)\int_0^{\pi}\frac{\sin^2(ks)}{2\pi\sin(s)}\,ds \\
&= \frac{-h_k(x)}{4\pi}\int_0^{\pi}\frac{1-\cos(2ks)}{\sin(s)}\,ds,
\end{aligned}$$

where we have used in the last step the definition of h_k, the formula $2\sin^2(\alpha)=1-\cos(2\alpha)$ and

$$\int_0^{\pi}\frac{\sin(ks)\cos(ks)}{2\pi\sin(s)}\,ds=0,$$

cf. [38, 3.612 1.]. With the indefinite integrals [38, 2.526 1. and 2.539 4.] we get

$$\int_0^{\pi}\frac{1-\cos(2ks)}{\sin(s)}\,ds=\left[\ln\left(\tan\frac{s}{2}\right)-2\sum_{j=1}^{k}\frac{\cos[(2j-1)s]}{2j-1}-\ln\left(\tan\frac{s}{2}\right)\right]_0^{\pi}=4\sum_{j=1}^{k}\frac{1}{2j-1}.$$

Hence

$$B_1^{\mathcal{T}}h_k=\left(k_0-\frac{1}{\pi}\sum_{j=1}^{k}\frac{1}{2j-1}\right)h_k=\left(\frac{\ln(4R)}{2\pi}-\frac{1}{\pi}\sum_{j=1}^{k}\frac{1}{2j-1}\right)h_k.$$

Analogously we get

$$B_1^{\mathcal{T}}\tilde{h}_k=\left(\frac{\ln(4R)}{2\pi}-\frac{1}{\pi}\sum_{j=1}^{k}\frac{1}{2j-1}\right)\tilde{h}_k$$

for the function $\tilde{h}_k$ defined by $\tilde{h}_k(x):=\cos(kt/R)$ with $t:=\tau^{-1}(x)$ and $k\in\mathbb{N}$. Therefore

$$\sigma_p\big(B_1^{\mathcal{T}}\big)\supseteq\left\{\frac{\ln(4R)}{2\pi}-\frac{1}{\pi}\sum_{j=1}^{k}\frac{1}{2j-1},k\in\mathbb{N}_0\right\}. \tag{4.40}$$

As the span of the functions $h_k,\tilde{h}_k$ with $k\in\mathbb{N}$ and the constant function $h\equiv 1$ is already dense in $L^2(\mathcal{T})$ and $B_1^{\mathcal{T}}$ is symmetric there are no other eigenfunctions and hence no other eigenvalues, i.e. in (4.40) equality holds. Note that the eigenvalue $\frac{\ln(4R)}{2\pi}$ has multiplicity one, while all other eigenvalues have multiplicity two.

Due to

$$(B_1^{\mathcal{T}} \pm i)h_k = \left(\frac{\ln(4R)}{2\pi} - \frac{1}{\pi} \sum_{j=1}^{k} \frac{1}{2j-1} \pm i \right) h_k \qquad \text{and}$$
$$(B_1^{\mathcal{T}} \pm i)\tilde{h}_k = \left(\frac{\ln(4R)}{2\pi} - \frac{1}{\pi} \sum_{j=1}^{k} \frac{1}{2j-1} \pm i \right) \tilde{h}_k$$

we observe that $\operatorname{ran}(B_1^{\mathcal{T}} \pm i)$ is dense in $L^2(\mathcal{T})$, hence $B_{\mathcal{T}}$ is essentially selfadjoint, cf. [68, Theorem 5.21]. □

Next we want to extend the results from Lemma 4.21 to all $\lambda \leq 1$ and to a general closed C^2-curve Σ of length $L = 2\pi R$, which is parametrized by its arc length parametrization $\sigma : [0,L] \to \mathbb{R}^3$. This is done by a perturbation of the operator $B_1^{\mathcal{T}}$. As a preparation we show the following lemma.

Lemma 4.22. *Let $\lambda \leq 1$. The operator $D_\lambda : L^2(\Sigma) \to L^2(\Sigma)$ defined by*

$$\big(D_\lambda h\big)(\sigma(t)) = \int_0^L h(\sigma(s)) \left[\frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|} - \frac{e^{-\sqrt{-(\lambda-1)}|\tau(t)-\tau(s)|}}{4\pi|\tau(t)-\tau(s)|} \right] ds \tag{4.41}$$

is compact and selfadjoint. Moreover there exists a λ-independent constant $C > 0$ such that $\|D_\lambda\| \leq C$ for all $\lambda \leq 1$.

Proof. In the following we will identify the parametrizations σ and τ of Σ and $\mathcal{T}$, respectively, with their L-periodic continuations on $\mathbb{R}$. Let $s,t \in \mathbb{R}$ with $|s-t| \leq \frac{L}{2}$. Define $f : (0,\infty) \to \mathbb{R}$ via $f(z) = \frac{e^{-\sqrt{-(\lambda-1)}z}}{4\pi z}$ for $z > 0$. Then

$$|f'(z)| = \frac{1}{4\pi} \left| \frac{-\sqrt{-(\lambda-1)}e^{-\sqrt{-(\lambda-1)}z}z - e^{-\sqrt{-(\lambda-1)}z}}{z^2} \right| = \frac{e^{-\sqrt{-(\lambda-1)}z}}{4\pi} \left[\frac{\sqrt{-(\lambda-1)}}{z} + \frac{1}{z^2} \right]. \tag{4.42}$$

Note that the functions $z \mapsto e^{-\sqrt{-(\lambda-1)}z}$, $z \mapsto \frac{1}{z}$ and $z \mapsto \frac{1}{z^2}$ are all monotonously nonincreasing on $(0,\infty)$, therefore the same is true for $|f'|$. Hence it follows

$$\left| \frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|} - \frac{e^{-\sqrt{-\lambda}|\tau(t)-\tau(s)|}}{4\pi|\tau(t)-\tau(s)|} \right| \leq |f'(\zeta_{\min})| \cdot \big| |\sigma(t)-\sigma(s)| - |\tau(t)-\tau(s)| \big| \tag{4.43}$$

with $\zeta_{\min} := \min\big\{ |\sigma(t)-\sigma(s)|, |\tau(t)-\tau(s)| \big\}$.

Note, that there exist $\varepsilon_\sigma > 0$ and $\varepsilon_\tau > 0$ such that for all $s,t \in \mathbb{R}$ with $|s-t| \leq \frac{L}{2}$

$$|\sigma(s)-\sigma(t)| \geq \varepsilon_\sigma |s-t| \quad \text{and} \quad |\tau(s)-\tau(t)| \geq \varepsilon_\tau |s-t|$$

holds. With $\varepsilon := \min\{\varepsilon_\sigma, \varepsilon_\tau\} > 0$ the estimate (4.43) can be simplified to

$$\left| \frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|} - \frac{e^{-\sqrt{-\lambda}|\tau(t)-\tau(s)|}}{4\pi|\tau(t)-\tau(s)|} \right| \leq |f'(\varepsilon|s-t|)| \big| |\sigma(t)-\sigma(s)| - |\tau(t)-\tau(s)| \big|. \tag{4.44}$$

As Σ is a C^2-curve we can apply Taylor's theorem to each component and get for some suitable ζ_1, ζ_2 and ζ_3

$$\sigma(t) = \begin{bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{bmatrix} = \sigma(s) + \sigma'(s)(t-s) + \begin{bmatrix} \sigma_1''(\zeta_1) \\ \sigma_2''(\zeta_2) \\ \sigma_3''(\zeta_3) \end{bmatrix} \frac{(t-s)^2}{2}.$$

With $C_\sigma := \sqrt{\|\sigma_1''\|_\infty^2 + \|\sigma_2''\|_\infty^2 + \|\sigma_3''\|_\infty^2}$ and $|\sigma'(s)| = 1$ it follows

$$|\sigma(t)-\sigma(s)| \leq |\sigma'(s)| \cdot |t-s| + \left| \begin{bmatrix} \sigma_1''(\zeta_1) \\ \sigma_2''(\zeta_2) \\ \sigma_3''(\zeta_3) \end{bmatrix} \right| \frac{(t-s)^2}{2} \leq |t-s| + \frac{C_\sigma}{2}|t-s|^2.$$

Analogously we get with $C_\tau := \sqrt{\|\tau_1''\|_\infty^2 + \|\tau_2''\|_\infty^2 + \|\tau_3''\|_\infty^2}$

$$|\tau(t)-\tau(s)| \geq |\tau'(s)| \cdot |t-s| - \left| \begin{bmatrix} \tau_1''(\xi_1) \\ \tau_2''(\xi_2) \\ \tau_3''(\xi_3) \end{bmatrix} \right| \frac{(t-s)^2}{2} \geq |t-s| - \frac{C_\tau}{2}|t-s|^2$$

for some suitable ξ_1, ξ_2 and ξ_3. Hence

$$|\sigma(t)-\sigma(s)| - |\tau(t)-\tau(s)| \leq \frac{C_\sigma + C_\tau}{2}|t-s|^2.$$

By changing the roles of σ and τ we observe

$$\big| |\sigma(t)-\sigma(s)| - |\tau(t)-\tau(s)| \big| \leq \frac{C_\sigma + C_\tau}{2}|t-s|^2. \tag{4.45}$$

Note that $e^{-x}(x+1) \leq 1$ for $x \geq 0$. Together with (4.42), (4.45) and $\widetilde{C} := \frac{C_\sigma + C_\tau}{8\pi\varepsilon^2}$ the estimate (4.44) implies

$$\begin{aligned} \left| \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|} - \frac{e^{-\sqrt{-(\lambda-1)}|\tau(t)-\tau(s)|}}{4\pi|\tau(t)-\tau(s)|} \right| &\leq \widetilde{C} e^{-\sqrt{-(\lambda-1)}\varepsilon|s-t|} \left[\sqrt{-(\lambda-1)}\varepsilon|s-t| + 1 \right] \\ &\leq \widetilde{C} \end{aligned} \tag{4.46}$$

for all $s,t \in \mathbb{R}$ with $|s-t| \le \frac{L}{2}$. For arbitrary $s,t \in \mathbb{R}$ there exists $k \in \mathbb{Z}$ such that $|(s+kL)-t| \le \frac{L}{2}$. As σ and τ are L-periodic it follows that (4.46) holds for all $s,t \in \mathbb{R}$. From (4.46) we conclude that the integral kernel given in the definition of D_λ in (4.41) is bounded and hence square-integrable on $[0,L]^2$. Therefore D_λ is a compact operator, cf. [56, Theorem VI.23]. Since the integral kernel of D_λ is real and symmetric it follows that D_λ is selfadjoint. Moreover we observe with the λ-independent constant $C := \widetilde{C}L$, the definition of D_λ in (4.41) and estimate (4.46)

$$\begin{aligned}\|D_\lambda h\|^2_{L^2(\Sigma)} &\le \|h\|^2_{L^2(\Sigma)} \int_0^L \int_0^L \left| \frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|} - \frac{e^{-\sqrt{-(\lambda-1)}|\tau(t)-\tau(s)|}}{4\pi|\tau(t)-\tau(s)|} \right|^2 ds\, dt \\ &\le C^2 \|h\|^2_{L^2(\Sigma)}\end{aligned}$$

for all $h \in L^2(\Sigma)$. □

Remark 4.23. The Weyl function M is just defined on $\mathbb{C} \setminus [1,\infty[$. However, in the present case the representation of M in (4.2) in Lemma 4.4 can be extended to the case $\lambda = 1$:

$$\big(M(1)h\big)(x) := \int_\Sigma h(y)\Big(G_0(|x-y|) - G_{-1}(|x-y|)\Big)\, d\sigma(y), \qquad h \in L^2(\Sigma), x \in \Sigma.$$

The operator $M(1)$ which we get in this way is again a selfadjoint compact operator in $L^2(\Sigma)$. Indeed, its integral kernel satisfies

$$0 \le G_0(|x-y|) - G_{-1}(|x-y|) = \frac{1-e^{-|x-y|}}{4\pi|x-y|} \le \frac{1}{4\pi},$$

and Σ is compact, which implies selfadjointness and compactness of $M(1)$, cf. [56, Theorem VI.23]. Furthermore, a direct computation shows that now Lemma 4.14 is also true for the case $\lambda = 1$.

Lemma 4.24. *Let $\lambda \le 1$. The operator B_λ is essentially selfadjoint in $L^2(\Sigma)$. Its closure $\overline{B_\lambda}$ is semibounded from above, has a compact resolvent, and its eigenvalues (ordered nonincreasingly and counted with multiplicity) satisfy*

$$\nu_k(\lambda) = -\frac{\ln k}{2\pi} + O(1) \quad \textit{as} \quad k \to \infty.$$

Moreover for every $k \in \mathbb{N}$ the function $\lambda \mapsto \nu_k(\lambda)$ is continuous and strictly increasing on the interval $(-\infty,1]$ and $\nu_k(\lambda) \to -\infty$ as $\lambda \to -\infty$.

Proof. Note that B_λ can be written as

$$B_\lambda = D_\lambda + J^* B_\lambda^{\mathcal{T}} J,$$

where $J : L^2(\Sigma) \to L^2(\mathcal{T})$ is the unitary operator defined by $Jh = h \circ \sigma \circ \tau^{-1}$ for $h \in L^2(\Sigma)$ and the operator $D_\lambda : L^2(\Sigma) \to L^2(\Sigma)$ is given by (4.41). Recall that $B_\lambda^{\mathcal{T}} = B_0^{\mathcal{T}} + M^{\mathcal{T}}(\lambda)$, cf. Lemma 4.14

and Remark 4.23, and that $M^{\mathcal{T}}(\lambda)$ and D_λ are compact and selfadjoint, cf. Lemma 4.4 (together with Remark 2.24) and Lemma 4.22. Hence we get

$$\begin{aligned}\overline{B_\lambda} &= \overline{D_\lambda + J^*\big(B_0^{\mathcal{T}} + M^{\mathcal{T}}(\lambda)\big)J} = D_\lambda + J^*\big(\overline{B_0^{\mathcal{T}}} + M^{\mathcal{T}}(\lambda)\big)J \\ &= D_\lambda + J^*\big(B_0^{\mathcal{T}} + M^{\mathcal{T}}(\lambda)\big)^*J = D_\lambda^* + \Big(J^*\big(B_0^{\mathcal{T}} + M^{\mathcal{T}}(\lambda)\big)J\Big)^* = B_\lambda^*,\end{aligned}$$

i.e. B_λ is essentially selfadjoint. As $\overline{B_\lambda^{\mathcal{T}}} = \overline{B_1^{\mathcal{T}}} + M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)$ we have for $u \in \operatorname{dom} \overline{B_\lambda}$

$$\begin{aligned}\langle \overline{B_\lambda} u, u\rangle_{L^2(\Sigma)} &= \langle D_\lambda u, u\rangle_{L^2(\Sigma)} + \langle J^*\big(\overline{B_1^{\mathcal{T}}} + M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)\big)Ju, u\rangle_{L^2(\Sigma)} \\ &= \langle D_\lambda u, u\rangle_{L^2(\Sigma)} + \langle \overline{B_1^{\mathcal{T}}} Ju, Ju\rangle_{L^2(\mathcal{T})} + \langle \big(M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)\big)Ju, Ju\rangle_{L^2(\mathcal{T})} \\ &\le \|D_\lambda\| \cdot \|u\|^2_{L^2(\Sigma)} + k_1^{\mathcal{T}} \|Ju\|^2_{L^2(\mathcal{T})} + \|M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)\| \cdot \|Ju\|^2_{L^2(\mathcal{T})} \\ &\le \big(C + k_1^{\mathcal{T}} + \|M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)\|\big)\|u\|^2_{L^2(\Sigma)},\end{aligned}$$

with the constants C from Lemma 4.22 and $k_1^{\mathcal{T}} = \frac{\ln(4R)}{2\pi}$, cf. Lemma 4.20. Hence $\overline{B_\lambda}$ is bounded from above. Moreover we have with Lemma 2.2

$$\begin{aligned}\nu_j(\lambda) :&= \max_{\substack{U \subseteq \operatorname{dom} \overline{B_\lambda} \\ \dim U = j}} \min_{u \in U\setminus\{0\}} \frac{\langle \overline{B_\lambda} u, u\rangle_{L^2(\Sigma)}}{\|u\|_{L^2(\Sigma)}} \\ &= \max_{\substack{U \subseteq \operatorname{dom} \overline{B_\lambda} \\ \dim U = j}} \min_{u \in U\setminus\{0\}} \frac{\langle D_\lambda u, u\rangle_{L^2(\Sigma)} + \langle \overline{B_1^{\mathcal{T}}} Ju, Ju\rangle_{L^2(\mathcal{T})} + \langle \big(M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)\big)Ju, Ju\rangle_{L^2(\mathcal{T})}}{\|u\|_{L^2(\Sigma)}} \\ &\le \max_{\substack{U \subseteq \operatorname{dom} \overline{B_\lambda} \\ \dim U = j}} \min_{u \in U\setminus\{0\}} \left\{ \frac{\langle \overline{B_1^{\mathcal{T}}} Ju, Ju\rangle_{L^2(\mathcal{T})}}{\|Ju\|_{L^2(\mathcal{T})}} + C + ||M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)|| \right\} \\ &= \max_{\substack{V \subseteq \operatorname{dom} \overline{B_1^{\mathcal{T}}} \\ \dim V = j}} \min_{v \in V\setminus\{0\}} \left\{ \frac{\langle \overline{B_1^{\mathcal{T}}} v, v\rangle_{L^2(\mathcal{T})}}{\|v\|_{L^2(\mathcal{T})}} \right\} + C + ||M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)|| \\ &= \nu_j^{\mathcal{T}}(1) + C + ||M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)||.\end{aligned}$$

Analogously we get $\nu_j(\lambda) \ge \nu_j^{\mathcal{T}}(1) - C - ||M^{\mathcal{T}}(\lambda) - M^{\mathcal{T}}(1)||$ which implies

$$\nu_j(\lambda) = \nu_j^{\mathcal{T}}(1) + O(1) \quad \text{as} \quad j \to \infty. \tag{4.47}$$

Recall that $\sum_{j=1}^k \frac{1}{j} = \ln k + O(1)$, see e.g. [1, Equation 4.1.32]. Hence

$$\sum_{j=1}^k \frac{1}{2j-1} = \sum_{j=1}^{2k} \frac{1}{j} - \frac{1}{2}\sum_{j=1}^k \frac{1}{j} = \ln(2k) - \frac{\ln(k)}{2} + O(1) = \frac{\ln(2k)}{2} + O(1) \quad \text{as} \quad k \to +\infty. \tag{4.48}$$

With equation (4.47), (4.48) and Lemma 4.21 we get

$$\nu_{2k}(\lambda) = \nu_{2k}^{\mathcal{T}}(1) + O(1) = \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{k}\frac{1}{2j-1} + O(1) = -\frac{\ln(2k)}{2\pi} + O(1) \quad \text{as} \quad k \to \infty.$$

Moreover we get

$$\nu_{2k+1}(\lambda) = \nu_{2k}(\lambda) = -\frac{\ln(2k)}{2\pi} + O(1) = -\frac{\ln(2k+1)}{2\pi} + O(1) \quad \text{as} \quad k \to \infty.$$

It remains to show that the eigenvalue functions $\lambda \mapsto \nu_k(\lambda)$ are continuous and strictly increasing for each $k \in \mathbb{N}$. For this let $\lambda, \mu < 1$ and define the operator $D_{\lambda,\mu} : L^2(\Sigma) \to L^2(\Sigma)$ by

$$(D_{\lambda,\mu}h)(x) = \int_{\Sigma} h(y)\frac{e^{-\sqrt{-(\lambda-1)}|x-y|} - e^{-\sqrt{-(\mu-1)}|x-y|}}{4\pi|x-y|}\, d\sigma(y).$$

As $|e^{-\alpha} - e^{-\beta}| \leq |\alpha - \beta|$ for all $\alpha, \beta \geq 0$ we for the integral kernel of $D_{\lambda,\mu}$ the estimate

$$\left|\frac{e^{-\sqrt{-(\lambda-1)}|x-y|} - e^{-\sqrt{-(\mu-1)}|x-y|}}{4\pi|x-y|}\right| \leq \frac{|\sqrt{-(\lambda-1)} - \sqrt{-(\mu-1)}|}{4\pi}. \tag{4.49}$$

Hence $D_{\lambda,\mu}$ is a compact operator and its norm can be estimated by $\frac{|\sqrt{-(\lambda-1)}-\sqrt{-(\mu-1)}|}{4\pi}L$, cf. [56, Theorem VI.23]. Since the integral kernel of $D_{\lambda,\mu}$ is real and symmetric it follows that $D_{\lambda,\mu}$ is selfadjoint.

It follows from the definition of $D_{\lambda,\mu}$ and the definition of B_λ and B_μ, that $B_\lambda h - B_\mu h = D_{\lambda,\mu}h$ holds for all $h \in C^{0,1}(\Sigma)$ and hence that

$$\overline{B_\lambda}h = \overline{B_\mu}h + D_{\lambda,\mu}h$$

holds for all $h \in \operatorname{dom}\overline{B_\lambda}$. With Lemma 2.2 we get

$$\begin{aligned}\nu_k(\lambda) &= \max_{\substack{U\subseteq \operatorname{dom}\overline{B_\lambda}\\ \dim U = k}}\ \min_{u\in U\setminus\{0\}} \frac{\langle \overline{B_\lambda}u,u\rangle_{L^2(\Sigma)}}{\|u\|_{L^2(\Sigma)}} = \max_{\substack{U\subseteq \operatorname{dom}\overline{B_\mu}\\ \dim U = k}}\ \min_{u\in U\setminus\{0\}} \frac{\langle \overline{B_\mu}u,u\rangle_{L^2(\Sigma)} + \langle D_{\lambda,\mu}u,u\rangle_{L^2(\Sigma)}}{\|u\|_{L^2(\Sigma)}}\\ &\leq \max_{\substack{U\subseteq \operatorname{dom}\overline{B_\mu}\\ \dim U = k}}\ \min_{u\in U\setminus\{0\}} \left\{\frac{\langle \overline{B_\mu}u,u\rangle_{L^2(\Sigma)}}{\|u\|_{L^2(\Sigma)}} + ||D_{\lambda,\mu}||\right\} = \nu_k(\mu) + ||D_{\lambda,\mu}||\end{aligned}$$

and analogously $\nu_k(\lambda) \geq \nu_k(\mu) - ||D_{\lambda,\mu}||$. Hence

$$|\nu_k(\lambda) - \nu_k(\mu)| \leq ||D_{\lambda,\mu}|| \leq \frac{|\sqrt{-(\lambda-1)} - \sqrt{-(\mu-1)}|}{4\pi}L \xrightarrow{\lambda\to\mu} 0,$$

i.e. $\lambda \mapsto \nu_k(\lambda)$ is continuous.

According to Lemma 4.14 and Lemma 2.7 we have

$$(\overline{B_\lambda} - \overline{B_\mu})h = (M(\lambda) - M(\mu))h = (\lambda - \mu)\gamma(\lambda)^*\gamma(\mu)h$$

for all $h \in \operatorname{dom}\overline{B_\lambda} = \operatorname{dom}\overline{B_\mu}$. Hence we get

$$\begin{aligned}\frac{d}{d\lambda}\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)} &= \lim_{\mu\to\lambda} \frac{\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)} - \langle \overline{B_\mu} h, h\rangle_{L^2(\Sigma)}}{\lambda - \mu} \\ &= \lim_{\mu\to\lambda} \langle \gamma(\lambda)^*\gamma(\mu)h, h\rangle_{L^2(\Sigma)} \\ &= \lim_{\mu\to\lambda} \langle \gamma(\mu)h, \gamma(\lambda)h\rangle_{L^2(\Sigma)} = \|\gamma(\lambda)h\|^2_{L^2(\Sigma)} > 0\end{aligned}$$

for all $h \in \operatorname{dom}\overline{B_\lambda} \setminus \{0\}$, i.e. the function $\lambda \mapsto \langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)}$ is strictly increasing on $(-\infty, 1]$. Furthermore, for every finite-dimensional subspace $U \subset \operatorname{dom}\overline{B_\mu}$ there exists $h_U \in U \setminus \{0\}$ such that

$$\frac{\langle \overline{B_\mu} h_U, h_U\rangle_{L^2(\Sigma)}}{\|h_U\|^2_{L^2(\Sigma)}} = \min_{h\in U\setminus\{0\}} \frac{\langle \overline{B_\mu} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}}.$$

As the function $\lambda \mapsto \langle \overline{B_\lambda} h_U, h_U\rangle_{L^2(\Sigma)}$ is strictly increasing we get for $\lambda < \mu \leq 1$

$$\min_{h\in U\setminus\{0\}} \frac{\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}} \leq \frac{\langle \overline{B_\lambda} h_U, h_U\rangle_{L^2(\Sigma)}}{\|h_U\|^2_{L^2(\Sigma)}} < \frac{\langle \overline{B_\mu} h_U, h_U\rangle_{L^2(\Sigma)}}{\|h_U\|^2_{L^2(\Sigma)}} = \min_{h\in U\setminus\{0\}} \frac{\langle \overline{B_\mu} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}}.$$

Denoting by U_k the k-dimensional subspace of $\operatorname{dom}\overline{B_\lambda}$ such that

$$\max_{\substack{U\subseteq \operatorname{dom}\overline{B_\lambda} \\ \dim U = k}} \min_{h\in U\setminus\{0\}} \frac{\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}} = \min_{h\in U_k\setminus\{0\}} \frac{\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}}$$

we get with the inequality above

$$\begin{aligned}\max_{\substack{U\subseteq \operatorname{dom}\overline{B_\lambda} \\ \dim U = k}} \min_{h\in U\setminus\{0\}} \frac{\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}} &= \min_{h\in U_k\setminus\{0\}} \frac{\langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}} \\ &< \min_{h\in U_k\setminus\{0\}} \frac{\langle \overline{B_\mu} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}} \leq \max_{\substack{U\subseteq \operatorname{dom}\overline{B_\mu} \\ \dim U = k}} \min_{h\in U\setminus\{0\}} \frac{\langle \overline{B_\mu} h, h\rangle_{L^2(\Sigma)}}{\|h\|^2_{L^2(\Sigma)}}.\end{aligned}$$

Thus Lemma 2.2 implies $\nu_k(\lambda) < \nu_k(\mu)$ for $\lambda < \mu \leq 1$. □

Now we are in the situation to improve the results about the spectrum of $-\Delta_{\Sigma,\alpha}$.

Theorem 4.25. *Let $\lambda \in \rho(-\Delta_{\Sigma,\alpha}) \cap \rho(-\Delta_{free})$ and $s_1(\lambda) \geq s_2(\lambda) \geq \dots$ be the singular values of the resolvent difference*

$$(-\Delta_{\Sigma,\alpha} - \lambda)^{-1} - (-\Delta_{free} - \lambda)^{-1}, \tag{4.50}$$

counted with multiplicities. Then

$$s_k(\lambda) = O\Big(\frac{1}{k^2 \ln k}\Big) \quad as \quad k \to \infty.$$

Proof. (In order to keep the notation simple we will choose $\lambda \in [\rho(-\Delta_{\Sigma,\alpha}) \cap \rho(-\Delta_{\text{free}})] + 1$ instead of $\lambda \in \rho(-\Delta_{\Sigma,\alpha}) \cap \rho(-\Delta_{\text{free}})$.) As in the proof of Theorem 4.18 we observe that the resolvent difference in (4.50) can be written as

$$(-\Delta_{\Sigma,\alpha} - \lambda + 1)^{-1} - (-\Delta_{\text{free}} - \lambda + 1)^{-1} = \gamma(\lambda)\big(\alpha - \overline{B_\lambda}\big)^{-1}\gamma(\lambda)^*.$$

With Lemma 4.4 and Remark 2.24) we get $\gamma(\lambda) \in \mathfrak{S}_p\big(L^2(\mathbb{R}^3), L^2(\Sigma)\big)$ for $p > 1$ and $s_j\big(\gamma(\lambda)\big) = O(1/j)$ for $j \to \infty$. Hence also $s_j\big(\gamma(\lambda)^*\big) = O(1/j)$ as $j \to \infty$, i.e. there exists a constant $C > 0$ such that

$$s_j\big(\gamma(\lambda)\big) \leq \frac{C}{j} \qquad \text{and} \qquad s_j\big(\gamma(\lambda)^*\big) \leq \frac{C}{j}.$$

Moreover it follows from Lemma 4.24 that the singular values (which coincide with the eigenvalues) of the selfadjoint operator $(\alpha - \overline{B_\lambda})^{-1}$ satisfy

$$s_j\big((\alpha - \overline{B_\lambda})^{-1}\big) \leq \frac{\tilde{C}}{\ln j}$$

for some suitable $\tilde{C}$. Without loss of generality we assume in the following $\tilde{C} = C$. With [37, Corollary 2.2, Chapter II] and $\ln j = \frac{1}{3}\ln(j^3) \geq \frac{1}{3}\ln(3j)$ for $j \geq 2$ we get

$$\begin{aligned} s_{3j-2}\left(\gamma(\lambda)(\alpha - \overline{B_\lambda})^{-1}\gamma(\lambda)^*\right) &\leq s_{2j-1}\left(\gamma(\lambda)(\alpha - \overline{B_\lambda})^{-1}\right) s_j\big(\gamma(\lambda)^*\big) \\ &\leq s_j\big(\gamma(\lambda)\big) s_j\left((\alpha - \overline{B_\lambda})^{-1}\right) s_j\big(\gamma(\lambda)^*\big) \leq \frac{C^3}{j^2 \ln j} \leq \frac{27C^3}{(3j)^2 \ln(3j)} \end{aligned}$$

for $j \geq 2$. Due to

$$s_{3j}\left(\gamma_{\lambda_0}(\alpha - \overline{B_\lambda})^{-1}\gamma_{\lambda_0}^*\right) \leq s_{3j-1}\left(\gamma_{\lambda_0}(\alpha - \overline{B_\lambda})^{-1}\gamma_{\lambda_0}^*\right) \leq s_{3j-2}\left(\gamma_{\lambda_0}(\alpha - \overline{B_\lambda})^{-1}\gamma_{\lambda_0}^*\right)$$

and

$$\frac{27C^3}{(3j)^2 \ln(3j)} \leq \frac{27C^3}{(3j-1)^2 \ln(3j-1)} \leq \frac{27C^3}{(3j-2)^2 \ln(3j-2)}$$

this implies

$$s_k\left(\gamma_{\lambda_0}(\alpha - \overline{B_\lambda})^{-1}\gamma_{\lambda_0}^*\right) \leq \frac{27C^3}{k^2 \ln k}$$

for all $k \in \mathbb{N}$, $k \geq 4$. □

A consequence of the compactness of the resolvent difference (4.50) is that the only possible accumulation point of the negative eigenvalues is 0. Moreover we know that $-\Delta_{\Sigma,\alpha}$ is semibounded from below, cf. Theorem 4.18. The following Theorem shows that there are even only finitely many negative eigenvalues and gives and estimate for its number. For this define

$$d_\Sigma := \left(\int_0^L \int_0^L \left| \frac{1}{4\pi|\sigma(t)-\sigma(s)|} - \frac{1}{4\pi|\tau(t)-\tau(s)|} \right|^2 dt\, ds \right)^{\frac{1}{2}}.$$

Note that $\|D_1\| \le d_\Sigma$ for the operator D_1 defined in equation (4.41) of Lemma 4.22. Furthermore define for $r \in \mathbb{N}_0$ the disjoint intervals

$$I_r := \left[\frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{r+1}\frac{1}{2j-1}, \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{r}\frac{1}{2j-1} \right) \quad \text{and} \quad I_{-1} := \left[\frac{\ln(4R)}{2\pi}, +\infty \right)$$

such that $\mathbb{R} = \bigcup_{r=-1}^{\infty} I_r$.

Theorem 4.26. *Let $\alpha \neq 0$ and $r,l \in \mathbb{N}_0 \cup \{-1\}$ such that $\alpha + d_\Sigma \in I_r$ and $\alpha - d_\Sigma \in I_l$. Denote by N_α the number of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$, counted with multiplicities. Then*

$$2r+1 \le N_\alpha \le \max\{2l+1, 0\}.$$

In particular $N_\alpha = 0$ if $\alpha - d_\Sigma \ge \frac{\ln(4R)}{2\pi}$.

Proof. Denote by $\nu_k(\lambda)$ the k-th eigenvalue (ordered nonincreasingly and counted with multiplicity) of $\overline{B_\lambda}$. Let $N \in \mathbb{N}_0$ be the number of eigenvalues of $\overline{B_1}$ (counted with multiplicity) which are larger than α (note that $N = \infty$ is not possible because the eigenvalues accumulate to $-\infty$, cf. Lemma 4.24):

$$\nu_1(1) \ge \nu_2(1) \ge \ldots \ge \nu_N(1) > \alpha \ge \nu_{N+1}(1) \ge \ldots.$$

Recall that the eigenvalues $\nu_k(1)$ of $\overline{B_1} = D_1 + J^* \overline{B_1^{\mathcal{T}}} J$ can be estimated by

$$\nu_k^{\mathcal{T}}(1) - \|D_1\| \le \nu_k(1) \le \nu_k^{\mathcal{T}}(1) + \|D_1\|, \qquad k \in \mathbb{N},$$

with the operator D_1 as defined in (4.41). By [56, Theorem VI.23] the norm of D_1 can be estimated by the L^2-norm of its integral kernel, i.e. $\|D_1\| \le d_\Sigma$. Hence, if $\alpha + d_\Sigma \in I_r$ we have

$$\alpha + d_\Sigma < \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{r}\frac{1}{2j-1} = \nu_{2r+1}^{\mathcal{T}}(1)$$

and therefore $\alpha < \nu_{2r+1}^{\mathcal{T}}(1) - d_\Sigma \le \nu_{2r+1}(1)$. This means that $\overline{B_1}$ has at least $2r+1$ eigenvalues larger than α, i.e. $2r+1 \le N$.

If $\alpha - d_\Sigma \in I_{-1}$ or, equivalently, $\alpha - d_\Sigma \geq \frac{\ln(4R)}{2\pi} = \nu_1^{\mathcal{T}}(1)$ we have $\nu_1(1) \leq \nu_1^{\mathcal{T}}(1) + d_\Sigma \leq \alpha$. This means that $\overline{B_1}$ has no eigenvalues larger than α, i.e. $N = 0$.

If $\alpha - d_\Sigma \in I_l$ for some $l \in \mathbb{N}_0$ we have

$$\alpha - d_\Sigma \geq \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{l+1} \frac{1}{2j-1} = \nu_{2l+2}^{\mathcal{T}}(1)$$

and therefore $\alpha \geq \nu_{2l+2}^{\mathcal{T}}(1) + d_\Sigma \geq \nu_{2l+2}(1)$. This means that $\overline{B_1}$ has at most $2l+1$ eigenvalues larger than α, i.e. $N \leq 2l+1$. So far we have shown

$$2r+1 \leq N \leq \max\{2l+1, 0\}$$

and it remains to show $N_\alpha = N$, i.e. the number of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$ coincides with the number of eigenvalues of $\overline{B_1}$ larger than α.

As seen in Lemma 4.24 the functions $\lambda \mapsto \nu_k(\lambda)$ are continuous and strictly increasing on $(-\infty, 1]$ and $\nu_k(\lambda) \to -\infty$ as $\lambda \to -\infty$. Hence for each $k \in \{1, \dots, N\}$ there exists $\lambda_k < 1$ such that $\nu_k(\lambda_k) = \alpha$ and $\nu_j(\lambda) < \alpha$ for all $j > N$ and all $\lambda < 1$.

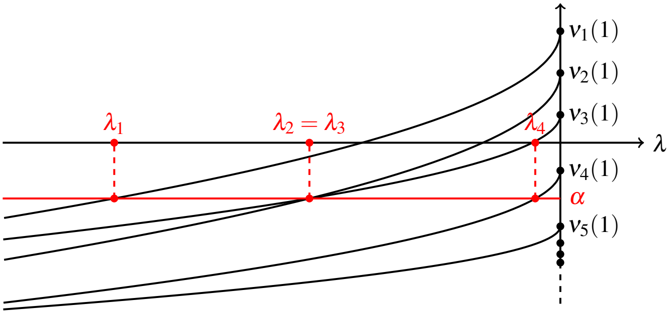
As $\gamma(\lambda)$ is for every $\lambda < 1$ an isomorphism between $\ker(\alpha - \overline{B_\lambda})$ and $\ker(A_\Theta - \lambda)$ we get

$$\begin{aligned}\dim\ker\big(-\Delta_{\Sigma,\alpha} - (\lambda_k - 1)\big) &= \dim\ker(A_\Theta - \lambda_k)\\ &= \dim\ker(\alpha - \overline{B_{\lambda_k}}) = \#\big\{j \in \{1, \dots, N\} : \nu_j(\lambda_k) = \alpha\big\}\end{aligned}$$

for each $k \in \{1, \dots, N\}$. In particular $\lambda_k - 1$ is a negative eigenvalue of $-\Delta_{\Sigma,\alpha}$. Moreover all negative eigenvalues of $-\Delta_{\Sigma,\alpha}$ are of the form $\lambda_j - 1$ for some $j \in \{1, \dots, N\}$.

If $\lambda_k - 1$ has multiplicity s (and $\lambda_k > \lambda_{k-1}$) then $\nu_k(\lambda_k) = \ldots = \nu_{k+s-1}(\lambda_k) = \alpha$. Hence the number of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$ counted with multiplicity coincides with the number of eigenvalues of $\overline{B_1}$ larger than α: $N_\alpha = N$. This completes the proof. □

The following picture illustrates the proof of Theorem 4.26. Each intersection of an eigenvalue function ν_j with the constant line α indicates a negative eigenvalue λ_j of $-\Delta_{\Sigma,\alpha} + 1$.



In the next corollary we give a more explicit estimate for the number of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$.

Corollary 4.27. *Let $\alpha \neq 0$. Then the number N_α of negative eigenvalues of $-\Delta_{\Sigma,\alpha}$, counted with multiplicities, can be estimated by*

$$2Rc^{-1}e^{-2\pi\alpha-\gamma} - 1 - 2(e^{\frac{1}{23}} - 1) < N_\alpha < 2Rce^{-2\pi\alpha-\gamma} + 1, \tag{4.51}$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant and $c := e^{2\pi d_\Sigma}$. In particular, $N_\alpha = e^{-2\pi\alpha+O(1)}$ as $\alpha \to -\infty$.

Proof. As in Theorem 4.26 let $r,l \in \mathbb{N}_0 \cup \{-1\}$ such that $\alpha + d_\Sigma \in I_r$ and $\alpha - d_\Sigma \in I_l$. The proof is based on the following estimate for the harmonic sum, which can be found for example in equation (9.89) in [39]:

$$\ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2} < \sum_{j=1}^{k} \frac{1}{j} < \ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4}, \quad k \in \mathbb{N}. \tag{4.52}$$

Equation (4.52) and $\sum_{j=1}^{k} \frac{1}{2j-1} = \sum_{j=1}^{2k} \frac{1}{2j-1} - \frac{1}{2}\sum_{j=1}^{k} \frac{1}{2j-1}$ imply

$$\begin{aligned}\sum_{j=1}^{k} \frac{1}{2j-1} &> \ln(2k) + \gamma + \frac{1}{4k} - \frac{1}{48k^2} - \frac{1}{2}\Big(\ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4}\Big) \\ &= \frac{\ln k + \ln 4 + \gamma}{2} + \frac{1}{48k^2} - \frac{1}{240k^4} > \frac{\ln k + \ln 4 + \gamma}{2}.\end{aligned}$$

Hence if $l \in \mathbb{N}$ then $\alpha - d_\Sigma \in I_l$ implies with the estimate above

$$\alpha - d_\Sigma < \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{l} \frac{1}{2j-1} < \frac{\ln(4R)}{2\pi} - \frac{\ln l + \ln 4 + \gamma}{2\pi}$$

and therefore $\ln l < -2\pi(\alpha - d_\Sigma) + \ln R - \gamma$. This implies together with the estimate $N_\alpha \leq 2l+1$ from Theorem 4.26

$$N_\alpha \leq 2l + 1 = 2e^{\ln l} + 1 < 2e^{-2\pi(\alpha-d_\Sigma)+\ln R-\gamma} + 1 = 2Rce^{-2\pi\alpha-\gamma} + 1,$$

which is the upper estimate in (4.51). If $l = -1$ or $l = 0$ then $N_\alpha \leq 1$, cf. Theorem 4.26, and the upper estimate in (4.51) follows immediately from the fact, that the exponential function is positive.

For the lower estimate in (4.51) we deduce from (4.52) in the same way as above

$$\begin{aligned}\sum_{j=1}^{k} \frac{1}{2j-1} &< \ln(2k) + \gamma + \frac{1}{4k} - \frac{1}{48k^2} + \frac{1}{1920k^4} - \frac{1}{2}\Big(\ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2}\Big) \\ &= \frac{\ln k + \ln 4 + \gamma}{2} + \frac{1}{48k^2} + \frac{1}{1920k^4} < \frac{\ln k + \ln 4 + \gamma + \frac{1}{23k^2}}{2}.\end{aligned}$$

Hence if $r \in \mathbb{N}_0$ then $\alpha + d_\Sigma \in I_r$ implies with the estimate above

$$\alpha + d_\Sigma \geq \frac{\ln(4R)}{2\pi} - \frac{1}{\pi}\sum_{j=1}^{r+1}\frac{1}{2j-1} > \frac{\ln(4R)}{2\pi} - \frac{\ln(r+1) + \ln 4 + \gamma + \frac{1}{23(r+1)^2}}{2\pi}$$

and therefore

$$\ln(r+1) + \frac{1}{23(r+1)^2} > -2\pi(\alpha + d_\Sigma) + \ln R - \gamma. \tag{4.53}$$

Equation (4.53) implies together with the estimate $N_\alpha \geq 2r+1$ from Theorem 4.26

$$\begin{aligned} N_\alpha - 2Rc^{-1}e^{-2\pi\alpha-\gamma} &\geq 2r+1-2e^{-2\pi(\alpha+d_\Sigma)+\ln R-\gamma} \\ &> 2r+1-2e^{\ln(r+1)+\frac{1}{23(r+1)^2}} \\ &= 2(r+1)-2(r+1)e^{\frac{1}{23(r+1)^2}}-1 \\ &= 2(r+1)(1-e^{\frac{1}{23(r+1)^2}})-1 =: g(r). \end{aligned}$$

As $g'(r) > 0$ for all $r > 0$, the minimum of g for $r \geq 0$ is attained at $r = 0$. Hence

$$N_\alpha - 2Rc^{-1}e^{-2\pi\alpha-\gamma} > 2\big(1-e^{\frac{1}{23}}\big)-1,$$

which gives the lower estimate in (4.51) for $r \in \mathbb{N}_0$.

For $r = -1$ we have $\alpha + d_\Sigma \geq \frac{\ln(4R)}{2\pi}$ and hence $2\pi(\alpha + d_\Sigma) \geq \ln(4R)$. Therefore

$$\begin{aligned} 2Rc^{-1}e^{-2\pi\alpha-\gamma}-1-2(e^{\frac{1}{23}}-1) &= 2Re^{-2\pi(\alpha+d_\Sigma)-\gamma}-1-2(e^{\frac{1}{23}}-1) \\ &\leq 2Re^{-\ln(4R)-\gamma}-1-2(e^{\frac{1}{23}}-1) \\ &= \frac{2R}{4R}e^{-\gamma}-1-2(e^{\frac{1}{23}}-1) < 0. \end{aligned}$$

Hence the lower estimate in (4.51) is also true for the case $r = -1$. □

Motivated by [28, 30] we prove finally the following theorem.

Theorem 4.28. *Let $\mathcal{T}$ be a circle in $\mathbb{R}^3$ of radius $R = \frac{L}{2\pi}$ and assume that Σ is not a circle. Let $\alpha < \frac{\ln(4R)}{2\pi}$. Then*

$$\min\sigma(-\Delta_{\Sigma,\alpha}) < \min\sigma(-\Delta_{\mathcal{T},\alpha}),$$

where $-\Delta_{\mathcal{T},\alpha}$ denotes the Schrödinger operator with δ-interaction of strength $\frac{1}{\alpha}$ supported on the circle $\mathcal{T}$.

Proof. The proof follows the ideas of [28, 30] and is based on the strict inequality

$$\int_0^L |\sigma(s+u)-\sigma(s)|\,ds < \frac{L^2}{\pi}\sin\frac{\pi u}{L}, \quad u\in(0,L/2], \tag{4.54}$$

cf. Proposition 2.1 and Theorem 2.2 in [30]. Here σ is again the parametrization of the curve Σ and is identified with its L-periodic extension to $\mathbb{R}$.

At first we will show that (4.54) holds also for $u \in (\frac{L}{2}, L)$ For this let $u \in (\frac{L}{2}, L)$. With the substitution $t := s+u$ and the fact, that σ is L-periodic, is L-periodic we get

$$\begin{aligned}
\int_0^L |\sigma(s+u)-\sigma(s)|\,ds &= \int_u^{L+u} |\sigma(t)-\sigma(t-u)|\,dt \\
&= \int_u^L |\sigma(t)-\sigma(t-u)|\,dt + \int_L^{L+u} |\sigma(t)-\sigma(t-u)|\,dt \\
&= \int_u^L |\sigma(t)-\sigma(t-u)|\,dt + \int_0^u |\sigma(t)-\sigma(t-u)|\,dt \\
&= \int_0^L |\sigma(t)-\sigma(t-u)|\,dt = \int_0^L |\sigma(t+[L-u])-\sigma(t)|\,dt.
\end{aligned} \tag{4.55}$$

As $L-u \in (0, \frac{L}{2})$ we can use (4.54) to estimate the last integral in (4.55) by

$$\int_0^L |\sigma(t+[L-u])-\sigma(t)|\,dt \le \frac{L^2}{\pi}\sin\frac{\pi[L-u]}{L} = \frac{L^2}{\pi}\sin\left(\pi-\frac{\pi u}{L}\right) = \frac{L^2}{\pi}\sin\frac{\pi u}{L}. \tag{4.56}$$

Combining (4.55) and (4.56) we observe that (4.54) holds for all $u \in (0,L)$.

Next we define for $\lambda < 1$ the function $G_\lambda : (0,\infty) \to \mathbb{R}$ via

$$G_\lambda(x) := \frac{e^{-\sqrt{-(\lambda-1)}x}}{x}, \qquad x>0.$$

It is easy to see, that G_λ is strictly monotonically decreasing and convex. As (4.54) holds for all $u \in (0,L)$ we get with the fact that G_λ is decreasing the inequality

$$G_\lambda\left(\frac{1}{L}\int_0^L |\sigma(s+u)-\sigma(s)|\,ds\right) > G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi u}{L}\right). \tag{4.57}$$

Using Jensen's Inequality (see e.g. [57, Theorem 3.3]) the convexity of G_λ implies

$$G_\lambda\left(\frac{1}{L}\int_0^L |\sigma(s+u)-\sigma(s)|\,ds\right) \leq \frac{1}{L}\int_0^L G_\lambda(|\sigma(s+u)-\sigma(s)|)\,ds \tag{4.58}$$

Combining (4.57) and (4.58) we observe

$$\begin{aligned}\int_0^L\int_0^L G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi u}{L}\right)du\,ds &< L\int_0^L G_\lambda\left(\frac{1}{L}\int_0^L |\sigma(s+u)-\sigma(s)|\,ds\right)du\\ &\leq \int_0^L\int_0^L G_\lambda(|\sigma(s+u)-\sigma(s)|)\,ds\,du.\end{aligned} \tag{4.59}$$

With the substitution $t := s+u$ and the formula $\sin\alpha = \sin(\pi-\alpha)$ we get

$$\begin{aligned}&\int_0^L G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi u}{L}\right)du = \int_s^{L+s} G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi(t-s)}{L}\right)dt\\ &= \int_s^L G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi(t-s)}{L}\right)dt + \int_0^s G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi(t+L-s)}{L}\right)dt\\ &= \int_s^L G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi(t-s)}{L}\right)dt + \int_0^s G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi(s-t)}{L}\right)dt = \int_0^L G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi|t-s|}{L}\right)dt\end{aligned}$$

and with the same substitution and the L-periodicity of σ we get

$$\begin{aligned}&\int_0^L G_\lambda(|\sigma(s+u)-\sigma(s)|)du = \int_s^{L+s} G_\lambda(|\sigma(t)-\sigma(s)|)dt\\ &\quad= \int_s^L G_\lambda(|\sigma(t)-\sigma(s)|)dt + \int_0^s G_\lambda(|\sigma(t+L)-\sigma(s)|)dt\\ &\quad= \int_s^L G_\lambda(|\sigma(t)-\sigma(s)|)dt + \int_0^s G_\lambda(|\sigma(t)-\sigma(s)|)dt = \int_0^L G_\lambda(|\sigma(t)-\sigma(s)|)dt.\end{aligned}$$

With these two equalities we observe from (4.59)

$$0 < \int_0^L\int_0^L G_\lambda(|\sigma(t)-\sigma(s)|) - G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi|t-s|}{L}\right)dt\,ds. \tag{4.60}$$

Next we recall that the operator B_λ can be written as

$$B_\lambda = D_\lambda + J^* B_\lambda^{\mathcal{T}} J,$$

with the selfadjoint compact operator $D_\lambda : L^2(\Sigma) \to L^2(\Sigma)$ and the unitary operator $J : L^2(\Sigma) \to L^2(\mathcal{T})$ defined in the proof of Lemma 4.24. According to the definition of D_λ in (4.41), equation (4.38) and (4.60) we have

$$\begin{aligned}
\langle D_\lambda \mathbb{1}, \mathbb{1}\rangle_{L^2(\Sigma)} &= \int_0^L \int_0^L \left[\frac{e^{-\sqrt{-(\lambda-1)}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|} - \frac{e^{-\sqrt{-(\lambda-1)}|\tau(t)-\tau(s)|}}{4\pi|\tau(t)-\tau(s)|} \right] ds\, dt \\
&= \int_0^L \int_0^L G_\lambda(|\sigma(t)-\sigma(s)|) - G_\lambda(|\tau(t)-\tau(s)|)\, ds\, dt \\
&= \int_0^L \int_0^L G_\lambda(|\sigma(t)-\sigma(s)|) - G_\lambda\left(\frac{L}{\pi}\sin\frac{\pi|t-s|}{L}\right) ds\, dt > 0.
\end{aligned}$$

Hence we have with the constant function $h = \frac{1}{\sqrt{L}}$ on Σ (which implies $\|h\|_{L^2(\Sigma)} = 1$)

$$\nu_1(\lambda) \geq \langle \overline{B_\lambda} h, h\rangle_{L^2(\Sigma)} = \langle D_\lambda h, h\rangle_{L^2(\Sigma)} + \langle \overline{B_\lambda^{\mathcal{T}}} Jh, Jh\rangle_{L^2(\mathcal{T})} > \langle \overline{B_\lambda^{\mathcal{T}}} Jh, Jh\rangle_{L^2(\mathcal{T})} = \nu_1^{\mathcal{T}}(\lambda). \quad (4.61)$$

Denote now by $\lambda_1 = \min \sigma(-\Delta_{\mathcal{T},\alpha}) < 0$ the smallest eigenvalue of $-\Delta_{\mathcal{T},\alpha}$. Due to

$$\dim\ker\big(-\Delta_{\Sigma,\alpha} - (\lambda-1)\big) = \dim\ker(A_\Theta - \lambda) = \dim\ker(\alpha - \overline{B_\lambda})$$

this means that α is an eigenvalue of $\overline{B_{\lambda_1+1}^{\mathcal{T}}}$. As $\nu_1^{\mathcal{T}}(\lambda_1+1)$ denotes the largest eigenvalue of $\overline{B_{\lambda_1+1}^{\mathcal{T}}}$ and due to (4.61) we get the estimate

$$\alpha \leq \nu_1^{\mathcal{T}}(\lambda_1+1) < \nu_1(\lambda_1+1).$$

According to Lemma 4.24 the function $\lambda \mapsto \nu_1(\lambda)$ is continuous and strictly increasing on $(-\infty, 0]$. Hence there exists $\lambda_2 < \lambda_1$ such that $\alpha = \nu_1(\lambda_2+1)$, i.e. λ_2+1 is an eigenvalue of $-\Delta_{\Sigma,\alpha}$. Hence $\min\sigma(-\Delta_{\Sigma,\alpha}) \leq \lambda_2+1 < \lambda_1+1 = \min\sigma(-\Delta_{\mathcal{T},\alpha})$. □

Finally, we will compare our operators $-\Delta_{\Sigma,\alpha}$ to the operators defined in [54, Example 3.5] and [65, § 3], which we consider as representatives of the class of Schrödinger operators with δ-interactions defined in the literature.

Lemma 4.29. *Let $-\Delta_\alpha^\Sigma$ be the singular perturbed Laplacian as defined in [54, Example 3.5]. Then $-\Delta_\alpha^\Sigma$ and $-\Delta_{\Sigma,\alpha}$ coincide.*

Proof. Let $\varepsilon > 0$. As in [54] we define in $L^2(\Sigma)$ the operator $\tilde{\Gamma}(\lambda)$ by

$$\begin{aligned}\big(\tilde{\Gamma}(\lambda)h\big)(x) :=& \int_\Sigma \big[h(x)-h(y)\big]\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)\\ &+h(x)\left[\frac{\ln(\varepsilon^{-1})}{2\pi}+\int_0^L \frac{\mathbb{1}_{[0,\varepsilon]}(|t-s|)}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\right],\\ \operatorname{dom}\tilde{\Gamma}(\lambda) :=& \{h\in C^1(\Sigma):\operatorname{supp} h\subset\Sigma\setminus\sigma^{-1}(0)\}\subseteq \operatorname{dom} B_{1+\lambda}.\end{aligned}$$

Note, that this operator is independent of ε, cf. equation (19) in [54]. Hence we get for all $h\in\operatorname{dom}\tilde{\Gamma}(\lambda)$ and all $x\in\Sigma$

$$\begin{aligned}&\big(B_{1+\lambda}h\big)(x)+\big(\tilde{\Gamma}(\lambda)h\big)(x)=\int_\Sigma[h(y)-h(x)]\frac{e^{-\sqrt{-\lambda}}|x-y|}{4\pi|x-y|}\,d\sigma(y)+h(x)k_\lambda(x)+\tilde{\Gamma}(\lambda)h(x)\\ &=h(x)\left[k_\lambda(x)+\frac{\ln(\varepsilon^{-1})}{2\pi}+\int_0^L\frac{\mathbb{1}_{[0,\varepsilon]}(|t-s|)}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\right]\\ &=h(x)\lim_{\delta\to 0}\left[\int_{\Sigma_\delta(x)}\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)+\frac{\ln\delta}{2\pi}\right.\\ &\qquad\left.+\frac{\ln(\varepsilon^{-1})}{2\pi}+\int_0^L\frac{\mathbb{1}_{[0,\varepsilon]}(|t-s|)}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\right]\\ &=h(x)\lim_{\delta\to 0}\left[\int_{\Sigma_\delta(x)}\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)+\int_0^L\frac{\mathbb{1}_{[0,\delta]}(|t-s|)}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\right],\end{aligned}$$

where we have chosen in the last step $\varepsilon=\delta$. Due to

$$\begin{aligned}&\int_{\Sigma_\delta(x)}\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)+\int_0^L\frac{\mathbb{1}_{[0,\delta]}(|t-s|)}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\\ &\quad=\int_0^L\big[1-\mathbb{1}_{[0,\delta]}(|t-s|)\big]\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds+\int_0^L\frac{\mathbb{1}_{[0,\delta]}(|t-s|)}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\\ &\quad=\int_0^L\mathbb{1}_{[0,\delta]}(|t-s|)\left[\frac{1}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\right]ds\\ &\quad=\int_{t-\delta}^{t+\delta}\frac{1}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\end{aligned}$$

we observe

$$\begin{aligned}&\big(B_{1+\lambda}h\big)(x)+\big(\tilde{\Gamma}(\lambda)h\big)(x)\\&=h(x)\lim_{\delta\to 0}\left[\int_{t-\delta}^{t+\delta}\frac{1}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|t-s|}}{4\pi|t-s|}\,ds+\int_{t-\delta}^{t+\delta}\frac{e^{-\sqrt{-\lambda}|t-s|}}{4\pi|t-s|}-\frac{e^{-\sqrt{-\lambda}|\sigma(t)-\sigma(s)|}}{4\pi|\sigma(t)-\sigma(s)|}\,ds\right].\end{aligned}$$

Analogously as in (4.49) on page 102 and in (4.43) on page 98 we see that both integrands are bounded. Hence, if we send δ to 0, the integrals converge to 0. Therefore $-\tilde{\Gamma}(\lambda)\subseteq B_{1+\lambda}$ and hence

$$\alpha+\overline{\tilde{\Gamma}(\lambda)}\subseteq\alpha-\overline{B_{1+\lambda}}=\alpha-\overline{B_0}-M(1+\lambda)=\Theta-M(1+\lambda)$$

with $\Theta:=\alpha-\overline{B_0}$, cf. Lemma 4.14. Hence we get for all sufficiently small $\lambda<1$

$$\begin{aligned}(-\Delta_{\Sigma,\alpha}-\lambda)^{-1}&=\big(A_\Theta-(1+\lambda)\big)^{-1}\\&=\big(A-(1+\lambda)\big)^{-1}+\gamma(1+\lambda)\big(\Theta-M(1+\lambda)\big)^{-1}\gamma(1+\lambda)^*\\&\supseteq\big(-\Delta_{\text{free}}-\lambda\big)^{-1}+\gamma(1+\lambda)\big(\alpha+\overline{\tilde{\Gamma}(\lambda)}\big)^{-1}\gamma(1+\lambda)^*.\end{aligned}$$

Keeping in mind $\gamma(1+\lambda)^*=\operatorname{tr}^2_\Sigma\big(A-(1+\lambda)\big)^{-1}$, cf. Lemma 4.3, we observe that the last expression coincides with $(-\Delta^\Sigma_\alpha-\lambda)^{-1}$, cf. the equation after (19) in [54]. Hence $(-\Delta_{\Sigma,\alpha}-\lambda)^{-1}\supseteq(-\Delta^\Sigma_\alpha-\lambda)^{-1}$ and therefore $-\Delta_{\Sigma,\alpha}\supseteq-\Delta^\Sigma_\alpha$. As both operators are selfadjoint they coincide. □

Lemma 4.30. *Denote by $-\triangle_{\alpha,\Sigma}$ the Schrödinger operator with δ-interaction of strength $\alpha\in\mathbb{R}\setminus\{0\}$ on Σ as defined in [65]. Then $-\triangle_{\alpha,\Sigma}$ and $-\Delta_{\Sigma,\alpha-\frac{\ln 2}{2\pi}}$ coincide.*

Proof. Let $\varepsilon>0$ be sufficiently small (in the sense of conditions **C-1** and **C-2** in [65]) and let $\lambda<0$ be arbitrary. For $h\in C^{0,1}(\Sigma)$ and $x\in\Sigma$ we define

$$\begin{aligned}&\big(\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)h\big)(x):=\int_\Sigma[h(x)-h(y)]\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)\\&+h(x)\left[\alpha-\int_{\Sigma_\varepsilon(x)}\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)+\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{4\pi|s-t|}-\frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}\,ds-\frac{\ln(2\varepsilon)}{2\pi}\right],\end{aligned}$$

cf. (3.2) and (3.9) in [65]. As above, $t\in[0,L]$ is chosen such that $\sigma(t)=x\in\Sigma$. Moreover, σ is again identified with its L-periodic continuation on $\mathbb{R}$. As we are interested in the limit $\delta\to 0$

we can assume in the following $\delta < \varepsilon$. At first note

$$
\begin{aligned}
&\big(B_{1+\lambda}h\big)(x) + \big(\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)h\big)(x) = h(x)\lim_{\delta\to 0}\left[\int\limits_{\Sigma_\delta(x)} \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y) + \frac{\ln\delta}{2\pi}\right]\\
&+h(x)\left[\alpha - \int\limits_{\Sigma_\varepsilon(x)} \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y) + \int\limits_{t-\varepsilon}^{t+\varepsilon} \frac{1}{4\pi|s-t|} - \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}\,ds - \frac{\ln(2\varepsilon)}{2\pi}\right]\\
&= h(x)\lim_{\delta\to 0}\left[\int\limits_{\Sigma_\delta(x)} \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y) - \int\limits_{\Sigma_\varepsilon(x)} \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y) + 2\frac{\ln\delta - \ln\varepsilon}{4\pi}\right]\\
&\qquad + h(x)\left[\int\limits_{t-\varepsilon}^{t+\varepsilon} \frac{1}{4\pi|s-t|} - \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}\,ds + \alpha - \frac{\ln 2}{2\pi}\right].
\end{aligned}
$$

Due to $\Sigma_{\delta(x)} \setminus \Sigma_{\varepsilon(x)} = \{\sigma(s) : \delta \le |s-t| < \varepsilon\} = \sigma\big(]t-\varepsilon, t-\delta] \cup [t+\delta, t+\varepsilon[\big)$ and $\ln\delta - \ln\varepsilon = -\int_{t-\varepsilon}^{t-\delta} \frac{1}{|s-t|}\,ds = -\int_{t+\delta}^{t+\varepsilon} \frac{1}{|s-t|}\,ds$ we observe

$$
\begin{aligned}
&\int\limits_{\Sigma_\delta(x)} \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y) - \int\limits_{\Sigma_\varepsilon(x)} \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y) + 2\frac{\ln\delta - \ln\varepsilon}{4\pi}\\
&\qquad = \int\limits_{t-\varepsilon}^{t-\delta} \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|} - \frac{1}{4\pi|s-t|}\,ds + \int\limits_{t+\delta}^{t+\varepsilon} \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|} - \frac{1}{4\pi|s-t|}\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big(B_{1+\lambda}h\big)(x) + \big(\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)h\big)(x)\\
&= h(x)\lim_{\delta\to 0}\left[\int\limits_{t-\varepsilon}^{t-\delta} \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|} - \frac{1}{4\pi|s-t|}\,ds + \int\limits_{t+\delta}^{t+\varepsilon} \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|} - \frac{1}{4\pi|s-t|}\,ds\right]\\
&\qquad + h(x)\left[\int\limits_{t-\varepsilon}^{t+\varepsilon} \frac{1}{4\pi|s-t|} - \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}\,ds + \alpha - \frac{\ln 2}{2\pi}\right]\\
&= h(x)\lim_{\delta\to 0}\left[\int\limits_{t-\delta}^{t+\delta} \frac{1}{4\pi|s-t|} - \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}\,ds\right] + h(x)\left[\alpha - \frac{\ln 2}{2\pi}\right]\\
&= h(x)\lim_{\delta\to 0}\left[\int\limits_{t-\delta}^{t+\delta} \frac{1}{4\pi|s-t|} - \frac{1}{4\pi|\sigma(s)-\sigma(t)|} + \frac{1-e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}\,ds\right] + h(x)\left[\alpha - \frac{\ln 2}{2\pi}\right].
\end{aligned}
$$

Analogously as in (4.43) and (4.49) we see that the integrand in the last line is bounded. Hence, if we send δ to zero the integral vanishes and we get

$$\big(B_{1+\lambda}h\big)(x) + \big(\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)h\big)(x) = h(x)\left[\alpha - \frac{\ln 2}{2\pi}\right]$$

for all $x \in \Sigma$ and all $h \in C^{0,1}(\Sigma)$. In particular we can consider $\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)$ as an essentially selfadjoint operator in $L^2(\Sigma)$ with $\operatorname{dom}\tilde{\Gamma}_{\alpha,\Sigma}(\lambda) = C^{0,1}(\Sigma)$ and $\tilde{\Gamma}_{\alpha,\Sigma}(\lambda) = \alpha - \frac{\ln 2}{2\pi} - B_{1+\lambda}$.

Let $\Gamma_{\alpha,\Sigma}(\lambda)$ be the representing operator of the lower bounded closed symmetric sesquilinear form $\Phi^\lambda_{\alpha,\Sigma}$ in $L^2(\Sigma)$ defined by

$$\begin{aligned}
&\operatorname{dom}\Phi^\lambda_{\alpha,\Sigma} := \{h \in L^2(\Sigma) : \Phi^\lambda_{\alpha,\Sigma}(h,h) < \infty\},\\
&\Phi^\lambda_{\alpha,\Sigma}(h,g) := \frac{1}{2}\int_\Sigma\int_\Sigma [h(x)-h(y)]\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\overline{[g(x)-g(y)]}\,d\sigma(y)\,d\sigma(x) +\\
&\int_\Sigma h(x)\overline{g(x)}\left[\alpha - \int_{\Sigma_\varepsilon(x)}\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}d\sigma(y) + \int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{4\pi|s-t|} - \frac{e^{-\sqrt{-\lambda}|\sigma(s)-\sigma(t)|}}{4\pi|\sigma(s)-\sigma(t)|}ds - \frac{\ln(2\varepsilon)}{2\pi}\right]d\sigma(x),
\end{aligned}$$

cf. (3.7) and (3.8) in [65]. Because of

$$\begin{aligned}
&\int_\Sigma\int_\Sigma [h(x)-h(y)]\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\overline{[g(x)-g(y)]}\,d\sigma(y)\,d\sigma(x)\\
&\qquad = 2\int_\Sigma\left(\int_\Sigma [h(x)-h(y)]\frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}\,d\sigma(y)\right)\overline{g(x)}\,d\sigma(x)
\end{aligned}$$

we get $\Phi^\lambda_{\alpha,\Sigma}(h,h) = \langle\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)h,h\rangle_{L^2(\Sigma)} < \infty$ for all $h \in \operatorname{dom}\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)$. Hence $\operatorname{dom}\tilde{\Gamma}_{\alpha,\Sigma}(\lambda) \subseteq \operatorname{dom}\Phi^\lambda_{\alpha,\Sigma}$. Moreover we get $\Phi^\lambda_{\alpha,\Sigma}(h,g) = \langle\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)h,g\rangle_{L^2(\Sigma)}$ for all $h \in \operatorname{dom}\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)$ and $g \in \operatorname{dom}\Phi^\lambda_{\alpha,\Sigma}$. According to Corollary 2.4 in [41, Chapter VI] this means

$$\tilde{\Gamma}_{\alpha,\Sigma}(\lambda) \subseteq \Gamma_{\alpha,\Sigma}(\lambda).$$

As $\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)$ is essentially selfadjoint and $\Gamma_{\alpha,\Sigma}(\lambda)$ is selfadjoint we conclude with Lemma 4.14

$$\Gamma_{\alpha,\Sigma}(\lambda) = \overline{\tilde{\Gamma}_{\alpha,\Sigma}(\lambda)} = \alpha - \frac{\ln 2}{2\pi} - \overline{B_{1+\lambda}} = \Theta + \overline{B_0} - \overline{B_{1+\lambda}} = \Theta - M(1+\lambda)$$

for $\Theta := \alpha - \frac{\ln 2}{2\pi} - \overline{B_0}$. Hence, with Proposition 5 in [65] and Lemma 4.3, we get

$$\begin{aligned}
&\big(-\triangle_{\alpha,\Sigma} - \lambda\big)^{-1} u \\
&\quad = (-\Delta_{\mathrm{free}} - \lambda)^{-1} u + \int\limits_{\Sigma} \Big[\Gamma_{\alpha,\Sigma}(\lambda)^{-1}\big(\mathrm{tr}^2_{\Sigma}(-\Delta_{\mathrm{free}} - \lambda)^{-1} u\big)\Big](y) \cdot G_{\lambda}(\cdot - y)\, d\sigma(y) \\
&\quad = \big(-\Delta_{\mathrm{free}} + 1 - (1+\lambda)\big)^{-1} u + \gamma(1+\lambda)\big(\Theta - M(1+\lambda)\big)^{-1} \gamma(1+\lambda)^{*} u \\
&\quad = \big(A_{\Theta} - (1+\lambda)\big)^{-1} u = (-\Delta_{\Sigma,\alpha} - \lambda)^{-1} u.
\end{aligned}$$

for all $u \in L^2(\mathbb{R}^3)$. Hence $\big(-\triangle_{\alpha,\Sigma} - \lambda\big)^{-1} = (-\Delta_{\Sigma,\alpha} - \lambda)^{-1}$ and $-\triangle_{\alpha,\Sigma} = -\Delta_{\Sigma,\alpha}$. $\square$

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