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Convergence Time Bounds for a Family of Second-Order Homogeneous State-Feedback Controllers

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***Abstract*—This paper studies a family of second-order homogeneous state-feedback controllers, which includes the well-known twisting algorithm as a special case. Upper bounds for the closed loop's convergence time are proposed that may be computed analytically for any values of the positive controller parameters. Numerical comparisons show that the bound approximates the actual convergence time to within a factor of two over a large parameter range.**

***Index Terms*—Homogeneity; Finite-time convergence; Lyapunov methods; Variable-structure/sliding-mode control**

I. Introduction

ASSESSING the performance of feedback control loops is an important part of control design. For this purpose, performance indicators such as overshoot, steady state accuracy, and settling time are typically considered. Some nonlinear controllers achieve closed-loop convergence in finite time. In this case, the finite convergence time, rather than the settling time, is of interest. Examples are sliding-mode controllers, see, e.g. [1], [2], and homogeneous state-feedback control laws with negative homogeneity degree, see, e.g. [3], [4]. Compared to linear state feedback, these controllers exhibit improved disturbance attenuation, and some of them are even able to completely reject certain classes of perturbations. Computing and bounding their convergence time has thus seen extensive research, see, e.g. [5], [6], [7], [8].

Here, homogeneous control of the perturbed double integrator $\ddot{\sigma} = u + w$ with control input u and disturbance w is considered. Such a system may arise from input-output linearization of plants with relative degree two, which are common, e.g., in mechanical setups. For this system, a family of homogeneous state-feedback controllers is considered that is parameterized by $\gamma \in (0, 1)$ and is given by

$$u = -k_1 |\sigma|^{\frac{\gamma}{2-\gamma}} \operatorname{sign}(\sigma) - k_2 |\dot{\sigma}|^{\gamma} \operatorname{sign}(\dot{\sigma}) \tag{1}$$

with positive gains k_1, k_2, see, e.g. [9], [3]. As γ tends to zero or one, this control law becomes the discontinuous twisting algorithm, see [10], or a linear state feedback controller, respectively; therefore, it can be considered an interpolation of these two cases.

There are few convergence time bounds for the resulting closed loop. A technique in [11] claims to provide an upper convergence time bound provided that γ is sufficiently close to one; a counterexample to that result is provided in [12], however. For $\gamma = 0$, i.e., for the twisting algorithm, the actual convergence time is bounded in [13], [14] and computed analytically in [15]. The present paper proposes an upper convergence time bound *for any value* of the parameter $\gamma \in (0, 1)$ and of the positive gains k_1, k_2 based on a family of Lyapunov functions presented in [16].

The paper is organized as follows: After the problem statement in Section II, Section III presents the main result: an upper bound for the convergence time of the homogeneous state-feedback control loop with and without perturbation. Section IV gives some technical preliminaries needed for finding the proposed bound, which is then derived and proven in Section V. Section VI shows how to choose a free parameter of the bound in an optimal way, and Section VII demonstrates the bound's accuracy by comparing it to numerical and simulation results. Section VIII concludes the paper.

II. Problem Statement

In the following and throughout the paper, the abbreviation $\lfloor y \rceil^p := |y|^p \operatorname{sign}(y)$ is used. With this notation and state variables $x_1 = \sigma$, $x_2 = \dot{\sigma}$, the closed loop obtained by applying (1) to the perturbed double integrator is given by

$$\dot{x}_1 = x_2, \tag{2a}$$

$$\dot{x}_2 = -k_1 \lfloor x_1 \rceil^{\frac{\gamma}{2-\gamma}} - k_2 \lfloor x_2 \rceil^{\gamma} + w. \tag{2b}$$

The state is aggregated in the vector $\mathbf{x} := [x_1 \quad x_2]^{\mathrm{T}}$. The disturbance is assumed to be bounded by

$$|w| \leq W(\mathbf{x}) := W_1 |x_1|^{\frac{\gamma}{2-\gamma}} + W_2 |x_2|^{\gamma} \tag{3}$$

with non-negative constants W_1, W_2. This bound is motivated by the following proposition, which is proven in the appendix:

Proposition 1: Consider system (2) with disturbance w bounded by $|w| \leq W_1 |x_1|^{c_1} + W_2 |x_2|^{c_2}$. If either $W_1 > 0$ and $c_1 < \frac{\gamma}{2-\gamma}$ or $W_2 > 0$ and $c_2 < \gamma$, then the perturbed system's origin is not attractive.

Remark 1: Similarly, it is also possible to show that attractivity cannot be global if $c_1 > \frac{\gamma}{2-\gamma}$ or $c_2 > \gamma$. In this case, results from this paper may still be used locally, however.

It is well-known that every trajectory of system (2)–(3) converges to the origin in finite time for all positive parameters k_1, k_2 and $\gamma \in (0, 1)$ with sufficiently small W_1, W_2. For a given initial state $\mathbf{x}_0$, the maximum time it takes to converge is given by the convergence time function. This function depends on the disturbance bounds W_1, W_2, is denoted by T_{W_1,W_2}, and is formally defined as

$$T_{W_1,W_2}(\mathbf{x}_0) := \sup_{\substack{w \\ \text{s.t. } (3)}} \inf\{\tau : \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(t) = \mathbf{0} \ \forall t \geq \tau\}. \tag{4}$$

The goal is to provide upper bounds for this function.

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III. Main Result

This section states the two main theorems that provide convergence time bounds for the unperturbed and the perturbed closed-loop system.

Theorem 1 (Unperturbed Convergence Time Bound): Consider the unperturbed system (2) with $w = 0$ and parameter $\gamma \in (0, 1)$. Let a, b be positive constants that satisfy either $0 < b < a \leq 1$ or $a = b = 1$ and define the positive constant c as

$$c = \frac{1}{3-\gamma}\left[(2-\gamma)a + b + (1-b)\left(\frac{1-b}{a-b}\right)^{\frac{2-\gamma}{\gamma}}\right]. \tag{5}$$

If the positive parameters k_1, k_2 satisfy

$$k_1^{\frac{2-\gamma}{2}} > (2-\gamma)\frac{\gamma^{\frac{\gamma}{2}}}{2} a k_2, \tag{6}$$

then the unperturbed system's convergence time function is bounded from above by

$$\overline{T}_{0,0}(\mathbf{x}) = \frac{3-\gamma}{1-\gamma}\frac{c}{bk_2}\frac{\left(\tilde{V}(\mathbf{x})^{\frac{3-\gamma}{2}} + \frac{(2-\gamma)ak_2}{c}x_1x_2\right)^{\frac{1-\gamma}{3-\gamma}}}{1-(2-\gamma)\frac{\gamma^{\frac{\gamma}{2}}}{2}ak_2k_1^{-\frac{2-\gamma}{2}}} \tag{7}$$

with the abbreviation

$$\tilde{V}(\mathbf{x}) = (2-\gamma)k_1 |x_1|^{\frac{2}{2-\gamma}} + |x_2|^2, \tag{8}$$

i.e., $T_{0,0}(\mathbf{x}_0) \leq \overline{T}_{0,0}(\mathbf{x}_0)$ holds for all $\mathbf{x}_0 \in \mathbb{R}^2$. ■

Proof: Given in Section V-C.

Remark 2: The parameter range (6), for which the bound is applicable, is determined by the choice of a. By suitably selecting it, any desired parameter range can be obtained. The selection of the parameter b is discussed in Section VI.

Remark 3: By introducing the parameter ratio

$$\rho = k_2 k_1^{-\frac{2-\gamma}{2}} < \frac{2}{(2-\gamma)\gamma^{\frac{\gamma}{2}}a}, \tag{9}$$

the bound (7) for $x_2 = 0$, i.e., for initial states with a velocity component of zero, can be written as

$$\overline{T}_{0,0}(x_1, 0) = \frac{c}{b}\frac{(3-\gamma)(2-\gamma)^{\frac{1-\gamma}{2}}}{(1-\gamma)\rho\sqrt{k_1}}\frac{|x_1|^{\frac{1-\gamma}{2-\gamma}}}{1-(2-\gamma)\frac{\gamma^{\frac{\gamma}{2}}}{2}a\rho}. \tag{10}$$

One can see that, for a fixed value of ρ, it is inversely proportional to $\sqrt{k_1}$. Using this fact, the convergence time for $x_2 = 0$ may be tuned by first selecting the ratio ρ, then tuning the parameter k_1, and finally computing k_2 from ρ.

Theorem 2 (Perturbed Convergence Time Bound): Consider the perturbed system (2)–(3) with positive parameters k_1, k_2, and $\gamma \in (0, 1)$. Suppose that positive constants a, b, c and parameters k_1, k_2 satisfy the conditions of Theorem 1, define the positive constants

$$m_0 = 1 - (2-\gamma)\frac{\gamma^{\frac{\gamma}{2}}}{2}ak_2k_1^{-\frac{2-\gamma}{2}}, \tag{11a}$$

$$m_1 = \frac{m_2}{(2-\gamma)^{\frac{\gamma}{2}}k_2k_1^{-\frac{2-\gamma}{2}}}, \tag{11b}$$

$$m_2 = \frac{(3-\gamma)\frac{c}{b}\left(3-\gamma+\frac{a}{c}(2-\gamma)^{\frac{\gamma}{2}}k_2k_1^{-\frac{2-\gamma}{2}}\right)}{\left((3-\gamma)^{\frac{3-\gamma}{2}} - \frac{a}{c}(2-\gamma)k_2k_1^{-\frac{2-\gamma}{2}}\right)^{\frac{2}{3-\gamma}}}, \tag{11c}$$

and let $\overline{T}_{0,0}(\mathbf{x})$ be defined as in (7). If the non-negative perturbation bounds W_1 and W_2 satisfy

$$m_1\frac{W_1}{k_1} + m_2\frac{W_2}{k_2} < m_0, \tag{12}$$

then the perturbed system's convergence time function is bounded from above by

$$\overline{T}_{W_1,W_2}(\mathbf{x}) = \frac{m_0}{m_0 - m_1\frac{W_1}{k_1} - m_2\frac{W_2}{k_2}}\overline{T}_{0,0}(\mathbf{x}), \tag{13}$$

i.e., $T_{W_1,W_2}(\mathbf{x}_0) \leq \overline{T}_{W_1,W_2}(\mathbf{x}_0)$ holds for all $\mathbf{x}_0 \in \mathbb{R}^2$.

Proof: Given in Section V-D. ■

Remark 4: Note that m_0 is equal to part of the denominator in the unperturbed bound (7) and that m_0, m_1, m_2 depend only on the parameter ratio ρ given in (9). Since $m_1^{-1}m_0k_1$ and $m_2^{-1}m_0k_2$ may be considered upper bounds for W_1 and W_2, respectively, the range of permissible perturbations in (12) may be increased by increasing k_1 (and thus also k_2) for a fixed ρ.

IV. Technical Lemmas

This section introduces two main technical lemmas that are used for deriving the bound. The first one is taken from [16] as a special case of Lemma 7 therein and is presented here along with an immediate corollary. The second one is proposed here and is proven in the appendix.

Lemma 1 ([16, Lemma 7]): For any positive constants α, β, κ, and $\gamma \in (0, 1)$, consider the function

$$V(\mathbf{x}) = \left(\alpha |x_1|^{\frac{2}{2-\gamma}} + \beta |x_2|^2\right)^{\frac{2+\kappa-\gamma}{2}} + \delta x_1 \lfloor x_2 \rceil^{\kappa}. \tag{14}$$

If $|\delta| < \overline{\delta}$ or $|\delta| = \overline{\delta}$ hold with

$$\overline{\delta} = \left(\frac{\alpha}{2-\gamma}\right)^{\frac{2-\gamma}{2}}\left(\frac{\beta}{\kappa}\right)^{\frac{\kappa}{2}}(2-\gamma+\kappa)^{\frac{2-\gamma+\kappa}{2}}, \tag{15}$$

then V is positive definite or semidefinite, respectively.

Corollary 1: Let α, β, κ be positive, let $\gamma \in (0, 1)$, and consider the set $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^2 : \alpha |x_1|^{\frac{2}{2-\gamma}} + \beta |x_2|^2 = 1\}$. Then,

$$\sup_{\mathbf{x}\in\mathcal{M}} x_1 \lfloor x_2 \rceil^{\kappa} = \frac{\left(\frac{2-\gamma}{\alpha}\right)^{\frac{2-\gamma}{2}}\left(\frac{\kappa}{\beta}\right)^{\frac{\kappa}{2}}}{(2-\gamma+\kappa)^{\frac{2-\gamma+\kappa}{2}}}. \tag{16}$$

Proof: Given in the appendix. ■

Lemma 2: Let positive constants α, β, κ, and Y be given. If $\beta \leq \alpha < \frac{1}{Y}$ and $\kappa < 1$, then

$$\inf_{\substack{|y_1|\leq Y \\ y_2\in[0,1]}} \frac{1+\alpha y_1}{(1+\beta y_1 y_2)^{\kappa}} = 1 - \alpha Y. \tag{17}$$

Proof: Given in the appendix. ■

V. Convergence Time Bound

This section derives and proves the presented convergence time bound. First, it briefly recapitulates the Lyapunov-based technique for convergence time bounding and the employed family of Lyapunov functions from [16]. Then, the choice of a suitable function from that family is discussed. Using the chosen Lyapunov function, the unperturbed and perturbed convergence time bounds are finally proven.

A. Lyapunov-Based Convergence Time Estimation

The convergence time bound is based on a family of strong, smooth Lyapunov functions for system (2) that were recently proposed in [16]. They have the form

$$V(\mathbf{x}) = \tilde{V}(\mathbf{x})^{\frac{3-\gamma}{2}} + \varepsilon x_1 x_2, \tag{18}$$

wherein ε is a positive constant and $\tilde{V}$ defined in (8) is a (well-known) weak Lyapunov function for (2), whose time derivative along the system's trajectories is given by $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{V} = -2k_2 |x_2|^{1+\gamma}$ for $w = 0$. Using this fact, the time derivative $\dot{V}$ of V is found as

$$\begin{aligned}\dot{V}(\mathbf{x}) = &-(3-\gamma)k_2\tilde{V}(\mathbf{x})^{\frac{1-\gamma}{2}} |x_2|^{1+\gamma} \\ &+ \varepsilon(|x_2|^2 - k_1 |x_1|^{\frac{2}{2-\gamma}} - k_2 x_1 \lfloor x_2 \rceil^{\gamma}).\end{aligned} \tag{19}$$

It is shown in [16] that for all positive values of the parameters k_1, k_2 there exists a sufficiently small value of ε such that V is positive definite and $\dot{V}$ is negative definite. For such values of ε, the Lyapunov function and its time derivative satisfy the inequality

$$\dot{V}(\mathbf{x}) \le -CV(\mathbf{x})^{\frac{2}{3-\gamma}} \tag{20}$$

with some $C > 0$ for all $\mathbf{x}$. Rewriting this differential inequality in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V^{\frac{1-\gamma}{3-\gamma}} \le -\frac{1-\gamma}{3-\gamma}C \tag{21}$$

and integrating it yields a bound for the convergence time:

Lemma 3: Consider the perturbed system (2)–(3) with given non-negative perturbation bounds W_1, W_2. Suppose that for some $\varepsilon > 0$, the function V defined in (18) is positive definite, and suppose that its time derivative $\dot{V}$ along the system's trajectories satisfies (20) for some $C > 0$. Then, the system's convergence time is bounded from above by $\overline{T}(\mathbf{x}_0) \ge T_{W_1,W_2}(\mathbf{x}_0)$ with

$$\overline{T}(\mathbf{x}) = \frac{3-\gamma}{1-\gamma}\frac{V(\mathbf{x})^{\frac{1-\gamma}{3-\gamma}}}{C} \tag{22}$$

for all initial states $\mathbf{x}_0 \in \mathbb{R}^2$.

Proof: Given in the appendix. ■

B. Choice of Lyapunov Function

In order to derive a convergence time bound, a Lyapunov function has to be chosen, i.e., the value of ε in (18) has to be selected. This has to be done as a function of the parameters k_1, k_2, γ without compromising the positive and negative definiteness of V and $\dot{V}$, respectively. Furthermore, the resulting $\dot{V}$ should allow for an analytic computation of C in the differential inequality (20) without requiring additional complicated or conservative parameter conditions.

In order to simplify considerations, homogeneity of most involved functions, in particular of V and $\dot{V}$, is used, see, e.g. [3], [4]. Any such homogeneous scalar-valued function $f(x_1, x_2)$ satisfies $f(\alpha^{2-\gamma}x_1, \alpha x_2) = \alpha^m f(x_1, x_2)$ for all $\alpha > 0$ and for some scalar m, which is called the homogeneity degree of f. As a consequence, any inequalities involving homogeneous functions hold for all $\mathbf{x}$ if and only if they hold on the compact set defined by $\tilde{V}(\mathbf{x}) = 1$. Therefore, the following obvious relations

$$\sup_{\tilde{V}=1} |x_1| = \frac{(2-\gamma)^{-\frac{2-\gamma}{2}}}{k_1^{\frac{2-\gamma}{2}}}, \qquad \sup_{\tilde{V}=1} |x_2|^2 = 1, \tag{23a}$$

and furthermore, due to Corollary 1, the relations

$$\sup_{\tilde{V}=1} x_1 x_2 = \frac{(3-\gamma)^{-\frac{3-\gamma}{2}}}{k_1^{\frac{2-\gamma}{2}}}, \quad \sup_{\tilde{V}=1} x_1 \lfloor x_2 \rceil^{\gamma} = \frac{\gamma^{\frac{\gamma}{2}}}{2k_1^{\frac{2-\gamma}{2}}} \tag{23b}$$

will prove to be useful.

Using these considerations, the next proposition gives a range of values for ε that ensures positive definiteness of V.

Proposition 2: Suppose that for some $a \in (0, 1]$ the positive parameters k_1, k_2, and $\gamma \in (0, 1)$ satisfy (6). Then, V in (18) is positive definite for all positive $\varepsilon \le (2-\gamma)ak_2$.

Proof: Restricting considerations to $\tilde{V}(\mathbf{x}) = 1$ due to homogeneity, one finds by using (23b) and applying (6) that

$$\begin{aligned}\inf_{\tilde{V}(\mathbf{x})=1} V(\mathbf{x}) = 1 - \varepsilon \sup_{\tilde{V}(\mathbf{x})=1} x_1 x_2 &\ge 1 - \frac{(2-\gamma)a}{(3-\gamma)^{\frac{3-\gamma}{2}}}\frac{k_2}{k_1^{\frac{2-\gamma}{2}}} \\ &\ge 1 - \frac{2}{(3-\gamma)^{\frac{3-\gamma}{2}}\gamma^{\frac{\gamma}{2}}} > 0\end{aligned} \tag{24}$$

holds, because the last expression is strictly decreasing with respect to γ and equal to zero for $\gamma = 1$. ■

The Lyapunov function's time derivative $\dot{V}$ is now considered. Taking into account that $\tilde{V}(\mathbf{x}) \ge |x_2|^2$, one obtains $\dot{V}(\mathbf{x}) \le -k_2 R(\mathbf{x})$ from (19) with

$$R(\mathbf{x}) = \big(3 - \gamma - \frac{\varepsilon}{k_2}\big)|x_2|^2 + \frac{\varepsilon}{k_2}k_1|x_1|^{\frac{2}{2-\gamma}} + \varepsilon x_1 \lfloor x_2 \rceil^{\gamma}. \tag{25}$$

A choice for ε is now proposed that yields a negative definite upper bound for $\dot{V}$ in addition to positive definiteness of V. Since it involves the constant c defined in (5), an auxiliary lemma is first given; then, the proposed choice for ε is stated.

Lemma 4: For a given $\gamma \in (0, 1)$, suppose that constants a, b, c satisfy the conditions of Theorem 1. Then, $c \ge 1$.

Proof: Given in the appendix. ■

Proposition 3: Suppose that parameters k_1, k_2, γ and constants a, b, c satisfy the conditions of Theorem 1. Then, choosing the positive constant ε as

$$\varepsilon = (2-\gamma)\frac{a}{c}k_2 \tag{26}$$

guarantees that V defined in (18) is positive definite and that $\dot{V}$ given in (19) satisfies

$$\frac{\dot{V}(\mathbf{x})}{V(\mathbf{x})^{\frac{2}{3-\gamma}}} \le -k_2\frac{b}{c}\frac{\tilde{V}(\mathbf{x}) + c\varepsilon x_1 \lfloor x_2 \rceil^{\gamma}}{\Big(\tilde{V}(\mathbf{x})^{\frac{3-\gamma}{2}} + \varepsilon x_1 x_2\Big)^{\frac{2}{3-\gamma}}} \tag{27}$$

for all $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Proof: Since $c \ge 1$ according to Lemma 4, positive definiteness of V follows from Proposition 2. By multiplying both sides of (27) with $\frac{c}{(1-b)k_2}V(\mathbf{x})^{\frac{2}{3-\gamma}}$ and using the fact that $\dot{V}(\mathbf{x}) \le -k_2 R(\mathbf{x})$ with $R(\mathbf{x})$ given in (25), one can see that it suffices to show that the function $H(\mathbf{x})$ defined as

$$H(\mathbf{x}) = \frac{cR(\mathbf{x}) - b\big(\tilde{V}(\mathbf{x}) + c\varepsilon x_1 \lfloor x_2 \rceil^{\gamma}\big)}{1-b} \tag{28}$$

is positive definite. Substituting ε from (26) and c from (5), this function can be written as

$$H(\mathbf{x}) = \alpha |x_1|^{\frac{2}{2-\gamma}} + \beta |x_2|^2 + \delta x_1 \lfloor x_2 \rceil^{\gamma} \tag{29}$$

with constants α, β, and δ given by

$$\alpha = (2-\gamma)k_1 \frac{a-b}{1-b}, \qquad \delta = (2-\gamma)ak_2, \tag{30}$$

$$\beta = \frac{(3-\gamma)c - (2-\gamma)a - b}{1-b} = \left(\frac{1-b}{a-b}\right)^{\frac{2-\gamma}{\gamma}}. \tag{31}$$

According to Lemma 1, $H(\mathbf{x})$ is positive definite, iff

$$(2-\gamma)ak_2 < 2\left(k_1 \frac{a-b}{1-b}\right)^{\frac{2-\gamma}{2}} \left[\frac{\left(\frac{1-b}{a-b}\right)^{\frac{2-\gamma}{\gamma}}}{\gamma}\right]^{\frac{\gamma}{2}} = \frac{2k_1^{\frac{2-\gamma}{2}}}{\gamma^{\frac{\gamma}{2}}} \tag{32}$$

holds, which is equivalent to condition (6) of Theorem 1. ∎

C. Bound for Unperturbed Case

The differential inequality in Proposition 3 may now be bounded from above by using Lemma 2. This allows to prove the unperturbed convergence time bound stated in Theorem 1.

Proof of Theorem 1: With the abbreviations $y_1 = x_1 \lfloor x_2 \rceil^{\gamma}$ and $y_2 = |x_2|^{1-\gamma}$, one may write the inequality (27) from Proposition 3 for $\tilde{V}(\mathbf{x}) = 1$ as

$$\left.\frac{\dot{V}(\mathbf{x})}{V(\mathbf{x})^{\frac{2}{3-\gamma}}}\right|_{\tilde{V}(\mathbf{x})=1} \le -k_2 \frac{b}{c} \frac{1 + c\varepsilon y_1}{\left(1 + \varepsilon y_1 y_2\right)^{\frac{2}{3-\gamma}}}. \tag{33}$$

By using (23), one finds that the variables y_1, y_2 satisfy

$$|y_1| \le \frac{\gamma^{\frac{\gamma}{2}}}{2k_1^{\frac{2-\gamma}{2}}}, \qquad 0 \le y_2 \le 1. \tag{34}$$

Thus, applying Lemma 2 and substituting ε from (26) yields

$$\begin{aligned}\frac{\dot{V}(\mathbf{x})}{V(\mathbf{x})^{\frac{2}{3-\gamma}}} &\le -k_2 \frac{b}{c}\Big(1 - c\varepsilon \frac{\gamma^{\frac{\gamma}{2}}}{2k_1^{\frac{2-\gamma}{2}}}\Big) \\ &= -k_2 \frac{b}{c}\Big(1 - \frac{(2-\gamma)ak_2\gamma^{\frac{\gamma}{2}}}{2k_1^{\frac{2-\gamma}{2}}}\Big) = -C. \end{aligned} \tag{35}$$

Applying Lemma 3 to this differential inequality yields the claimed convergence time bound (7). ∎

D. Bound for Perturbed Case

The perturbed case is now considered, and the result in Theorem 2 is derived and proven. Taking the time derivative of the Lyapunov function V defined in (18) along the trajectories of the perturbed system (2)–(3) yields

$$\begin{aligned}\dot{V}(\mathbf{x}) &\le -CV(\mathbf{x})^{\frac{2}{3-\gamma}} + \frac{\partial V}{\partial x_2}(\mathbf{x}) w \\ &\le -CV(\mathbf{x})^{\frac{2}{3-\gamma}} + \left|\frac{\partial V}{\partial x_2}(\mathbf{x})\right| W(\mathbf{x}). \end{aligned} \tag{36}$$

with C given in (35) and the partial derivative given by

$$\frac{\partial V}{\partial x_2}(\mathbf{x}) = (3-\gamma)\tilde{V}(\mathbf{x})^{\frac{1-\gamma}{2}} x_2 + \varepsilon x_1. \tag{37}$$

A differential inequality of the form $\dot{V} \le -(C-D)V^{\frac{2}{3-\gamma}}$ may be obtained by bounding the second term in (36) as

$$\left|\frac{\partial V}{\partial x_2}(\mathbf{x})\right| W(\mathbf{x}) \le DV(\mathbf{x})^{\frac{2}{3-\gamma}}, \tag{38}$$

with a positive constant D. This allows to prove Theorem 2.

Proof of Theorem 2: As before, it suffices to consider (38) only for $\tilde{V}(\mathbf{x}) = 1$. By using (23), one obtains

$$\inf_{\tilde{V}(\mathbf{x})=1} V(\mathbf{x}) = 1 - \varepsilon \frac{(3-\gamma)^{-\frac{3-\gamma}{2}}}{k_1^{\frac{2-\gamma}{2}}} = F_1 \tag{39a}$$

$$\sup_{\tilde{V}(\mathbf{x})=1} \left|\frac{\partial V}{\partial x_2}(\mathbf{x})\right| \le (3-\gamma) + \varepsilon \frac{(2-\gamma)^{-\frac{2-\gamma}{2}}}{k_1^{\frac{2-\gamma}{2}}} = F_2 \tag{39b}$$

$$\sup_{\tilde{V}(\mathbf{x})=1} W(\mathbf{x}) \le W_1 \frac{(2-\gamma)^{-\frac{\gamma}{2}}}{k_1^{\frac{\gamma}{2}}} + W_2 \tag{39c}$$

with positive constants F_1, F_2. Therefore, by substituting ε from (26), one finds that

$$\begin{aligned}\frac{\left|\frac{\partial V}{\partial x_2}(\mathbf{x})\right| W(\mathbf{x})}{V(\mathbf{x})^{\frac{2}{3-\gamma}}} &\le \frac{F_2}{F_1^{\frac{2}{3-\gamma}}}\left[\frac{(2-\gamma)^{-\frac{\gamma}{2}}}{k_1^{\frac{\gamma}{2}}} W_1 + W_2\right] \\ &= \frac{bk_2}{c}\big(\frac{m_1}{k_1}W_1 + \frac{m_2}{k_2}W_2\big) = D \end{aligned} \tag{40}$$

and $\frac{bk_2}{c} m_0 = C$ hold with m_0, m_1, m_2 in (11) and C in (35). Since (12) implies $C > D$, Lemma 3 may be applied to the resulting differential inequality $\dot{V} \le -(C-D)V^{\frac{2}{3-\gamma}}$ to yield the convergence time bound (13). ∎

VI. Parameter Selection

As pointed out in Remark 2, the value of a in Theorem 1 determines the range (6) of k_1, k_2 or, equivalently, of the parameter ratio ρ in (9), for which the bound is applicable. Once a is fixed, the main quantity of interest is the scaling factor $\frac{c}{b}$ that appears in (7) and (10). In general, the smaller this factor is, the smaller (and better) the bound is. Thus, the parameter b should be selected to minimize this factor. The following proposition provides a way to do this by computing the root of a generalized polynomial.

Proposition 4 (Choice of b): Let constants $a \in (0, 1]$ and $\gamma \in (0, 1)$ be given. Define c as in Theorem 1 and consider the unique solution $q \in (\frac{1}{a}, \infty)$ of the equation

$$q^{\frac{2}{\gamma}} - \frac{2}{(2-\gamma)a} q^{\frac{2-\gamma}{\gamma}} - \gamma = 0. \tag{41}$$

Then, $\frac{c}{b} \ge 1$ holds for all $b \in (0, a)$ and selecting

$$b = \frac{1-aq}{1-q} \tag{42}$$

minimizes the value of $\frac{c}{b}$ with respect to b.

Proof: Given in the appendix. ∎

Remark 5: Note that $b = c = 1$ is obtained from this proposition for $a = 1$, regardless of γ.

Fig. 1 plots the optimal value of b and the corresponding scaling factor $\frac{c}{b}$ as a function of a for different values of γ. One can see that the scaling factor increases with decreasing

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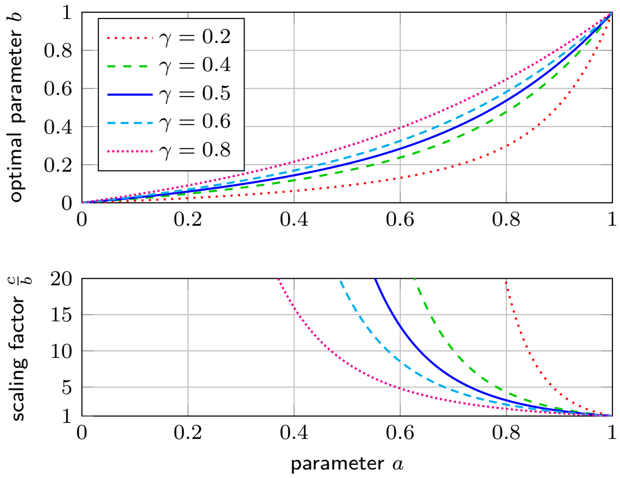

Fig. 1. Optimal parameter values of b obtained from Proposition 4 (top) and corresponding minimum scaling factor $\frac{c}{b}$ in Theorem 1 (bottom) as a function of the parameter a for different values of γ.

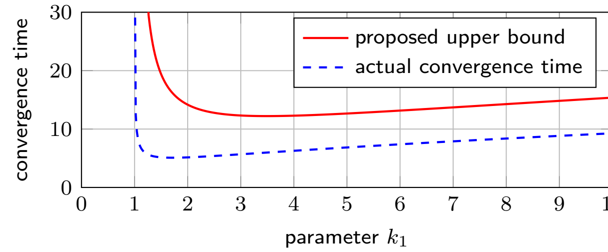

Fig. 2. Proposed bound's limit for $\gamma \to 0$ with $a = b = 1$ and actual convergence time of the twisting algorithm computed in [14] (or also obtained from [13] with optimally chosen parameter $\overline{k}$ therein) with $k_2 = 1$ and initial state $x_0 = [1 \quad 1]^{\mathrm{T}}$ as a function of k_1.

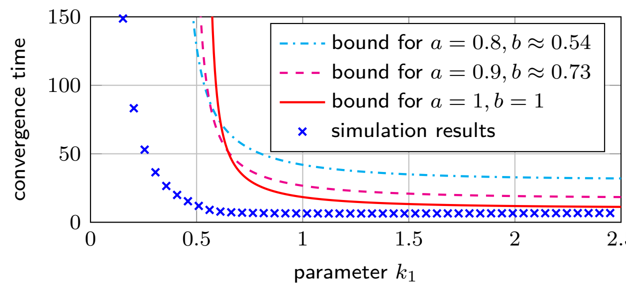

Fig. 3. Upper convergence time bounds from Theorem 1 for different values of a with optimally chosen parameter b, and actual convergence time of system (2) with $w = 0$ from a simulation with initial condition $\mathbf{x}_0 = [1 \quad 1]^{\mathrm{T}}$ and parameters $\gamma = 0.5$, $k_2 = 1$ as a function of k_1.

values of a and γ. With decreasing γ, it thus becomes harder to obtain small bounds for values of k_1 that require $a < 1$.

The reason for this effect can be seen by considering the limit $\gamma \to 0$. Condition (6) then tends to $k_1 > ak_2$ and system (2) becomes the well-known twisting algorithm. The twisting algorithm is only finite-time stable for $k_1 > k_2$, however. Therefore, all convergence time bounds with $a < 1$ must diverge at this point, which explains the increase in scaling factors with decreasing γ.

VII. Comparisons and Simulation Results

This section compares the proposed convergence time bound to the actual convergence time of the twisting algorithm, which is obtained for $\gamma \to 0$, and to simulation results with $\gamma = 0.5$, which is a common parameter choice.

A. Comparison for $\gamma \to 0$: Twisting Algorithm

In the limit as γ tends to zero, system (2) becomes the well-known twisting algorithm. Although $\gamma = 0$ is not considered explicitly in Theorem 1, one may take the limit $\gamma \to 0$ there, provided that $a = b = 1$ is selected. The corresponding limits of condition (6) and bound (7) are studied as a benchmark of the proposed result.

For vanishing γ, the bound (7) tends to

$$\lim_{\gamma \to 0} \overline{T}_{0,0}(\mathbf{x}) = \frac{3}{k_2} \frac{\left(\left(2k_2 |x_1| + |x_2|^2\right)^{\frac{3}{2}} + 2k_2 x_1 x_2 \right)^{\frac{1}{3}}}{1 - k_2 k_1^{-1}}, \tag{43}$$

while condition (6) tends to $k_1 > k_2$, a well-known necessary and sufficient stability condition for the twisting algorithm. Due to continuity reasons, expression (43) is an upper convergence time bound for system (2) with $\gamma = 0$ and $w = 0$. Fig. 2 compares this bound to the convergence time function of the twisting algorithm that is computed analytically in [15] or also obtained from [13] by optimally choosing $\overline{k}$ therein. One can see that the proposed bound is qualitatively similar to the actual convergence time; for large k_1, in particular, it is conservative only by a fixed offset.

B. Simulation Results for $\gamma = 0.5$

Fig. 3 studies the behavior of the bound for $k_2 = 1$, $\gamma = 0.5$ as a function of k_1 for different values of a. For each a, the parameter b is chosen optimally according to Proposition 4. The results are compared to actual convergence times obtained from a simulation. One can see that by decreasing a, the asymptote may be shifted to the left, though at the cost of an increasingly larger bound. Nevertheless, the bounds show the same qualitative behavior as for the twisting algorithm, and for large values of k_1, the bound approximates the actual convergence time to within a factor of two.

VIII. Conclusion

An upper bound for the convergence time of a family of second-order homogeneous state-feedback controllers was presented. The range of controller parameters, for which the bound is applicable, may be tuned using a scalar parameter and may be made arbitrarily large. The bound was compared to convergence times obtained from simulations and to the convergence time of the twisting algorithm, which is obtained as a special case in the limit. These comparisons showed that the bound approximates the actual convergence time to within a factor of two over a large parameter range.

Appendix

Proof of Proposition 1: In the first case, $w = \delta \lfloor x_1 \rceil^{c_1}$ with any $\delta \in (0, W_1]$ yields the nontrivial equilibrium

$$x_1 = \left|k_1^{-1}\delta\right|^{\frac{1}{\frac{\gamma}{2-\gamma} - c_1}}, \qquad x_2 = 0 \tag{44}$$

arbitrarily close to the origin. In the second case, choose $w = W_2 \lfloor x_2 \rceil^{c_2}$; then, the time derivative of the positive definite function $\tilde{V}$ defined in (8) along the system's trajectories

$$\dot{\tilde{V}}(\mathbf{x}) = -2k_2 |x_2|^{1+\gamma} + 2W_2 |x_2|^{1+c_2}, \tag{45}$$

is non-negative in every open neighborhood of the origin, wherein $|x_2| < \left|k_2^{-1} W_2\right|^{\frac{1}{\gamma - c_2}}$ holds. ■

Proof of Corollary 1: Consider $V(\mathbf{x})$ in (14) with $\delta = -\overline{\delta}$ from (15). Since V is only positive semidefinite, its infimum is zero. Keeping $\delta < 0$ in mind, one has

$$\begin{aligned} 0 = \inf_{\mathbf{x} \in \mathcal{M}} V(\mathbf{x}) &= 1 + \inf_{\mathbf{x} \in \mathcal{M}} \delta x_1 \lfloor x_2 \rceil^{\kappa} \\ &= 1 - \overline{\delta} \sup_{\mathbf{x} \in \mathcal{M}} x_1 \lfloor x_2 \rceil^{\kappa}, \end{aligned} \tag{46}$$

and thus, $\sup_{\mathbf{x} \in \mathcal{M}} x_1 \lfloor x_2 \rceil^{\kappa} = -\overline{\delta}^{-1}$. ■

Proof of Lemma 2: Using $\beta y_2 \le \alpha$, one finds that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y_1} \frac{1 + \alpha y_1}{(1 + \beta y_1 y_2)^{\kappa}} &= \frac{(\alpha - \kappa \beta y_2) + \alpha \beta (1 - \kappa) y_1 y_2}{(1 + \beta y_1 y_2)^{\kappa + 1}} \\ &\ge \frac{\alpha (1 - \kappa)}{(1 + \beta y_1 y_2)^{\kappa}} \ge 0, \end{aligned} \tag{47}$$

i.e., that the expression in (17) is non-decreasing with respect to y_1 for every value of $y_2 \in [0, 1]$. Thus, the minimum is obtained for $y_1 = -Y$ and, since $1 - \alpha Y > 0$, $y_2 = 0$. ■

Proof of Lemma 3: Suppose to the contrary that there is a trajectory $\mathbf{x}(t)$ of system (2)–(3) with initial state $\mathbf{x}_0 = \mathbf{x}(0)$ and $\mathbf{x}_\mathrm{T} = \mathbf{x}(\overline{T}(\mathbf{x}_0)) \neq \mathbf{0}$. Then, integration of (21) yields

$$V(\mathbf{x}_\mathrm{T})^{\frac{1-\gamma}{3-\gamma}} - V(\mathbf{x}_0)^{\frac{1-\gamma}{3-\gamma}} \le -\frac{1-\gamma}{3-\gamma} C \overline{T}(\mathbf{x}_0) = -V(\mathbf{x}_0)^{\frac{1-\gamma}{3-\gamma}}. \tag{48}$$

Hence, $V(\mathbf{x}_\mathrm{T}) \le 0$ but $\mathbf{x}_\mathrm{T} \neq \mathbf{0}$, which contradicts positive definiteness of V. ■

Proof of Lemma 4: Rewrite (5) as

$$c = \frac{(3-\gamma) b + (1 - b)\left[(2 - \gamma)\frac{a-b}{1-b} + \left(\frac{1-b}{a-b}\right)^{\frac{2-\gamma}{\gamma}}\right]}{3 - \gamma}. \tag{49}$$

Introducing the abbreviation $y = \frac{a-b}{1-b} \in (0, 1]$, one has

$$\inf_{y \in (0,1]} \left[(2-\gamma) y + \frac{1}{y^{\frac{2-\gamma}{\gamma}}}\right] = 3 - \gamma \tag{50}$$

with the infimum being obtained for $y = 1$, because the expression's derivative is negative for all $y \in (0, 1]$. Therefore, the right-hand side of (49) may be bounded from below as

$$c \ge \frac{(3-\gamma) b + (1 - b)(3 - \gamma)}{3 - \gamma} = 1, \tag{51}$$

which completes the proof. ■

Proof of Proposition 4: With $z = \frac{a}{b} \in [1, \infty)$, (5) yields

$$\frac{(3-\gamma)c}{b} = \left[\frac{\left(\frac{z}{a} - 1\right)^{\frac{2}{\gamma}}}{(z-1)^{\frac{2-\gamma}{\gamma}}} + (2-\gamma) z + 1\right] \ge 3 - \gamma. \tag{52}$$

To see that the solution of (41) is unique for $q \in [\frac{1}{a}, \infty)$, note that the left-hand side of (41) is negative for $qa = 1$, positive for sufficiently large q, and strictly increasing, because

$$\frac{\mathrm{d}}{\mathrm{d}q}\left(q^{\frac{2}{\gamma}} - \frac{2}{(2-\gamma)a} q^{\frac{2-\gamma}{\gamma}} - \gamma\right) = \frac{2}{\gamma} q^{\frac{2-\gamma}{\gamma}} \left(1 - \frac{1}{qa}\right) > 0 \tag{53}$$

for $qa > 1$. It will be shown that (41) is the first order optimality condition for a minimum of $\frac{c}{b}$ given in (52). Differentiation with respect to z yields

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{(3-\gamma)c}{b} = \frac{2}{a\gamma}\left(\frac{\frac{z}{a} - 1}{z - 1}\right)^{\frac{2-\gamma}{\gamma}} - \frac{2-\gamma}{\gamma}\left(\frac{\frac{z}{a} - 1}{z - 1}\right)^{\frac{2}{\gamma}} + (2 - \gamma). \tag{54}$$

By multiplying this expression with $\frac{\gamma}{\gamma - 2}$ and introducing

$$q = \frac{\frac{z}{a} - 1}{z - 1} = \frac{1 - b}{a - b}, \tag{55}$$

the left-hand side of (41) is obtained, and inverting (55) yields the value of b in (42). Since $\frac{c}{b}$ in (52) tends to infinity for $z \to 1$ and $z \to \infty$ and the solution of (41) is unique, the unique global minimum is obtained this way. ■

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