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An integral extension technique for continuous homogeneous state-feedback control laws preserving nominal performance

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Summary
This paper proposes a technique to extend a nominal homogeneous state-feedback control law by continuous or discontinuous integral terms. Compared to pure state feedback, this permits to suppress non-vanishing perturbations that are either constant or Lipschitz continuous with respect to time. The proposed technique seeks to do this while maintaining nominal performance in the sense that the nominal control signal and closed-loop behavior is not modified in the unperturbed case. The class of controllers thus obtained is shown to include the well-known super-twisting algorithm as a special case. Simulations comparing the technique to other approaches demonstrate its intuitive tuning and show a performance preserving effect also in the perturbed case.

KEYWORDS
finite-time convergence, integral extension, nonlinear control, sliding mode control, state feedback, weighted homogeneity

1 | INTRODUCTION

Rejecting disturbances acting on a plant is one of the main goals of feedback control. Linear state feedback can usually attenuate very small disturbances or disturbances that vanish with vanishing state to a satisfactory degree, but fails to suppress large nonvanishing disturbances. Homogeneous state-feedback control laws improve on the disturbance rejection capability of linear controllers due to their increase in (linearized) gain close to the equilibrium.[1] Nevertheless, also they fail to completely reject nonvanishing (even constant) disturbances. Handling such disturbances typically requires techniques such as disturbance observers or integral control.

This contribution proposes a technique for adding an integral part to any homogeneous state-feedback controller that stabilizes a nominal, that is, unperturbed plant in finite time. The integral controller is constructed in such a way that the nominal performance is recovered in the unperturbed case. It is shown that global finite-time stability of the closed loop is achieved in the presence of constant or slope bounded, that is, Lipschitz continuous perturbations. For the latter disturbance class in particular, the technique yields integral state-feedback control laws with discontinuous integrands.

In the context of sliding-mode control,[2] the proposed technique may be used to obtain higher order sliding-mode control laws for sliding surfaces of arbitrary relative degree. In contrast to the integral sliding mode technique,[3] which has a similar objective of preserving the performance of a nominal controller, no dynamic extension or additional sliding surface is used. Rather, a whole class of state-feedback controllers with integral part is obtained, which contain the well-known super-twisting algorithm as a special case.

The design of state-feedback controllers with discontinuous integrands has been studied, for example, using integral sliding mode control,[4] passivity-based techniques,[5] explicit Lyapunov functions,[6-8] and implicit Lyapunov functions.[9] Thereof, some of the nonpassive Lyapunov-based approaches[6,7] allow for the design of pure output-feedback controllers. They are not designed to extend a given nominal control law, however. The passivity-based approach[5] is able to do so, but applying it requires a Lyapunov function for the nominal closed loop, which is not always available. The implicit Lyapunov function approach[9] remedies this to some extent, but requires a compatible nominal control law. With the technique proposed here, a Lyapunov function for the nominal case is needed only for obtaining quantitative (rather than qualitative) closed-loop stability conditions, and in further contrast to the approaches from the literature, not only homogeneous but also nonhomogeneous closed loops may be obtained using it. Though none of the existing approaches aim to preserve a nominal controller behavior, they have several other interesting features. Some of those are illustrated by comparing two approaches[5,6] from literature with the proposed technique in the course of a simulation.

The paper is structured as follows: After some preliminaries and notational details in Section 2, the considered problem is stated in Section 3. Section 4 then discusses basic assumptions and properties of the nominal homogeneous state-feedback control law, which forms the basis of the proposed technique. The integral extension technique is proposed in Section 5, and its structural and performance preserving properties are shown. Closed-loop stability is analyzed in Section 6 for both constant and Lipschitz continuous perturbations, and conditions for global asymptotic and finite-time stability are derived. Section 7, finally, compares the proposed technique with two approaches found in the literature, and Section 8 gives concluding remarks. An Appendix contains the proofs of all lemmas, whereas theorems and propositions are proven in the main text.

2 | PRELIMINARIES

The abbreviations

$$\lfloor y \rceil^p = |y|^p \operatorname{sign}(y) \tag{1}$$

for $y \in \mathbb{R}$ and $p \neq 0$, as well as $\lfloor y \rceil^0 = \operatorname{sign}(y)$ are used throughout the paper. Matrices and vectors are written as boldface symbols, and $\mathbb{R}$ denotes the field of real numbers. The Euclidian norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is written as $\|\mathbf{x}\|$.

Some notions of generalized homogeneity of scalar valued functions and vector fields are now discussed. Let a vector $\mathbf{r} = [r_1 \quad r_2 \quad \dots \quad r_n]^T$ of weights $r_i > 0$, $i = 1, \dots, n$ be given. The associated dilation matrix $\mathbf{D}_\alpha^\mathbf{r}$ is defined as

$$\mathbf{D}_\alpha^\mathbf{r} = \begin{bmatrix} \alpha^{r_1} & & \\ & \ddots & \\ & & \alpha^{r_n} \end{bmatrix}. \tag{2}$$

The scalar valued function $V : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $m \in \mathbb{R}$ with respect to the weights $\mathbf{r}$, denoted by $\deg_\mathbf{r} V = m$, if

$$V(\mathbf{D}_\alpha^\mathbf{r}\mathbf{x}) = \alpha^m V(\mathbf{x}) \tag{3}$$

holds for all $\alpha > 0$ and for all $\mathbf{x} \in \mathbb{R}^n$. Furthermore, $\deg_\mathbf{r} x_i$ is written for the weight of a single variable x_i, that is, $\deg_\mathbf{r} x_i = r_i$. The vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous of degree m with respect to the weights $\mathbf{r}$ if

$$\mathbf{f}(\mathbf{D}_\alpha^\mathbf{r}\mathbf{x}) = \alpha^m \mathbf{D}_\alpha^\mathbf{r}\mathbf{f}(\mathbf{x}) \tag{4}$$

holds for all $\alpha > 0$ and for all $\mathbf{x} \in \mathbb{R}^n$, that is, if $\deg_\mathbf{r} f_i = r_i + m$ holds for the components of $\mathbf{f}$. By appropriate choice of the weight vector $\mathbf{r}$ one can always normalize a vector field's homogeneity degree to either—in case of a negative degree—minus one,—in case of positive degree—plus one, or zero otherwise.

3 | PROBLEM STATEMENT

To simplify considerations, the technique is demonstrated using a perturbed integrator chain as a plant. Such a system is routinely obtained by applying exact linearization techniques to a nonlinear plant,[10] or when considering sliding variable dynamics in the context of sliding mode control.[2] Therefore, the proposed technique may straightforwardly be applied to problems of sliding-mode or other nonlinear control problems with nonlinear plant dynamics.

Consider a perturbed nth order integrator chain governed by the differential equation

$$\frac{\mathrm{d}^n\sigma}{\mathrm{d}t^n} = u + w \tag{5}$$

with output σ, control input u, and a matched disturbance w. The disturbance is assumed to be Lipschitz continuous with its time derivative $\dot{w}$ bounded by

$$|\dot{w}| \leq L. \tag{6}$$

The control task is to steer σ to zero in finite time and keep it there by means of a continuous control signal u, that is, to achieve $\sigma(t) = 0$ for all $t \geq T$ after some finite time T depending on the initial conditions.

Without the disturbance, that is, for $w = 0$, the system may be stabilized in finite time by means of continuous homogeneous state-feedback control. One such control law for $n = 2$, for example, is proposed by Bacciotti and Rosier[1] based on an example due to Bhat and Bernstein[11] as

$$u = -k_1 \lfloor \sigma \rceil^{\frac{1}{3}} - k_2 \lfloor \dot{\sigma} \rceil^{\frac{1}{2}} \tag{7}$$

with positive parameters k_1,k_2. Similarly structured control laws exist also for higher plant order.[12] For nonvanishing w, however, such state-feedback control laws can achieve neither finite-time nor asymptotic convergence, because they cannot compensate for nonzero disturbances in equilibrium.

In this paper, the following problem is considered: Given any finite-time stabilizing homogeneous state-feedback control law, extend it by an integral part in such a way that (i) finite-time convergence is achieved in the perturbed case, and (ii) performance is preserved in the sense that, in the unperturbed case, the closed loop behaves as the original state feedback control law.

4 | FINITE-TIME STABILIZING HOMOGENEOUS CONTROL LAWS

In state-space form, the perturbed integrator chain may be written as

$$\dot{x}_i = x_{i+1} \quad \text{for} \ \ i = 1, \ldots, n-1 \tag{8a}$$

$$\dot{x}_n = u + w \tag{8b}$$

with state variables $x_1, \ldots, x_n$ aggregated in the vector $\mathbf{x} := [x_1 \quad \ldots \quad x_n]^\mathrm{T}$ and with the bound

$$|\dot{w}| \leq L \tag{8c}$$

on the perturbation's slope $\dot{w}$. Homogeneous control laws for this system are considered, which stabilize it in finite-time by means of a continuous control signal. This implies that the closed loop's homogeneity degree is negative. In order to simplify the further analysis by removing all redundant degrees of freedom, this homogeneity degree is fixed at $m = -1$, without restriction of generality. This requires the homogeneity weights to satisfy

$$r_{i+1} = \deg_{\mathbf{r}} x_{i+1} = \deg_{\mathbf{r}} \dot{x}_i = r_i + m = r_i - 1 \tag{9}$$

for $i = 1, \ldots, n-1$. Therefore, the weight vector's structure is fully determined as

$$\mathbf{r} = [\beta + (n-1) \quad \beta + (n-2) \quad \dots \quad \beta + 1 \quad \beta]^{\mathrm{T}} \tag{10}$$

with a single scalar parameter $\beta > 1$. Since this parameter β fully determines the weights in the dilation matrix (2), it is called the *dilation generator* in the following. Throughout the paper, the short-hand notation $\deg_\beta f := \deg_{\mathbf{r}} f$, with $\mathbf{r}$ as defined in (10), is used for the homogeneity degree of some function f with respect to the weights $\mathbf{r}$ generated by β. Note that this notation is consistent also for $n = 1$, because then $\mathbf{r} = \beta$.

In the following, a nominal state-feedback control law is assumed to be given by

$$u = -k(\mathbf{x}) \tag{11}$$

with a continuous homogeneous function k, which stabilizes the unperturbed plant. This is reflected in the following definition.

Definition 1. A continuous homogeneous function $k : \mathbb{R}^n \to \mathbb{R}$ is called a nominal stabilizing feedback, if the origin of the closed-loop system, formed by the control law $u = -k(\mathbf{x})$ and the plant (8) with $w = 0$, is globally asymptotically stable.

For the overall closed loop to be homogeneous with degree $m = -1$, the homogeneity degree of this function has to be $\deg_\beta k = \beta - 1 > 0$. In this case, asymptotic stability is equivalent to global finite-time stability. One family of such control laws for arbitrary order n is proposed by Bhat and Bernstein.[12] With the introduced notation—the originally proposed controller is parametrized using $\frac{\beta-1}{\beta} \in (0, 1)$ rather than β—it is given by

$$u = -k_1 \lfloor x_1 \rceil^{\frac{\beta-1}{\beta+n-1}} - k_2 \lfloor x_2 \rceil^{\frac{\beta-1}{\beta+n-2}} - \dots - k_{n-1} \lfloor x_{n-1} \rceil^{\frac{\beta-1}{\beta+1}} - k_n \lfloor x_n \rceil^{\frac{\beta-1}{\beta}} \tag{12}$$

with positive parameters $k_1, \dots, k_n$ and $\beta > 1$. It is shown[12] that if $k_1 + k_2 s + \dots + k_n s^{n-1} + s^n$ is a Hurwitz polynomial, then the resulting closed loop is globally finite-time stable for sufficiently large values of β, that is, for all $\beta > \beta^*$ with some sufficiently large $\beta^* \geq 1$. It may be noted that in this case the exponents in (12) are (sufficiently) close to one.

5 | PERFORMANCE PRESERVING INTEGRAL EXTENSIONS

This section introduces the proposed integral extension technique and discusses the choice of its design parameters. Section 5.1 shows how to construct the proposed controller using a homogeneous function and a scalar positive gain as parameters. This yields an entire family of control laws with continuous and discontinuous integrands. The resulting closed-loop structure and a performance preserving property of the proposed controller is then shown in Section 5.2. Section 5.3 discusses the choice of the homogeneous function parametrizing the family of controllers.

5.1 | Integral extension

Consider a dilation generator $\beta > 1$ and a control law $u = -k(\mathbf{x})$ with a nominal stabilizing feedback function k of degree $\deg_\beta k = \beta - 1$. An integral extension of this controller is proposed as

$$u = -k(\mathbf{x}) - k_{\mathrm{I}} h(\mathbf{x}) + k_{\mathrm{I}} v \tag{13a}$$

$$\dot{v} = g(\mathbf{x}). \tag{13b}$$

Therein, k_{I} is a positive parameter, h is a homogeneous function to be chosen, its homogeneity degree $\alpha = \deg_\beta h > 1$ represents another important parameter, and the homogeneous function g is given by

$$g(\mathbf{x}) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2} x_3 + \dots + \frac{\partial h}{\partial x_{n-1}} x_n - \frac{\partial h}{\partial x_n} k(\mathbf{x}) = -\frac{\partial h}{\partial x_n} k(\mathbf{x}) + \sum_{i=1}^{n-1} \frac{\partial h}{\partial x_i} x_{i+1}. \tag{13c}$$

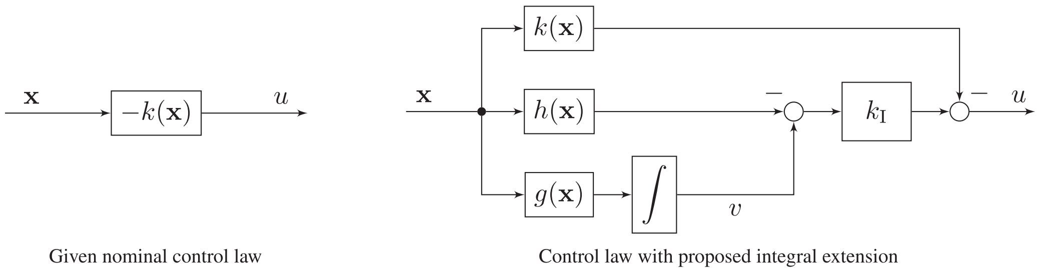

FIGURE 1 Block diagrams of the nominal controller and of the proposed integral extension

TABLE 1 Structural properties of the closed loop obtained with the extended control law (13) for different values of the generating feedback's homogeneity degree $\alpha \geq 1$ and the dilation generator $\beta > 1$

Conditions	Homogeneity of Closed-Loop System	Continuity of Integrand $g(\mathbf{x})$
$\alpha > 1, \beta \neq 1 + \alpha$	Nonhomogeneous	Continuous
$\alpha > 1, \beta = 1 + \alpha$	Homogeneous	Continuous
$\alpha = 1, \beta \neq 2$	Nonhomogeneous	Discontinuous
$\alpha = 1, \beta = 2$	Homogeneous	Discontinuous

Note that the control law's parameter k_I may be interpreted as an integrator gain. The values of α and β determine the closed loop's structural properties; this will be discussed in more detail later on. Figure 1illustrates the structure of the control law.

For the proposed control law to be well-defined and in order to guarantee closed-loop stability later on, the function h, which generates it, has to have some properties that are summarized in the following definition. Its first item ensures that the control signal is continuous and that the integrand g is locally bounded and continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, which is reasonable from a practical point of view. The second item will be shown to be important for ensuring closed-loop stability with the proposed control law.

Definition 2. A homogeneous function $h : \mathbb{R}^n \to \mathbb{R}$ is called an admissible generating feedback, if it has the following two properties:

(i) h is continuous on $\mathbb{R}^n$ and continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$;
(ii) the partial derivative of h with respect to x_n is strictly positive on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, that is, $\frac{\partial h}{\partial x_n}(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

Remark 1. Note that item (i) in particular implies that partial derivatives of h are continuous and thus locally bounded on $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

The generating feedback's homogeneity degree α and the dilation generator β determine whether the extended control law and thus the overall closed loop are homogeneous and whether the integrand $\dot{v}$ is continuous or discontinuous with respect to the state $\mathbf{x}$. This is stated in the following proposition and is also summarized in Table 1.

Proposition 1 (Structural properties of the controller). *Let a dilation generator $\beta > 1$, an admissible generating feedback h with $\deg_\beta h = \alpha \geq 1$, and a nominal stabilizing feedback k with $\deg_\beta k = \beta - 1$ be given and consider the control law (13). Then, the following statements hold:*

(i) the control input u as a function of $\mathbf{x}$ and v in (13a) is homogeneous (with $\deg_\beta v = \alpha$) if and only if $\beta = 1 + \alpha$;
(ii) the integrand $g(\mathbf{x})$ in (13b) is locally bounded on $\mathbb{R}^n$ and continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$;
(iii) the integrand $g(\mathbf{x})$ in (13b) is globally bounded on $\mathbb{R}^n$ and discontinuous at the origin if and only if $\alpha = 1$.

Proof. From (13c) one can see that g is the time derivative of h along the nominal closed loop, and thus $\deg_\beta g = \alpha - 1 \geq 0$. The first statement is then clear from the fact that $k_I v - k_I h - k$ is homogeneous if and only if $\deg_\beta h = \deg_\beta k$ and the weight of v is chosen as $\deg_\beta v = \deg_\beta h$.

For the second statement, note that all partial derivatives of h and, thus also g, are locally bounded on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ due to item (i) of Definition 2. Since g is homogeneous with nonnegative homogeneity degree, it is then locally bounded also at the origin. For the third statement, note that homogeneity implies (Reference 12, theorem 4.1) continuity of g unless $\deg_\beta g \leq 0$, which shows necessity of $\alpha = 1$. To show also sufficiency, note that g being locally bounded with $\deg_\beta g = 0$ implies that g is globally bounded and either discontinuous or constant. Assume now that g is constant and equal to $G_0 \in \mathbb{R}$. Since $\frac{\partial h}{\partial x_n}$ is sign definite and $h(\mathbf{0}) = 0$, there exists a vector $\mathbf{x}_0 \in \mathbb{R}^n$ satisfying $h(\mathbf{x}_0) \neq 0$ and $h(\mathbf{x}_0)G_0 \geq 0$. Then, $|h(\mathbf{x}(t))|$ is positive and nondecreasing along the trajectory $\mathbf{x}(t)$ of the nominal closed loop with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, because $\frac{d}{dt}h(\mathbf{x}(t)) = g(\mathbf{x}(t)) = G_0$, but $\mathbf{x}(t)$ converges to zero in finite time. This contradicts the fact $h(\mathbf{0}) = 0$; therefore, g is discontinuous. ■

Example 1 (Controller for the first order integrator—super-twisting algorithm). As an example, consider a first-order integrator $\dot{x} = u + w$. For a given dilation generator β, the only admissible generating feedback with homogeneity degree $\alpha \geq 1$—up to a scaling, which is however redundant with k_I— is $h(x) = \frac{\beta}{\alpha} \lfloor x \rceil^{\frac{\alpha}{\beta}}$. One verifies that its derivative $\frac{\partial h}{\partial x} = |x|^{\frac{\alpha-\beta}{\beta}}$ is strictly positive and locally bounded on $\mathbb{R} \setminus \{\mathbf{0}\}$. Consider the nominal stabilizing state feedback $k(x) = k_\mathrm{S} \lfloor x \rceil^{\frac{\beta-1}{\beta}}$ with constant positive gain k_S. The function g in (13c) is then given by

$$g(x) = -\frac{\partial h}{\partial x} k(x) = -k_\mathrm{S} \lfloor x \rceil^{\frac{\alpha-1}{\beta}}, \tag{14}$$

and thus the proposed extended control law (13) is

$$u = -k_\mathrm{S} \lfloor x \rceil^{\frac{\beta-1}{\beta}} - k_\mathrm{I} \frac{\beta}{\alpha} \lfloor x \rceil^{\frac{\alpha}{\beta}} + k_\mathrm{I} v, \quad \dot{v} = -k_\mathrm{S} \lfloor x \rceil^{\frac{\alpha-1}{\beta}}. \tag{15}$$

One can see that the integrand is discontinuous if $\alpha = 1$, and that in general the control law is not homogeneous. For the special case $\beta = 1 + \alpha \geq 2$ and with the abbreviations $\nu = k_\mathrm{I} v$, $k_1 = k_\mathrm{S} + k_\mathrm{I} \frac{\beta}{\alpha}$, and $k_2 = k_\mathrm{S} k_\mathrm{I}$ one obtains the homogeneous control law

$$u = -k_1 \lfloor x \rceil^{\frac{\alpha}{\alpha+1}} + \nu, \qquad \dot{\nu} = -k_2 \lfloor x \rceil^{\frac{\alpha-1}{\alpha+1}}. \tag{16}$$

For $\alpha = 1$, the integral term is discontinuous and the control law corresponds to the well-known super-twisting algorithm.[13]

5.2 | Closed-loop system and performance preserving property

The proposed state-feedback controller with integral part permits to preserve the performance of the nominal closed loop. In order to show this, and to investigate stability later on, the closed-loop system is considered. Introducing the state variable $q = k_\mathrm{I} v + w$, one finds that the closed loop formed by the plant (8) and control law (13) is governed by the differential equations

$$\dot{x}_i = x_{i+1} \quad \text{for } i = 1, \ldots, n-1 \tag{17a}$$

$$\dot{x}_n = -k(\mathbf{x}) - k_\mathrm{I} h(\mathbf{x}) + q \tag{17b}$$

$$\dot{q} = k_\mathrm{I} g(\mathbf{x}) + \dot{w}. \tag{17c}$$

Since g may be discontinuous in the origin if $\alpha = 1$ and since $\dot{w}$ may take any value in the interval $[-L, L]$, solutions of this system are defined in the sense of Filippov.[14] In particular, (17c) with $|\dot{w}| \leq L$ is to be understood as the differential inclusion

$$\dot{q} \in \begin{cases} [k_\mathrm{I} g(\mathbf{x}) - L, k_\mathrm{I} g(\mathbf{x}) + L] & \mathbf{x} \neq \mathbf{0} \\ [-k_\mathrm{I} G_- - L, k_\mathrm{I} G_+ + L] & \mathbf{x} = \mathbf{0}, \end{cases} \tag{18a}$$

with

$$G_- = -\liminf_{\mathbf{x}\to\mathbf{0}} g(\mathbf{x}), \quad G_+ = \limsup_{\mathbf{x}\to\mathbf{0}} g(\mathbf{x}). \tag{18b}$$

If $\alpha > 1$ and thus $\deg_\beta g > 0$, then g is continuous in the origin and these quantities are $G_- = G_+ = g(\mathbf{0}) = 0$. Otherwise, $G_- = -\inf_{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} g(\mathbf{x})$ and $G_+ = \sup_{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} g(\mathbf{x})$ by homogeneity, and the infimum and supremum can be found by computing the minimum and maximum of $g(\mathbf{x})$ on the unit sphere, for example, where g is continuous.

While (17) is a well-defined Filippov inclusion describing the dynamics of the closed-loop system, it does not provide any immediate insight into its properties. A more insightful description is obtained by choosing the state variable

$$z = -k_\mathrm{I} h(\mathbf{x}) + k_\mathrm{I} v + w, \tag{19}$$

which for an admissible generating feedback h is differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, but not necessarily in the origin. Since the time derivative of h along the closed-loop trajectories satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}h(\mathbf{x}) = \frac{\partial h}{\partial x_1}x_2 + \frac{\partial h}{\partial x_2}x_3 + \ldots + \frac{\partial h}{\partial x_{n-1}}x_n + \frac{\partial h}{\partial x_n}(-k(\mathbf{x}) - k_\mathrm{I} h(\mathbf{x}) + k_\mathrm{I} v + w) = g(\mathbf{x}) + \frac{\partial h}{\partial x_n}z \tag{20}$$

when it is differentiable, one finds that for $\mathbf{x} \neq \mathbf{0}$ the closed loop is equivalently governed by the differential equations

$$\dot{x}_i = x_{i+1} \quad \text{for } i = 1, \ldots, n-1 \tag{21a}$$

$$\dot{x}_n = -k(\mathbf{x}) + z \tag{21b}$$

$$\dot{z} = -k_\mathrm{I}\frac{\partial h}{\partial x_n}z + \dot{w}. \tag{21c}$$

As noted, $\frac{\partial h}{\partial x_n}$ may not be differentiable for $\mathbf{x} = \mathbf{0}$ and even has a singularity (ie, an essential discontinuity) in the origin if $\alpha < \beta$, because $\deg_\beta \frac{\partial h}{\partial x_n} = \alpha - \beta$. Therefore, (21) does not necessarily correspond to a well-defined Filippov inclusion. Nevertheless, the differential equations may be used to compute time derivatives for $\mathbf{x} \neq \mathbf{0}$ later on.

For constant perturbations, that is, for $\dot{w} = 0$, relation (21c) suggests that z will decay to zero for $k_\mathrm{I} > 0$, because $\frac{\partial h}{\partial x_n}$ is strictly positive. In particular, if the initial condition $z(0)$ is zero, one may expect that $z(t) = 0$ will hold indefinitely, thus preserving the nominal performance obtained with $u = -k(\mathbf{x})$. The following proposition and the stability analysis in Section 6 will show that both of these intuitions are correct and, moreover, that $\mathbf{x}$ and z even converge to zero in finite time under certain conditions.

Proposition 2 (Performance preserving property). *Let a dilation generator $\beta > 1$, a nominal stabilizing feedback k with $\deg_\beta k = \beta - 1$, and an admissible generating feedback h with $\deg_\beta h = \alpha > 1$ be given. If $k_\mathrm{I} \geq 0$, the integrator's initial condition is chosen as $v(0) = h(\mathbf{x}(0))$, and the perturbation satisfies $w(t) = 0$ for all $t \geq 0$, then applying either the nominal controller $u = -k(\mathbf{x})$ or the extended controller (13) to the plant (8) both yields the same closed-loop trajectory $\mathbf{x}(t)$.*

Remark 2. Although the proposition only considers the unperturbed case, a performance preserving effect is observed also in the perturbed case for sufficiently large values of the integrator gain k_I. This is demonstrated in Section 7 in the course of simulation studies with different values of k_I. It can also be seen intuitively from the definition of z in (19) and the corresponding differential equation (21c). As z tends to zero, the term $k_\mathrm{I} h(\mathbf{x}) - k_\mathrm{I} v$ reconstructs (ie, tends to) the perturbation w and the nominal system is recovered from (21). This happens the faster the larger the gain k_I is.

Proof. Let $\mathbf{x}(t)$ be a trajectory of the nominal closed loop obtained from (8) with $w = 0$, $u = -k(\mathbf{x})$ and denote by T the corresponding finite convergence time. Then, since $\mathbf{x}(t) \neq \mathbf{0}$ for all $t < T$, one can use (21) to verify that this trajectory along with $z(t) = 0$ satisfies the differential inclusion (17) for $t < T$. For $t \geq T$, finally, one can verify that $\mathbf{x}(t) = 0$ and $z(t) = q(t) - k_\mathrm{I} h(\mathbf{x}(t)) = q(t) = 0$ also satisfies (17). Therefore, the trajectories of both systems coincide, which concludes the proof. ∎

5.3 | Design of the generating feedback

For $\alpha = \beta$, the relation $\deg_\beta h = \beta = \deg_\beta x_n$ suggests to choose the generating feedback $h(\mathbf{x}) = x_n$, which is indeed admissible. In the general case, however, it is not immediately obvious how to find a generating feedback h satisfying all conditions of Definition 2. The following lemma, which is proven in the appendix, gives a sufficient condition for such a function:

Lemma 1. *Let a dilation generator $\beta > 1$ and two homogeneous functions $p, q : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be given, whose homogeneity degrees satisfy $\deg_\beta p < \deg_\beta q$. Suppose that p and q are continuously differentiable, that p is strictly positive, and that*

$$p(\mathbf{x})\frac{\partial q}{\partial x_n} > q(\mathbf{x})\frac{\partial p}{\partial x_n} \tag{22}$$

holds for all $\mathbf{x} \neq 0$. Then, the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by $h(\mathbf{0}) = 0$ and

$$h(\mathbf{x}) = \frac{q(\mathbf{x})}{p(\mathbf{x})} \tag{23}$$

for $\mathbf{x} \neq \mathbf{0}$ is an admissible generating feedback.

Based on this lemma, the following proposition gives a useful class of generating feedback functions for any given homogeneity degree $\alpha \geq 1$.

Proposition 3 (Class of admissible generating feedbacks). *Let a dilation generator $\beta > 1$, a desired homogeneity degree $\alpha \geq 1$, and a real-valued constant $\gamma > \beta + n - 1$ be given. Then, for all positive constants $a_1, \ldots, a_n$ the function $h : \mathbb{R}^n \to \mathbb{R}$ given by*

$$h(\mathbf{x}) = \frac{x_n}{\left(a_1|x_1|^{\frac{\gamma}{\beta+n-1}} + a_2|x_2|^{\frac{\gamma}{\beta+n-2}} + \ldots + a_{n-1}|x_{n-1}|^{\frac{\gamma}{\beta+1}} + a_n|x_n|^{\frac{\gamma}{\beta}}\right)^{\frac{\beta-\alpha}{\gamma}}} = \frac{x_n}{\left(\sum_{i=1}^n a_i|x_i|^{\frac{\gamma}{\beta+n-i}}\right)^{\frac{\beta-\alpha}{\gamma}}} \tag{24}$$

is an admissible generating feedback with homogeneity degree $\deg_\beta h = \alpha$.

Remark 3. Note that there are no restrictions on the sign of $\alpha - \beta$, and for $\alpha = \beta$ in particular, $h(\mathbf{x}) = x_n$ is obtained.

Proof. Comparing h with (23), one can see that the denominator p and the numerator q are given by the homogeneous functions

$$p(\mathbf{x}) = \left(\sum_{i=1}^n a_i|x_i|^{\frac{\gamma}{\beta+n-i}}\right)^{\frac{\beta-\alpha}{\gamma}}, \quad q(\mathbf{x}) = x_n \tag{25}$$

with degrees $\deg_\beta p = \beta - \alpha < \beta = \deg_\beta q$. Therefore, h is homogeneous with degree $\deg_\beta h = \deg_\beta q - \deg_\beta p = \alpha$ and it has the form required by Lemma 1.

It will be shown that the functions p,q also satisfies the other conditions of that lemma. Obviously, q is continuous and continuously differentiable and p is continuous and strictly positive on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Furthermore, p is continuously differentiable for all nonzero $\mathbf{x}$, because $\frac{\gamma}{\beta+n-i} > 1$ for all $i = 1, \ldots, n$, that is, all exponents of single variable expressions are greater than one. Finally, one finds that

$$\frac{q(\mathbf{x})\frac{\partial p}{\partial x_n}}{p(\mathbf{x})\frac{\partial q}{\partial x_n}} = \frac{x_n\frac{\partial p}{\partial x_n}}{p(\mathbf{x})} = \frac{\frac{\beta-\alpha}{\beta}\left(\sum_{i=1}^n a_i|x_i|^{\frac{\gamma}{\beta+n-i}}\right)^{\frac{\beta-\alpha}{\gamma}-1} a_n|x_n|^{\frac{\gamma}{\beta}}}{\left(\sum_{i=1}^n a_i|x_i|^{\frac{\gamma}{\beta+n-i}}\right)^{\frac{\beta-\alpha}{\gamma}}} = \frac{\frac{\beta-\alpha}{\beta}a_n|x_n|^{\frac{\gamma}{\beta}}}{\sum_{i=1}^n a_i|x_i|^{\frac{\gamma}{\beta+n-i}}} \leq \frac{\beta-\alpha}{\beta} < 1 \tag{26}$$

and thus (22) holds for all $\mathbf{x} \neq \mathbf{0}$. Therefore, h is an admissible generating feedback. ■

Example 2 (Controller for the second order integrator chain). It is now shown how an integral controller previously studied in a conference paper[15] is obtained. Consider a second order integrator chain $\dot{x}_1 = x_2$, $\dot{x}_2 = u + w$ with dilation generator $\beta = 2$ and nominal stabilizing feedback

$$k(x_1, x_2) = k_1 \lfloor x_1 \rceil^{\frac{1}{3}} + k_2 \lfloor x_2 \rceil^{\frac{1}{2}} \tag{27}$$

with positive parameters k_1, k_2. The corresponding control law $u = -k(x_1, x_2)$ stabilizes the unperturbed double integrator in finite time for all positive parameter values.[1] Using Proposition 3 with $\gamma = 4$, $a_1 = \frac{3k_1}{2}$, $a_2 = 2$, the function

$$h(x_1, x_2) = \frac{x_2}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{2-\alpha}{4}}} \tag{28}$$

is found to be an admissible generating feedback with homogeneity degree $\deg_\beta h = \alpha \geq 1$. Its partial derivatives are given by

$$\frac{\partial h}{\partial x_1} = -\frac{\frac{(2-\alpha)k_1}{2} x_2 \lfloor x_1 \rceil^{\frac{1}{3}}}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{6-\alpha}{4}}}, \qquad \frac{\partial h}{\partial x_2} = \frac{\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + \alpha|x_2|^2}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{6-\alpha}{4}}}. \tag{29}$$

One can see that they are locally bounded on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and that $\frac{\partial h}{\partial x_2}$ is furthermore strictly positive. The function g in (13c) is computed as

$$g(x_1, x_2) = \frac{\partial h}{\partial x_1} x_2 - \frac{\partial h}{\partial x_2} k(x_1, x_2) = -\frac{\frac{(2+\alpha)k_1}{2}|x_2|^2 \lfloor x_1 \rceil^{\frac{1}{3}} + \frac{3k_1k_2}{2}\lfloor x_2 \rceil^{\frac{1}{2}}|x_1|^{\frac{4}{3}} + \frac{3k_1^2}{2}\lfloor x_1 \rceil^{\frac{5}{3}} + \alpha k_2 \lfloor x_2 \rceil^{\frac{5}{2}}}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{6-\alpha}{4}}}, \tag{30}$$

and from $g(0, x_2) = -2^{-\frac{5}{4}} \alpha k_2 \lfloor x_2 \rceil^{\alpha - 1}$ one can see that it is discontinuous in the origin if $\alpha = 1$. For this latter value of α and after simplifying the above expression, the proposed control law (13) is given by

$$u = -k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}} - \frac{k_I x_2}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{1}{4}}} + k_I v \tag{31a}$$

$$\dot{v} = -\frac{k_1 \lfloor x_1 \rceil^{\frac{1}{3}} + k_2 \lfloor x_2 \rceil^{\frac{1}{2}}}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{1}{4}}} + \frac{\frac{k_1}{2}\lfloor x_1 \rceil^{\frac{1}{3}}|x_2|^2 + k_2 \lfloor x_2 \rceil^{\frac{5}{2}}}{\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + 2|x_2|^2\right)^{\frac{5}{4}}}. \tag{31b}$$

It is worth to point out that, similar to the so-called quasi-continuous[16] sliding-mode controllers, the right-hand side of (31b) is discontinuous only in the origin, but is continuous everywhere else.

6 | STABILITY ANALYSIS

In the following, the stability properties of the perturbed closed loop system (17) are studied. Like the structural properties of the control law in Table 1, also the closed loop's stability and robustness properties depend on the dilation generator β and the generating feedback's homogeneity degree α. If the function g is continuous, that is, if $\alpha \neq 1$, stability can only be guaranteed for constant perturbations with $L = 0$, because the origin is not even an equilibrium otherwise. Section 6.2 discusses this case, showing that choosing $\alpha < \beta$ permits to ensure finite-time stability, while only asymptotic convergence can be guaranteed otherwise. For $\alpha = 1$, on the other hand, it will be shown in Section 6.3 that Lipschitz continuous perturbations with arbitrarily large Lipschitz constant L can be handled by appropriate tuning of the parameter k_I. Table 2 shows an overview of the stability and robustness properties that will be proven in the following.

Conditions	Closed-Loop Stability	Perturbation Class	Stability Condition
$\alpha \geq \beta > 1$	Asymptotic	Constant ($L = 0$)	$k_\mathrm{I} > 0$
$\beta > \alpha > 1$	Finite-time	Constant ($L = 0$)	$k_\mathrm{I} > 0$
$\beta > \alpha = 1$	Finite-time	Lipschitz ($L \geq 0$)	$k_\mathrm{I} > \mu L$

TABLE 2 Global stability and robustness properties of the closed loop formed by plant (8) with extended control law (13) for different values of the generating feedback's homogeneity degree $\alpha \geq 1$ and the dilation generator $\beta > 1$ (with $\mu > 0$ sufficiently large, see Theorem 2 in Section 6.3.1 and its Corollary 1 in Section 6.3.2)

6.1 | Nominal Lyapunov function

The stability analysis is based on a Lyapunov function for the *nominal* closed loop, which is obtained by applying the nominal state-feedback control law $u = -k(\mathbf{x})$ to the unperturbed plant (8). Its dynamics are governed by

$$\dot{x}_i = x_{i+1} \quad \text{for } i = 1, \ldots, n-1 \tag{32a}$$

$$\dot{x}_n = -k(\mathbf{x}). \tag{32b}$$

The existence of a Lyapunov function for this system is guaranteed by standard converse Lyapunov results.[12] This is stated in the following lemma and proven in the appendix.

Lemma 2. *Let a dilation generator $\beta > 1$, a nominal stabilizing feedback k with $\deg_\beta k = \beta - 1$, and a constant $\alpha \geq 1$ be given. Then, there exist constants c_1,c_2 and a continuous, homogeneous, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ with homogeneity degree $\deg_\beta V = \alpha$, which is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and whose time derivative $\dot{V}$ along the trajectories of the nominal closed loop (32) and partial derivative $\frac{\partial V}{\partial x_n}$ satisfy*

$$\dot{V} \leq -c_1 V(\mathbf{x})^{\frac{\alpha-1}{\alpha}}, \qquad \left|\frac{\partial V}{\partial x_n}\right| \leq c_2 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \tag{33}$$

for all $\mathbf{x} \neq \mathbf{0}$.

6.2 | Constant perturbation

The case of a constant perturbation, that is, $L = 0$, is considered first. For this class of perturbations, global asymptotic stability may be guaranteed regardless of α and β, and global finite-time stability is achieved if $\alpha < \beta$. The former is proven by showing that, with z as in (19) and a nominal Lyapunov function V as in Lemma 2,

$$\overline{V}(\mathbf{x}, z) = V(\mathbf{x}) + \delta|z| \tag{34}$$

is a Lyapunov function for the actual closed loop (17) for sufficiently large values of $\delta > 0$. In order to show finite-time stability, a contraction property of the nonnegative expressions $V(\mathbf{x})$ and $|z|$ is furthermore employed. Proving this requires several technical arguments that also have to deal with the fact that $\overline{V}$ is not everywhere differentiable. Since these offer little additional insight, the main statements are encapsulated in the following technical lemma, whose proof the interested reader finds in the Appendix.

Lemma 3. *Let a dilation generator $\beta > 1$ and an admissible generating feedback h with $\deg_\beta h = \alpha \geq 1$ be given. Let the function $V : \mathbb{R}^n \to \mathbb{R}$ be continuous, positive definite with homogeneity degree $\deg_\beta V = \alpha$ and denote by $Q : \mathbb{R}^n \to \mathbb{R}$ its time derivative along the trajectories of the nominal closed loop (32). Suppose that there exist positive constants c_1,c_2,c_3 such that the inequalities*

$$Q(\mathbf{x}) \leq -c_1 V(\mathbf{x})^{\frac{\alpha-1}{\alpha}}, \qquad \left|\frac{\partial V}{\partial x_n}\right| \leq c_2 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}}, \qquad \frac{\partial h}{\partial x_n} \geq c_3 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \tag{35}$$

hold for all $\mathbf{x} \neq \mathbf{0}$. For any trajectory $\mathbf{x}(t)$ and $z(t) = q(t) - k_\mathrm{I} h(\mathbf{x}(t))$ of the closed-loop system (17) with $\dot{w}(t) = 0$, define functions $\overline{V}, \zeta_1, \zeta_2 : \mathbb{R} \to \mathbb{R}$ as

$$\overline{V}(t) = V(\mathbf{x}(t)) + \frac{c_2}{k_\mathrm{I} c_3}|z(t)|, \quad \zeta_1(t) = V(\mathbf{x}(t))^{\frac{\beta-1}{\alpha}}, \quad \zeta_2(t) = \frac{2c_2}{c_1}|z(t)|. \tag{36}$$

Then, $\overline{V}$ is nonincreasing and $\lim_{t\to\infty}\overline{V}(t) = 0$. Moreover, if $\alpha < \beta$, then the implication

$$\zeta_1(t_0) \le D, \zeta_2(t_0) \le D \quad \Rightarrow \quad \zeta_1(t_0+\tau) \le \frac{1}{2}D, \zeta_2(t_0+\tau) \le \frac{1}{2}D \quad \text{for all } \tau \ge \tau_D = \frac{\ln 2}{k_\mathrm{I} c_3} D^{\frac{\beta-\alpha}{\beta-1}} + \frac{2\alpha}{c_1} D^{\beta-1} \tag{37}$$

holds for all t_0 and all nonnegative constants D.

Using this lemma, the main stability result for constant perturbations—asymptotic stability for every $k_\mathrm{I} > 0$ and, additionally, finite-time stability for $\alpha < \beta$—is now shown.

Theorem 1 (Closed-loop stability with constant perturbations). *Let a dilation generator $\beta > 1$, an admissible generating feedback h with $\deg_\beta h = \alpha \ge 1$, and a nominal stabilizing feedback k with $\deg_\beta k = \beta - 1$ be given. Consider the closed loop (17) formed by the interconnection of the plant (8) and the control law (13) with integrator gain k_I and constant perturbation w, that is, $L = 0$. Then, the following statements are true:*

(i) if $\alpha \ge \beta$ and $k_\mathrm{I} > 0$, then the closed loop's origin is globally asymptotically stable;
(ii) if $\alpha < \beta$ and $k_\mathrm{I} > 0$, then the closed loop's origin is globally finite-time stable.

Proof. Let V be the nominal Lyapunov function with $\deg_\beta V = \alpha$, whose existence Lemma 2 guarantees. Since $\frac{\partial h}{\partial x_n}$ and V are both strictly positive and continuous $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and since $\deg_\beta \frac{\partial h}{\partial x_n} = \alpha - \beta$, the function

$$\ell(\mathbf{x}) = \frac{\frac{\partial h}{\partial x_n}}{V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}}} \tag{38}$$

is homogeneous of degree zero and is continuous and strictly positive on the unit sphere characterized by $\|\mathbf{x}\| = 1$. Hence, there exists a positive constant c_3 given by $c_3 = \min_{\|\mathbf{x}\|=1} \ell(\mathbf{x})$ such that

$$\frac{\partial h}{\partial x_n} \ge c_3 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \tag{39}$$

holds for all $\mathbf{x} \neq \mathbf{0}$. Consequently, the conditions of Lemma 3 are fulfilled and the positive definite, radially unbounded function $\overline{V}(\mathbf{x}, z) = V(\mathbf{x}) + \frac{c_2}{k_\mathrm{I} c_3}|z|$ is nonincreasing and converges to zero along every closed-loop trajectory. This implies global asymptotic stability of the origin.

Consider now the case $\alpha < \beta$ and define

$$D_0 = \max\left(V(\mathbf{x}(0))^{\frac{\beta-1}{\alpha}}, \frac{2c_2}{c_1}|z(0)|\right). \tag{40}$$

Repeatedly applying implication (37) from Lemma 3 then shows that the time T for V and $|z|$ to converge to zero is bounded from above by the sum of two geometric series as

$$T \le \sum_{i=0}^{\infty} \tau_{2^{-i}D_0} = \frac{\ln 2}{c_3} \sum_{i=0}^{\infty} \left(\frac{D_0}{2^i}\right)^{\frac{\beta-\alpha}{\beta-1}} + \frac{2\alpha}{c_1} \sum_{i=0}^{\infty} \left(\frac{D_0}{2^i}\right)^{\beta-1} = \frac{\ln 2}{c_3} \frac{D_0^{\frac{\beta-\alpha}{\beta-1}}}{1 - 2^{\frac{\alpha-\beta}{\beta-1}}} + \frac{2\alpha}{c_1} \frac{D_0^{\beta-1}}{1 - 2^{1-\beta}}. \tag{41}$$

Therefore, the origin is globally finite-time stable. ∎

Example 3 (Robustifying homogeneous state feedback under constant perturbations). Consider any homogeneous state-feedback control law $u = -k(\mathbf{x})$ with dilation generator $\beta > 1$ and a nominal stabilizing feedback k. Selecting $\alpha = \beta$ permits to choose the admissible generating feedback $h(\mathbf{x}) = x_n$. According to Theorem 1, applying the resulting control law

$$u = -k(\mathbf{x}) - k_{\mathrm{I}} x_n + k_{\mathrm{I}} v \tag{42a}$$

$$\dot{v} = -k(\mathbf{x}) \tag{42b}$$

to the plant (8) with constant disturbance w yields an asymptotically stable closed loop for every $k_{\mathrm{I}} > 0$. Although finite-time stability is lost compared to the nominal controller, this control law offers an easy way to make the closed loop robust with respect to constant disturbances. At the same time, nominal performance—and thus also the nominal controller's capability to attenuate other disturbances—is preserved in the sense of Proposition 2 by selecting the integrator's initial condition as $v(0) = x_n(0)$.

6.3 | Lipschitz continuous perturbation

The case of (nonconstant) Lipschitz continuous perturbations, that is, $L > 0$ is now considered. With these perturbations, stability can only be guaranteed for $\alpha = 1$; otherwise, the integrand is not discontinuous and therefore the origin is not an equilibrium of the Filippov inclusion (17). For $\alpha = 1$ and sufficiently large k_{I}, global finite-time stability of the closed loop will be shown using the Lyapunov function (34) with an appropriately selected δ. For this purpose, the following technical lemma is needed, which is proven in the Appendix.

Lemma 4. *Consider the closed-loop system (17). Let $\overline{V} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a continuous, positive definite and radially unbounded function of $\mathbf{x}$ and q and denote by $\dot{\overline{V}}$ its time derivative along the closed loop's trajectories. If a positive constant D exists such that $\dot{\overline{V}}(\mathbf{x}, q)$ is defined and $\dot{\overline{V}} \leq -D$ holds for all $\mathbf{x} \neq \mathbf{0}$, then the closed loop is globally finite-time stable and the convergence time is bounded from above by $\overline{T} = D^{-1} V(\mathbf{x}(0), q(0))$.*

6.3.1 | Qualitative stability condition

The following qualitative stability condition shows that appropriate tuning of k_{I} guarantees global finite-time stability for perturbations with any arbitrarily large Lipschitz constant L.

Theorem 2 (Closed-loop stability with Lipschitz continuous perturbations). *Let a dilation generator $\beta > 1$, an admissible generating feedback h with $\deg_\beta h = \alpha = 1$, and a nominal stabilizing state feedback k with $\deg_\beta k = \beta - 1$ be given. Consider the closed loop (17) formed by the interconnection of the plant (8) and the extended control law (13). Then, there exists a positive constant μ such that for any Lipschitz constant $L \geq 0$ the condition*

$$k_{\mathrm{I}} > \mu L \tag{43}$$

implies that the origin of the closed loop with $|\dot{w}| \leq L$ is globally finite-time stable.

Proof. Let V be the nominal Lyapunov function with $\deg_\beta V = \alpha = 1$, whose existence Lemma 2 guarantees and denote by Q its time derivative along the nominal closed loop (32). Since $\frac{\partial h}{\partial x_n}$ and V both are strictly positive and locally bounded on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and since $\deg_\beta \frac{\partial h}{\partial x_n} = 1 - \beta$, there exists a positive constant c_3 such that

$$\frac{\partial h}{\partial x_n} \geq c_3 V(\mathbf{x})^{1-\beta} \geq \frac{c_3}{c_2} \left| \frac{\partial V}{\partial x_n} \right| \tag{44}$$

holds for all $\mathbf{x} \neq \mathbf{0}$. Consider the positive definite function $\overline{V}(\mathbf{x}, z) = V(\mathbf{x}) + \frac{c_2}{k_{\mathrm{I}} c_3} |z|$ as a candidate Lyapunov function, with z as in (19). For $z = 0$, one has $\overline{V}(\mathbf{x}, z) = V(\mathbf{x})$ and, since the closed loop is governed by (32) in this case, its time derivative along the closed-loop trajectories satisfies $\dot{\overline{V}} = Q(\mathbf{x}) \leq -c_1$. Otherwise, if $z \neq 0$ and $\mathbf{x} \neq \mathbf{0}$, the time derivative of $\overline{V}$ is bounded by

$$\dot{\overline{V}} = Q(\mathbf{x}) + \frac{\partial V}{\partial x_n} z + \frac{c_2}{k_{\mathrm{I}} c_3} \left(-k_{\mathrm{I}} \frac{\partial h}{\partial x_n} |z| + \lfloor z \rceil^0 \dot{w} \right) \leq -c_1 + \frac{\partial V}{\partial x_n} z + \frac{c_2}{k_{\mathrm{I}} c_3} \left(-\frac{k_{\mathrm{I}} c_3}{c_2} \left| \frac{\partial V}{\partial x_n} \right| |z| + L \right) \leq -c_1 + \frac{c_2 L}{k_{\mathrm{I}} c_3}. \tag{45}$$

Thus, one concludes using Lemma 4 that condition (43) with $\mu = \frac{c_2}{c_1 c_3}$ implies global finite-time stability of the closed loop's origin. ∎

6.3.2 | Quantitative stability condition

If a nominal Lyapunov function is available, then a quantitative rather than the qualitative stability condition in the previous theorem may even be obtained. Additionally, it enables the computation of a convergence time bound. This is shown in the following corollary of Theorem 2.

Corollary 1 (Closed-loop stability conditions with Lipschitz continuous perturbations). *Suppose that the conditions of Theorem 2 are fulfilled and let $\tilde{V}$ be a continuous, positive definite function with homogeneity degree $\deg_\beta \tilde{V} = \gamma \geq 1$, which is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and denote by $\tilde{Q}$ its time derivative along the trajectories of the nominal closed loop (32). Let C, F be positive constants, such that the differential inequalities*

$$\tilde{Q}(\mathbf{x}) \leq -C\tilde{V}(\mathbf{x})^{\frac{\gamma-1}{\gamma}}, \qquad \left|\frac{\partial \tilde{V}}{\partial x_n}\right| \leq F\tilde{V}(\mathbf{x})^{\frac{\gamma-1}{\gamma}} \frac{\partial h}{\partial x_n} \tag{46}$$

hold for all $\mathbf{x} \neq \mathbf{0}$. Then, the condition

$$k_\mathrm{I} > \frac{F}{C} L \tag{47}$$

implies that the origin of the actual closed loop (17) with $|\dot{w}| \leq L$ is globally finite-time stable. Furthermore, if the integrator's initial condition is given as in Proposition 2, that is, by $v(0) = h(\mathbf{x}(0))$, and the initial disturbance satisfies $|w(0)| \leq W$, then the closed loop's convergence time is bounded from above by

$$\overline{T} = \frac{\gamma \tilde{V}(\mathbf{x}(0))^{\frac{1}{\gamma}} + \frac{F}{k_\mathrm{I}} W}{C - \frac{F}{k_\mathrm{I}} L}. \tag{48}$$

Proof. Define the function $V(\mathbf{x}) = \tilde{V}(\mathbf{x})^{\frac{1}{\gamma}}$. Denoting by $Q(\mathbf{x})$ its time derivative along the nominal closed loop's trajectories, one finds by means of the chain rule for $\mathbf{x} \neq \mathbf{0}$

$$Q(\mathbf{x}) = \frac{1}{\gamma}\tilde{V}(\mathbf{x})^{\frac{1-\gamma}{\gamma}}\tilde{Q}(\mathbf{x}) \leq -\frac{C}{\gamma}, \quad \left|\frac{\partial V}{\partial x_n}\right| = \left|\frac{1}{\gamma}\tilde{V}(\mathbf{x})^{\frac{1-\gamma}{\gamma}} \frac{\partial \tilde{V}}{\partial x_n}\right| \leq \frac{F}{\gamma}\frac{\partial h}{\partial x_n}. \tag{49}$$

With this V, one has $c_1 = \frac{C}{\gamma}$ and $\frac{c_2}{c_3} = \frac{F}{\gamma}$ in the proof of Theorem 2, which yields $\mu = \frac{F}{C}$ and thus condition (47). The convergence time estimate is furthermore obtained from Lemma 4 using (45) and the fact that the initial conditions satisfy $|z(0)| = |w(0) - k_\mathrm{I} h(\mathbf{x}(0)) + k_\mathrm{I} v(0)| \leq W$. ∎

Example 4 (Quantitative stability conditions for discontinuous integral control). Consider the control law from Example 2 in Section 5.3 for $\alpha = 1$, which is given in (31). A candidate Lyapunov function for the *nominal* closed loop $\dot{x}_1 = x_2, \dot{x}_2 = -k_1\lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}}$ is[15,17]

$$\tilde{V}(x_1, x_2) = \left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + |x_2|^2\right)^{\frac{5}{4}} + \frac{3k_2}{2}x_1 x_2. \tag{50}$$

Its homogeneity degree is $\deg \tilde{V} = \gamma = 5$, and its time derivative $\tilde{Q}$ along the nominal closed loop's trajectories and partial derivative $\frac{\partial \tilde{V}}{\partial x_2}$ can be shown (Reference 15, proposition 8 and lemma 9) to satisfy

$$\tilde{Q}(x_1, x_2) \leq -\left(1 - \frac{2k_2}{3k_1^{\frac{3}{4}}}\right) k_2 \tilde{V}(x_1, x_2)^{\frac{4}{5}}, \qquad \left|\frac{\partial \tilde{V}}{\partial x_2}\right| \leq \frac{19}{3}\tilde{V}(x_1, x_2)^{\frac{4}{5}}\frac{\partial h}{\partial x_2} \tag{51}$$

if $3k_1^{\frac{3}{4}} > 2k_2$. Applying Corollary 1 yields the sufficient conditions

$$3k_1^{\frac{3}{4}} > 2k_2, \quad k_\mathrm{I} > \frac{F}{C}L = \frac{\frac{19}{3}L}{\left(1 - \frac{2k_2}{3k_1^{\frac{3}{4}}}\right)k_2} \tag{52}$$

for finite-time stability of the closed loop's origin with the considered control law. For $x_2(0) = 0$, $v(0) = h(\mathbf{x}(0)) = 0$ and $|w(0)| \leq W$, for example, the convergence time bound

$$\overline{T} = \frac{5\left(\frac{3k_1}{2}|x_1(0)|^{\frac{4}{3}}\right)^{\frac{1}{4}} + \frac{19}{3k_\mathrm{I}}W}{\left(1 - \frac{2k_2}{3k_1^{\frac{3}{4}}}\right)k_2 - \frac{19}{3k_\mathrm{I}}L} \tag{53}$$

is obtained.

7 | SIMULATION RESULTS AND COMPARISONS TO EXISTING APPROACHES

This section shows simulation results with the proposed approach and compares them to two other techniques for designing discontinuous integral controllers from the literature. In particular, a passivity based approach proposed by Laghrouche et al and output feedback-based approaches proposed by Moreno et al are investigated. For each approach an integral extension of the nominal state-feedback control law

$$u = -k_1 \lfloor x_1 \rceil^{\frac{1}{3}} - k_2 \lfloor x_2 \rceil^{\frac{1}{2}} \tag{54}$$

is considered. The corresponding dilation generator is $\beta = 2$. For the proposed performance preserving technique, the extended control law is derived in Example 2 and is given by (31) in Section 5.3. For the two other approaches, the basic technique and the considered control law are briefly explained in the following.

7.1 | Passivity-based integral extension

Laghrouche et al[5] propose an integral extension technique for homogeneous state-feedback controllers that is constructed using passivity properties. For a given dilation generator β and a given nominal stabilizing feedback k, that control law (with some minor notational adaptations) is given by

$$u = -k(\mathbf{x}) + k_\mathrm{I} v \tag{55a}$$

$$\dot{v} = -\frac{\partial V}{\partial x_n}. \tag{55b}$$

Therein, k_I is a positive design parameter and $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the nominal closed loop formed by the unperturbed plant (8) and the nominal control law $u = -k(\mathbf{x})$. It is constructed by feeding back the passive output $\frac{\partial V}{\partial x_n}$ of the nominal closed loop with storage function V. If the homogeneity degree of V is $\deg_\beta V = \beta$, then the integral part's homogeneity degree is given by $\deg_\beta \frac{\partial V}{\partial x_n} = 0$ and the integrand is discontinuous. Like the performance preserving integral extension presented here, this passivity-based integral extension has the advantage that it guarantees finite-time stability in the unperturbed case for any value of k_I that is positive (or sufficiently large in the perturbed case).

For comparison purposes, this technique is applied to the nominal state feedback (54) with

$$V(x_1,x_2) = 5\left[\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + |x_2|^2\right)^{\frac{5}{4}} + \frac{3k_2}{2}x_1x_2\right]^{\frac{2}{5}}, \tag{56}$$

which is a strict Lyapunov function[15] for $3k_1^{\frac{3}{4}} > 2k_2$. Since $\deg_\beta V = 2 = \beta$, one obtains the following controller with a discontinuous term in the integral:

$$u = -k_1\lfloor x_1\rceil^{\frac{1}{3}} - k_2\lfloor x_2\rceil^{\frac{1}{2}} + k_I v \tag{57a}$$

$$\dot{v} = -\frac{5\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + |x_2|^2\right)^{\frac{1}{4}}x_2 + 3k_2x_1}{\left[\left(\frac{3k_1}{2}|x_1|^{\frac{4}{3}} + |x_2|^2\right)^{\frac{5}{4}} + \frac{3k_2}{2}x_1x_2\right]^{\frac{3}{5}}}. \tag{57b}$$

Similar to (31b), the right-hand side of (57b) is discontinuous in the origin, as can be seen for $x_1 = 0$, for example, from $\dot{v} = -\frac{\partial V}{\partial x_2}(0,x_2) = -5\lfloor x_2\rceil^0$.

7.2 | Output feedback integral extension

The approaches considered up to now apply a discontinuous function of the *full* state to the integrator. Moreno et al[6,7] propose and study discontinuous integral controllers that permit to make this function depend only on the plant's output x_1. This has the advantage of being more useful when the plant's state is not known exactly; the schemes furthermore exploit this to propose pure output feedback integral controllers. These include an observer with continuous right-hand side and unlike the concepts studied in this paper do not require any knowledge of the full state.

The observer-based variants are not considered here, because the additional observer parameters limit the comparability with the other approaches. Therefore, an integral controller with full state feedback is selected for comparison purposes, whose integrand can be chosen to depend only on the plant's output.[6] With some minor notational adaptations it is given by

$$u = -k_1\lfloor x_1\rceil^{\frac{1}{3}} - k_2\lfloor x_2\rceil^{\frac{1}{2}} + k_I v \tag{58a}$$

$$\dot{v} = -\left\lfloor x_1 + k_3\lfloor x_2\rceil^{\frac{3}{2}}\right\rceil^0, \tag{58b}$$

where k_3 is arbitrary and k_1,k_2,k_I are positive design parameters. One can see that the discontinuous function in the integrator depends only on x_1 for $k_3 = 0$, which is selected in the following. Furthermore, compared to the two other control laws in (31) and (57), it is structurally simpler and thus easier to implement. These advantages come at the cost, however, that k_I can not be chosen arbitrarily large nor (in the perturbed case) arbitrarily small, which makes tuning more difficult.

7.3 | Comparisons

For all approaches, the parameters $k_1 = 6$, $k_2 = 3$ are chosen, leading to identical nominal behavior with all approaches for $k_I = 0$ and $w = 0$. Furthermore, $k_3 = 0$ is selected in (58), such that the integrand depends only on the plant's output x_1. Figure 2 compares the output $x_1(t)$ obtained using the three approaches with initial state $\mathbf{x}(0) = [4 \quad 0]^T$, $v(0) = 0$ for different values of k_I. The smallest value $k_I = 0.5$ is chosen such that all three approaches show a similar behavior. The proposed extension tends toward the nominal behavior with growing k_I, thus recovering the performance of the nominal state-feedback controller. The same is true initially for the output feedback approach up to $k_I = 2$, while the performance of the passivity-based approach improves only slightly. Nonetheless, the latter maintains finite-time stability also for arbitrarily large values of k_I, whereas the output feedback approach eventually becomes unstable.

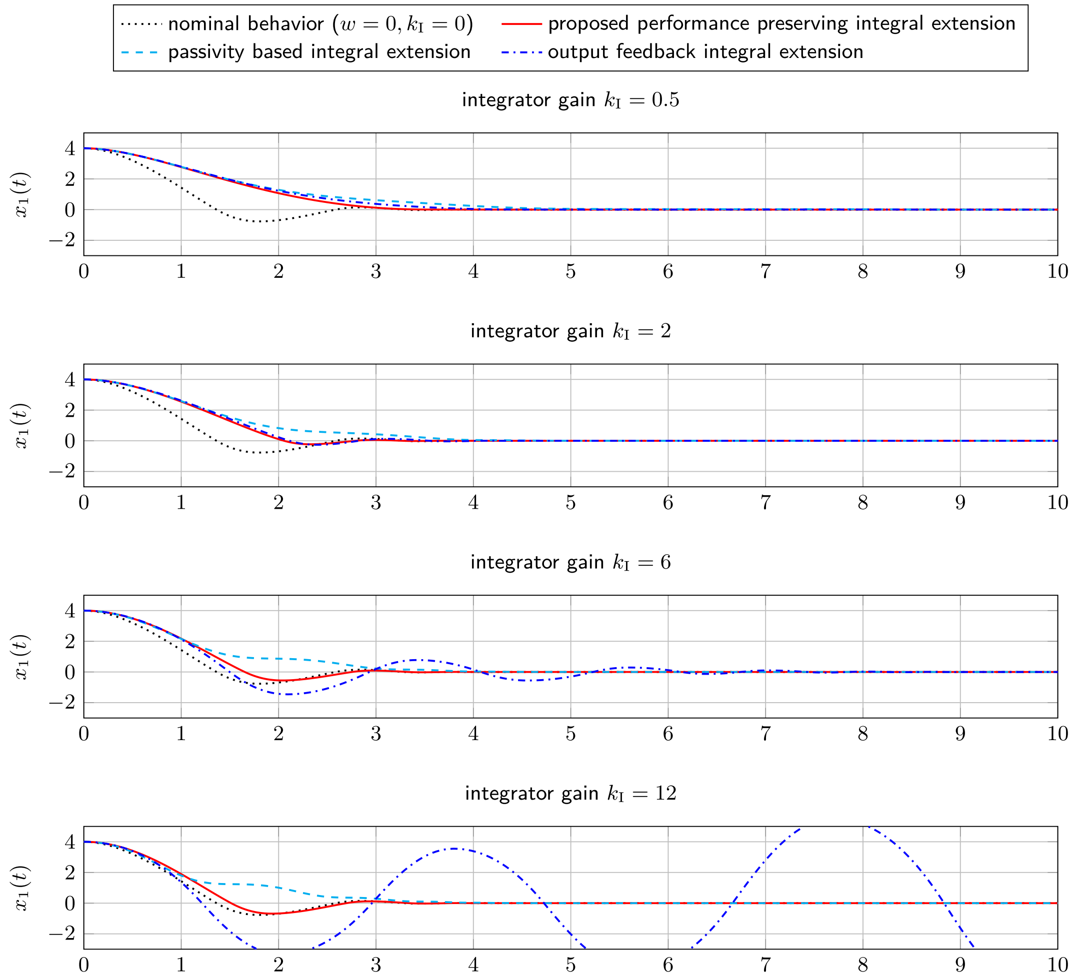

FIGURE 2 Comparison of the proposed performance preserving approach (——), the passivity based approach (- - -), the output feedback approach (-·-·) and the nominal, unperturbed trajectory (.....) for increasing integrator gains $k_I \in \{0.5, 2, 6, 12\}$ from top to bottom, with disturbance $w(t) = 4 + \sin t$, nominal controller parameters $k_1 = 6$, $k_2 = 3$ and initial conditions $\mathbf{x}(0) = [4 \quad 0]^T$ and $v(0) = 0$ [Colour figure can be viewed at wileyonlinelibrary.com]

8 | CONCLUSION

A technique for extending homogeneous state-feedback control laws by an integrator was proposed. It can feature either a continuous or a discontinuous integrand and consequently is able to reject either constant or Lipschitz continuous perturbations, respectively. It furthermore recovers the behavior of the nominal state-feedback controller in the unperturbed case, thus preserving the nominal performance in some sense. The type and speed of closed-loop convergence was shown be be determined by two tuning parameters, a desired homogeneity degree α and the integrator gain k_I, respectively. In particular, it was shown that global asymptotic or finite-time stability may be guaranteed by appropriate selection of α for every sufficiently large value of the gain k_I. Comparisons with two other approaches from literature demonstrated the intuitive tuning of the proposed approach and showed that nominal performance is preserved to some extent also in the perturbed case by increasing the integrator gain. In the future, extending the approach for plants with unknown control coefficient may be studied.

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APPENDIX

Proof of Lemma 1. Since p is strictly positive, the function h given in (23) is well-defined and continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Furthermore, $\lim_{\mathbf{x}\to\mathbf{0}} h(\mathbf{x}) = 0$ holds, because $\deg_\beta q > \deg_\beta p$, that is, the numerator's homogeneity degree exceeds that of the denominator. The partial derivatives of h are given by

$$\frac{\partial h}{\partial x_i} = \frac{p(\mathbf{x})\frac{\partial q}{\partial x_i} - q(\mathbf{x})\frac{\partial p}{\partial x_i}}{p(\mathbf{x})^2} \tag{A1}$$

for $i = 1, \dots, n$. Continuous differentiability of p and q therefore implies that h has the same property on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Finally, (22) and (A1) for $i = n$ imply strict positivity of $\frac{\partial h}{\partial x_n}$ on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, which completes the proof. ■

Proof of Lemma 2. The stabilizing state feedback k applied to the nominal plant (8) with $w = 0$ yields an asymptotically stable closed loop with homogeneity degree $m = -1$. Then (Reference 12, theorem 7.2), this nominal closed loop is finite-time stable and there exist a constant $\tilde{c}_1$ and a continuous, homogeneous, positive definite function $\tilde{V}$ with $\deg_\beta \tilde{V} = \alpha + 1 > -m$, which is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and whose time derivative $\dot{\tilde{V}}$ along the trajectories of the nominal closed loop (32) satisfies

$$\dot{\tilde{V}} \leq -\tilde{c}_1 \tilde{V}^{\frac{\alpha}{\alpha+1}}. \tag{A2}$$

Since $\frac{\partial \tilde{V}}{\partial x_n}$ is continuous on the compact set of all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\tilde{V}(\mathbf{x}) = 1$, it is also bounded on this set and

$$\tilde{c}_2 = \max_{\tilde{V}(\mathbf{x})=1} \left| \frac{\partial \tilde{V}}{\partial x_n} \right| \tag{A3}$$

exists and is finite. Since its homogeneity degree is $\deg_\beta \frac{\partial \tilde{V}}{\partial x_n} = \alpha + 1 - \beta$, one has (Reference 12, lemma 4.2)

$$\left| \frac{\partial \tilde{V}}{\partial x_n} \right| \leq c_2 \tilde{V}^{\frac{\alpha+1-\beta}{\alpha+1}}. \tag{A4}$$

Defining the function V by $V(\mathbf{x}) = \tilde{V}(\mathbf{x})^{\frac{\alpha}{\alpha+1}}$ preserves continuous differentiability for $\mathbf{x} \neq \mathbf{0}$ and yields

$$\dot{V} = \frac{\alpha}{\alpha+1} \tilde{V}^{-\frac{1}{\alpha+1}} \dot{\tilde{V}} \leq -\frac{\tilde{c}_1 \alpha}{\alpha+1} \tilde{V}^{\frac{\alpha-1}{\alpha+1}} = -\frac{\tilde{c}_1 \alpha}{\alpha+1} V^{\frac{\alpha-1}{\alpha}}, \tag{A5}$$

$$\left| \frac{\partial V}{\partial x_n} \right| = \frac{\alpha}{\alpha+1} \tilde{V}^{-\frac{1}{\alpha+1}} \left| \frac{\partial \tilde{V}}{\partial x_n} \right| \leq -\frac{\tilde{c}_2 \alpha}{\alpha+1} \tilde{V}^{\frac{\alpha-\beta}{\alpha+1}} = -\frac{\tilde{c}_2 \alpha}{\alpha+1} V^{\frac{\alpha-\beta}{\alpha}}, \tag{A6}$$

that is, (33) hold with $c_1 = \tilde{c}_1 \alpha (\alpha+1)^{-1}$ and $c_2 = \tilde{c}_2 \alpha (\alpha+1)^{-1}$. ■

Proof of Lemma 3. Consider any closed-loop trajectory in the form of functions $\mathbf{x}(t)$ and $z(t)$ and denote by T its finite convergence time (with $T = \infty$, if no such time exists). It is first shown that zero crossings of $\mathbf{x}(t)$, that is, time instants τ with $\mathbf{x}(\tau) = \mathbf{0}$, are isolated and finite on every compact subinterval of $[0,T)$. At such time instants, one has $\dot{x}_n = z$; if $z(\tau) \neq 0$, then the zero crossing is isolated because x_n is strictly increasing or decreasing, whereas if $z(\tau) = 0$, then the system is already in equilibrium at $t = \tau$ and thus $\tau \geq T$. Therefore, $\mathbf{x}(t)$, $z(t)$ satisfy the differential equations (21) for almost all t on every compact subinterval of $[0,T)$.

The implication (37) for $\alpha < \beta$ is shown first. Using (35), the time derivative of V and ζ_2 along the trajectories of the closed-loop system (21) for $\dot{w} = 0$ is found to be bounded by

$$\begin{aligned} \dot{V} &= Q(\mathbf{x}) + \frac{\partial V}{\partial x_n} z \leq -c_1 V(\mathbf{x})^{\frac{\alpha-1}{\alpha}} + \frac{\partial V}{\partial x_n} z \leq -c_1 V(\mathbf{x})^{\frac{\alpha-1}{\alpha}} + c_2 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \frac{c_1}{2c_2} \zeta_2 \\ &\leq -c_1 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \left(V(\mathbf{x})^{\frac{\beta-1}{\alpha}} - \frac{1}{2}\zeta_2 \right), \end{aligned} \tag{A7a}$$

$$\begin{aligned} \dot{\zeta}_2 &= -\frac{2c_2 k_{\mathrm{I}}}{c_1} \frac{\partial h}{\partial x_n} |z| = -k_{\mathrm{I}} \frac{\partial h}{\partial x_n} \zeta_2 \\ &\leq -k_{\mathrm{I}} c_3 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \zeta_2. \end{aligned} \tag{A7b}$$

Since the implication is trivially fulfilled for $t_0 + \tau \geq T$, considerations may be restricted to a compact subinterval of $[0,T)$. Therefore, the differential inequalities (A7) hold for all but finitely many time instants and they may be integrated in the following.

Assume now that $\zeta_1(t_0) = V(\mathbf{x}(t_0))^{\frac{\beta-1}{\alpha}} \leq D$, $\zeta_2(t_0) \leq D$ holds and observe from (A7b) that ζ_2 is non-increasing. Thus, $\zeta_2(t)$ and consequently also $V(\mathbf{x}(t))^{\frac{\beta-1}{\alpha}}$ cannot exceed D for $t \geq t_0$, because according to (A7a) one has $\dot{V} < 0$ for $V^{\frac{\beta-1}{\alpha}} = D$. Since $\alpha - \beta$ is negative, the differential inequality

$$\dot{\zeta}_2 \leq -k_{\mathrm{I}} c_3 D^{\frac{\alpha-\beta}{\beta-1}} \zeta_2 \tag{A8}$$

is therefore satisfied for $t \geq t_0$, and integrating it yields the inequality

$$\zeta_2(t) \leq \zeta_2(t_0) \exp\left(-k_{\mathrm{I}} c_3 D^{\frac{\alpha-\beta}{\beta-1}} (t - t_0) \right) \leq D \exp\left(-k_{\mathrm{I}} c_3 D^{\frac{\alpha-\beta}{\beta-1}} (t - t_0) \right). \tag{A9}$$

In particular, $\zeta_2(t_0+\tau) \leq 0.5D$ holds for $\tau \geq \tau_{D,1} = \frac{\ln 2}{k_I c_3} D^{\frac{\beta-\alpha}{\beta-1}}$. Consider now $V(\mathbf{x}(t))$ for $t \geq t_1 := t_0 + \tau_{D,1}$, where the differential inequality

$$\dot{V} \leq -c_1 V^{\frac{\alpha-\beta}{\alpha}} (V^{\frac{\beta-1}{\alpha}} - \frac{1}{2}\zeta_2) \leq -\frac{c_1}{2} V^{\frac{\alpha-1}{\alpha}} \tag{A10}$$

is satisfied as long as $\zeta_2(t) < V(\mathbf{x}(t))^{\frac{\beta-1}{\alpha}} = \zeta_1(t)$ holds. Integrating this inequality and using the fact that $\zeta_1(t_1)$ does not exceed D shows that

$$V(\mathbf{x}(t))^{\frac{1}{\alpha}} \leq V(\mathbf{x}(t_1))^{\frac{1}{\alpha}} - \frac{c_1(t-t_1)}{2\alpha} \leq D^{\beta-1} - \frac{c_1(t-t_1)}{2\alpha} \tag{A11}$$

holds if $t \geq t_1$ and $\zeta_1(t) > \zeta_2(t)$. Therefore, by contradiction, $\zeta_1(t_1+\tau) \leq \zeta_2(t_1+\tau)$ holds for all $\tau \geq \tau_{D,2} = \frac{2\alpha}{c_1} D^{\beta-1}$, because the right-hand side of (A11) becomes zero for $t = t_1 + \tau_{D,2}$. Since ζ_2 is nonincreasing, one has $\zeta_1(t_0+\tau) \leq \zeta_2(t_0+\tau) \leq 0.5D$ for all $\tau \geq \tau_D = \tau_{D,1} + \tau_{D,2}$, which proves the implication (37).

It will now be shown that $\overline{V}$ as defined in (36) is nonincreasing and tends to zero for all $\alpha \geq 1$, $\beta > 1$. Computing its time derivative along the trajectories of (21) with $\dot{w} = 0$ yields

$$\dot{\overline{V}} = Q(\mathbf{x}) + \frac{\partial V}{\partial x_n} z - \frac{c_2}{c_3}\frac{\partial h}{\partial x_n}|z| \leq -c_1 V(\mathbf{x})^{\frac{\alpha-1}{\alpha}} + c_2 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}}|z| - \frac{c_2}{c_3}\frac{\partial h}{\partial x_n}|z| \leq -c_1 V(\mathbf{x})^{\frac{\alpha-1}{\alpha}}. \tag{A12}$$

Since this differential inequality holds for all but finitely many time instants on every compact subinterval of $[0,T)$, one may integrate it to see that $\overline{V}$ is nonincreasing on every such subinterval. Using continuity of $\overline{V}$ one then concludes that it is nonincreasing on $[0,T]$ and, because $\mathbf{x}(t) = \mathbf{0}$ and $z(t) = 0$ for $t \geq T$, also on $[0,\infty)$.

Since $\overline{V}(t)$ and $\zeta_2(t)$ are nonnegative and nonincreasing, they tend to constant limits as t tends to infinity. Therefore, also $V(\mathbf{x}(t))$ tends to a constant $c_4 = \lim_{t\to\infty} V(\mathbf{x}(t)) \geq 0$. It will be shown that this limit is zero, which implies that also $z(t)$ and therefore $\overline{V}(t)$ tends to zero, because $\mathbf{x}=\mathbf{0}$, $z \neq 0$ is not an equilibrium of the closed loop (17). To show $c_4 = 0$, assume to the contrary that $c_4 > 0$, which implies $T = \infty$ and $V(\mathbf{x}(t)) \geq c_4$ for all t. If $\alpha < \beta$, repeatedly applying the implication (37) eventually leads to a contradiction. Thus $\alpha \geq \beta$ may be assumed and one can conclude from (A7b) that

$$\dot{\zeta}_2 \leq -k_I c_3 V(\mathbf{x})^{\frac{\alpha-\beta}{\alpha}} \zeta_2 \leq -k_I c_3 c_4^{\frac{\alpha-\beta}{\alpha}} \zeta_2 \tag{A13}$$

holds for all but countably many, isolated time instants on $[0,\infty)$. Therefore, $\zeta_2(t)$ tend to zero exponentially for $t \to \infty$ and one can see from (A7a) that $V(\mathbf{x}(t))$ then also tends to zero. This contradicts $V(\mathbf{x}(t)) \geq c_4$ for sufficiently large t, which concludes the proof. ■

Proof of Lemma 4. Consider any trajectory $\mathbf{x}(t)$ and $q(t)$ of the closed loop (17) and denote by T its convergence time (with $T = \infty$, if no such time exists). In order to show that $T \leq \overline{T}$, assume to the contrary that $T > \overline{T}$. Then, as argued in the proof of Lemma 3, there are only a finite number of time instants t in the interval $[0, \overline{T}]$ where $\mathbf{x}(t) = \mathbf{0}$. Thus, one may integrate the differential inequality $\dot{\overline{V}} \leq -D$ on this interval to obtain

$$\overline{V}(\mathbf{x}(\overline{T}), q(\overline{T})) \leq \overline{V}(\mathbf{x}(0), q(0)) - D\overline{T} = 0 \tag{A14}$$

and thus $\mathbf{x}(\overline{T}) = \mathbf{0}$, $q(\overline{T}) = 0$, which yields the contradiction $T \leq \overline{T}$. In a similar way one may show by contradiction that $\overline{V}$ is nonincreasing along every closed-loop trajectory. Radial unboundedness of $\overline{V}$ then implies Lyapunov stability and thus finite-time stability. ■