Proc. Annual Meeting of the Austrian Society for Biomedical Engineering 2021

DOI: 10.3217/978-3-85125-826-4-24

ANALYSIS OF A MATHEMATICAL MODEL OF THE SPREAD OF THE SARS-COV-2 PANDEMIC DETERMINED BY THE TRANSFER MATRIX

J. Lozanović Šajić[1,2], S. Langthaler[1], S. Stoppacher[1], C. Baumgartner[1]

[1] Institute of Health Care Engineering with European Testing Center of Medical Devices, Graz University of Technology, Austria

[2] Innovation Center of the Faculty of Mechanical Engineering, University of Belgrade, Serbia

j.lozanovicsajic@tugraz.at

Abstract — *This paper presents the development of a system theory-based model of the COVID-19 pandemic spread based on weekly reports available on the EU Open Data Portal (EUODP). The considered mathematical model will be represented in the S complex domain as a transfer matrix. System identification methodology, well known in control system theory, was applied. Stability analysis with possibilities of controllability and observability was considered. The spread of a pandemic can be controlled by proportional (P) action in an open loop.*

Keywords — *SARS-CoV-2, transfer matrix, transfer function, mathematical modeling, system identification*

Introduction

The spread of the SARS-Cov-2 pandemic can be described as a mathematical model of a system that is a unique function that maps the input vector to the output vector according to the mathematical law. The system model is an idealized, imaginary system, which retains the properties of the real system essential for system analysis [1]. Having in mind the definitions of the system [1], [2], the spread of the coronavirus pandemic can be considered as a dynamic model.

A large number of mathematical models of disease spread can be found in [3-8] with the aim to predict the pandemic's next moves. However, available models [9-14] are based on data representing the output function.

Mathematical modeling of a biological system based on system theory and control system engineering concepts however enable to determine and characterize the system model of the spreading pandemic by considering various aspects of the system including stability, observability, and controllability.

Methods

As known from systems dynamics and control theory, a dynamic system can be described by a behavioral differential equation which can be stochastic and deterministic [1]. The assumption is introduced that the spread of a pandemic system can be described as a dynamic system.

In previous considerations [15], a so-called SISO (single input – single output) system was considered. A similar approach is applied in this work, however, the system here will be defined as a multiple transfer system. Such a system is described by one input and two outputs (SIMO – single input – multiple outputs), that is why it is necessary to define the transfer matrix of the system instead of the transfer function.

The behavioral differential equation of the system with multiple inputs and multiple outputs is defined according to [16], [17]

$$\sum_{k=0}^{l} \mathbf{A}_k \mathbf{Y}(t) = \sum_{k=0}^{m} \mathbf{B}_k \mathbf{U}(t)\,, m \leq l \tag{1}$$

where $A \in R^{N \times N}$ and $B \in R^{N \times M}$ are matrices with constant coefficients, and l, m the highest derivations that occur between output and input variables. Based on the ordinary differential equation of the system behavior, the transfer matrix of a system is defined as

$$\mathbf{G}(s) = \begin{pmatrix} G_{11} & \cdots & G_{1M} \\ \vdots & \ddots & \vdots \\ G_{N1} & \cdots & G_{NM} \end{pmatrix} \tag{2}$$

where N is the dimension of the output vector and M is the dimension of the input vector.

For the considered system describing the spread SARS-CoV-2 virus we assumed a system with one input and two outputs. The general transfer matrix is:

$$\mathbf{G}(s) = \begin{pmatrix} G_{11}(s) \\ G_{21}(s) \end{pmatrix} \tag{3}$$

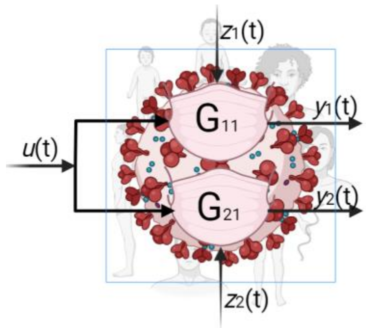

Figure 1. Illustrated block diagram of the system.

A block diagram of the system is given in Figure 1 based on Equation 3. Following Figure 1, $u(t)$ is the input variable, $\mathbf{y}(t) = (y_1(t) \quad y_2(t))^T$ is the output

Published by Verlag der TU Graz
Graz University of Technology

DOI: 10.3217/978-3-85125-826-4-24

vector and $\boldsymbol{z}(t) = (z_1(t) \quad z_2(t))^T$ is the disturbance vector. The disturbance vector can also be defined as an input vector, but can also be considered separately. Note that the disturbance vector is needed to be defined according to the definition of the input of a system [16]. For modeling the spreading pandemic, disturbances were not taken into consideration.

To determine the transfer matrix of the system, the concept of system identification that has been widely used in the field of control system engineering was applied. If it is not possible to determine a mathematical model based on physical laws, e.g. Newton's law, Bernoulli's equation, Kirchhoff's law, etc. the system identification methodology can be subsequently applied in case of a known response of the system, represented by the input and output vectors.

However, there are different approaches for system identification including parametric, nonparametric, linear, nonlinear, stochastic or deterministic modeling concepts. In general, the identification of a system is performed according to a flow chart with multiple key elements in the system identification cycle as defined in [18]. An adapted flow chart for this process is given in Figure 2.

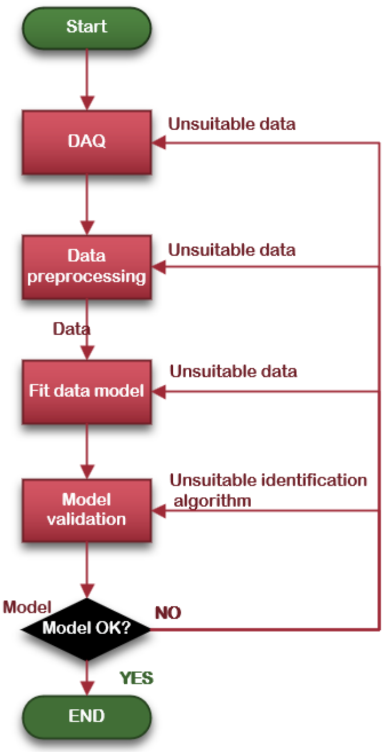

Figure 2. Adapted flow chart in system identification of the spread of SARS-CoV-2.

To determine the mathematical model of the system (spread of SARS-CoV-2) we used publicly collected data available on the EU Open Data Portal [19]. In a previous study pre-published in [15], mathematical models were provided based on data collected daily. It should be noted that in one period (14.12.2020 to 11.03.2021), the availability of data changed from a daily to weekly periods, as countries have begun to adopt anti-pandemic strategies on a two-week basis. Therefore, in this paper we present models based on the number of infected and dead persons weekly. Observed from the aspect of control theory, the sampling time was finally defined to be one week.

The basic hypothesis of this work was to define and predict the input vector, i.e. the number of new cases on a weekly level based on the output variable. In our recent work [15] we were able to demonstrate that the methodology can be applied similarly for different countries when the sampling time is daily. Model validation therein was performed for multiple countries such as Austria, Italy, Germany and Serbia. In this work, vectors were defined on a weekly basis, sample rate was set to one week, and results for Austria and Germany are presented in more detail.

Results

Models for Austria and Germany were exemplarily developed and evaluated in order to determine whether the used methodology can be applied with regard to a diminished (weekly) sampling time by maintaining a sufficient prediction accuracy.

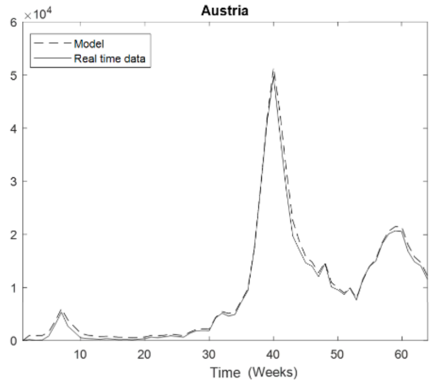

Figure 3. Identified transfer function model in relation to the number of new cases in Austria

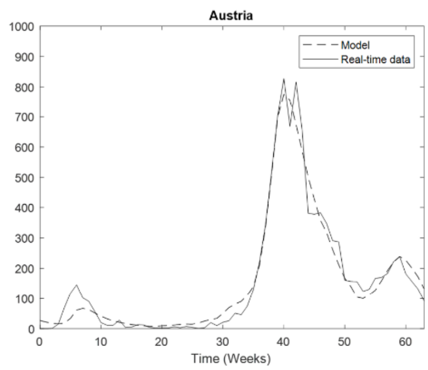

Figure 4. Identified transfer function model in relation to the number of deaths in Austria

In accordance with the definition of a behavioral differential equation, the ordinary differential equation (ODE) should be of lowest order, fully describing the

 DOI: 10.3217/978-3-85125-826-4-24

dynamic characteristics of the system. Therefore, we decided to adopt a second-order system based on the conducted analysis for systems from 1st to 5th order.

The spread of pandemic was assumed as a continuous-time model, and therefore a continuous-time transfer matrix was identified. Parameterization of the model included that each transfer function has two poles and one zero, the number of free coefficients is 4. The transfer matrix of the system $\mathbf{G}(s)$, which responses are shown in Figures 3 and 4, is:

$$\mathbf{G}(s) = \begin{pmatrix} G_{11}(s) \\ G_{21}(s) \end{pmatrix} = \begin{pmatrix} \frac{17{,}53s+17{,}71}{s^2+10{,}59s+18{,}85} \\ \frac{0.007816s-0.0001566}{s^2+0{,}3026s+0{,}000003505} \end{pmatrix} \quad (4)$$

Transfer matrices for other countries can be determined similarly. Figures 5 and 6 show that the transfer matrix for Germany with a ten times higher population than Austria can also be defined as a system of second-order in relation to the number of new cases and deaths.

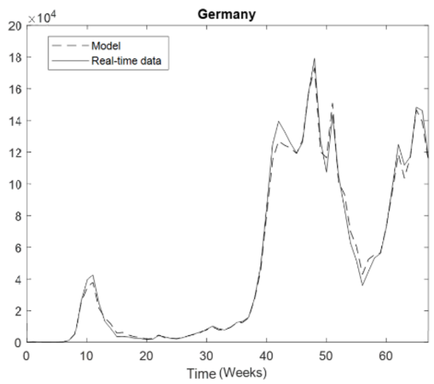

Figure 5. Identified transfer function model in relation to the number of new cases in Germany.

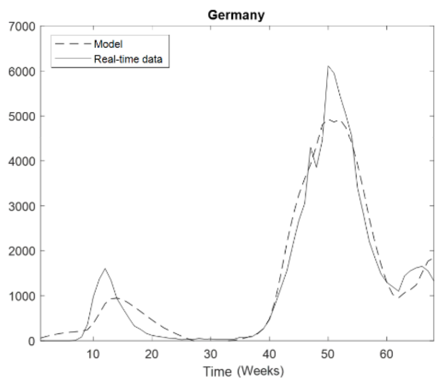

Figure 6. Identified transfer function model in relation to the number of deaths in Germany.

The transfer matrix determined for Germany, in relation to the number of new infected and the number of deaths as shown in Figures 5 and 6, is given by Eq. 5.

$$\mathbf{G}(s) = \begin{pmatrix} G_{11}(s) \\ G_{21}(s) \end{pmatrix} = \begin{pmatrix} \frac{6{,}397s+8{,}704}{s^2+2{,}787s+9{,}023} \\ \frac{0.007961s+0.0001296}{s^2+0{,}1518s+0{,}01323} \end{pmatrix} \quad (5)$$

The slight time delay of the model simulations observed in Figures 4 and 6 can be explained by effects of the disturbance and input vector. It should be noted that for both countries the spread of the pandemic can be determined by the transfer matrix which transfer functions are of second order. Note that the coefficients are different as a consequence of different values of the input and output vectors. If other countries were analyzed similarly, transfer matrices would be determined with appropriate transmission functions of second order.

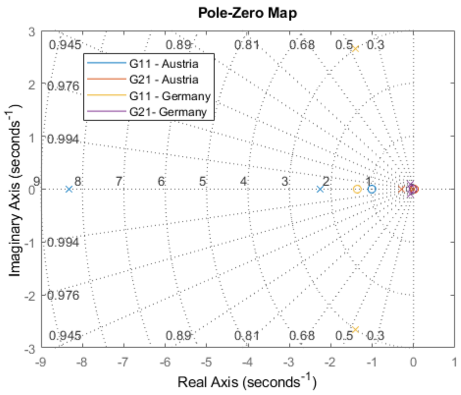

Figure 7. Pole-Zero Map for Austria and Germany.

Based on the pole distribution of transfer matrices for both systems as shown in Figure 7, it is reasonable to evaluate controllability and observability, because the systems of both countries are concluded to be stable. The same conclusion about stability can be made from the analysis of transient responses of the systems. For both systems, controllability and observability matrices were determined, and pandemics in both countries were observable and controllable, which could also be confirmed for the other countries.

When transfer matrices are known, behavioral differential equations are also known. By applying the inverse Laplace transform, the time responses of the systems are obtained, which also represent the solutions of the behavioral differential equations of those systems.

Discussion

By applying the system identification methodology from control theory, it is possible to determine a mathematical model of pandemic spread, as demonstrated on the example of SARS-CoV-2. Linear models of a second-order system at a satisfying level can be used to describe mathematical models representing a dynamic system. Deviations of the actual values from the model can be explained by the effect of disturbance on the system.

Theoretically it is possible to consider different types of control algorithms, e.g., predictive control model,

DOI: 10.3217/978-3-85125-826-4-24

natural tracking control, exponential tracking control or fuzzy control so that in the future, in the case of similar epidemics, epidemic control can be exerted.
The models which are applied for both countries from the aspect of system control theory are represented by an application which contains a proportional gain of the control system. The system is considered as an open-loop control system without compensation of the effects of disturbances. A lockdown represents the application of a P (proportional) action in the open loop of controlling the spreading of the disease (see Figure 8).

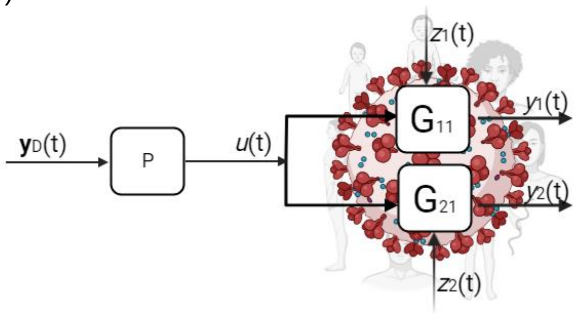

Figure 8. Control in open-loop with P-action.

Models were able to be determined even though the diminished weekly sampling time and provide sufficient information "on the basis of which" it is possible to predict the further spread of the pandemic since the mathematical model of the system is known. One of the conditions that need to be considered in further research is the impact of the number of vaccinated persons and the determination of mathematical models of pandemic spread, considering the data of the population.

References

[1] Lazic D. Nelinearni sistemi (Nonlinear systems), Faculty of Mechanical Engineering, University of Belgrade, 2005
[2] Sayama H. Introduction to the Modeling and Analysis of Complex Systems, Binghamton University, State University of New York, OpenSUNY, 2019
[3] Giordano G, Blanchini F, Bruno R. et al. Modelling the COVID-19 epidemic and implementation of population-wide interventions in Italy. Nat Med 26, 855–860, 2020 https://doi.org/10.1038/s41591-020-0883-7
[4] Chang SL, Harding N, Zachreson C. et al. Modelling transmission and control of the COVID-19 pandemic in Australia. Nat Commun 11, 5710 2020, doi.org/10.1038/s41467-020-19393-6
[5] Kyrychko YN, Blyuss KB & Brovchenko I. Mathematical modelling of the dynamics and containment of COVID-19 in Ukraine. Sci Rep 10, 19662, 2020, doi.org/10.1038/s41598-020-76710-1
[6] Ahmed I, Modu GU, Yusuf A, Kumam P, Yusuf I. A mathematical model of Coronavirus Disease (COVID-19) containing asymptomatic and symptomatic classes. Results Phys. 2021, doi: 10.1016/j.rinp.2020.103776
[7] Zeb A, Alzahrani E, Suat Erturk V, Zaman G, Mathematical Model for Coronavirus Disease 2019 (COVID-19) Containing Isolation Class, BioMed Research International, vol. 2020, Article ID 3452402, 7 pages, 2020 https://doi.org/10.1155/2020/3452402
[8] Logeswari K, Ravichandran C, Sooppy Nisar K, Mathematical model for spreading of COVID-19 virus with the Mittag–Leffler kernel, Numerical Methods for Partial Differential equations, ISSN:1098-2426, 2020, doi.org/10.1002/num.22652
[9] Ivorra B, Ferrández MR, Vela-Pérez M, Ramos AM. Mathematical modeling of the spread of the coronavirus disease 2019 (COVID-19) taking into account the undetected infections. The case of China. Communications in Nonlinear Science and Numerical Simulation, 2020
[10] Gostic K, Gomez ACR, Mummah RO, Kucharski AJ, Lloyd-Smith JO. Estimated effectiveness of symptom and risk screening to prevent the spread of COVID-19, Gostic et al. eLife 2020, doi: doi.org/10.7554/eLife.55570
[11] Pal D, Ghosh D, Santra PK, Mahapatra GS. Mathematical Analysis of a COVID-19 Epidemic Model by using Data Driven Epidemiological Parameters of Diseases Spread in India, doi.org/10.1101/2020.04.25.20079111, Preprint
[12] Din A, Li Y, Khan T, Zaman G. Mathematical analysis of spread and control of the novel corona virus (COVID-19) in China, Chaos, Solitons & Fractals Volume 141, December 2020
[13] Atangana A & Araz SI. Mathematical model of COVID-19 spread in Turkey and South Africa: theory, methods, and applications, Advances in Difference Equations volume 2020, Article number: 659, 2020
[14] Dehning J, Zierenberg J, Spitzner FP, Wibral M, Neto JP, Wilczek M, Priesemann V, Inferring change points in the spread of COVID-19 reveals the effectiveness of interventions. Science 369, 160, 2020
[15] Lozanovic Sajic J, Langthaler S, Stoppacher S, Baumgartner C. The Transfer Function of the SARS-CoV-2 Spread in Different Countries with Stability, Observability, and Controllability Analysis, doi.org/10.21203–/rs.3.rs-514785/v1, May 2021, Preprint
[16] Lazic D. Osnove automatskog upravljama, Univerzitet u Beogradu, Masinski fakultet, 2015
[17] Ogata K. Modern Control Engineering, Prentice Hall, NJ, 2009
[18] Verhaegen M and Verdult V. Filtering and System Identification: A Least Squares Approach, ISBN:9780511618888, 2007
[19] EU Open Data Portal, data.europa.eu, accessed on 24.05.2021

Published by Verlag der TU Graz
Graz University of Technology